Sobolev space theory for the elliptic and parabolic equations with the fractional Laplacian on $C^{1,1}$ open sets

Kyeong-Hun Kim

Korea University, South Korea

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Joint work with Jae-Hwan Choi and Junhee Ryu

Outline

I will introduce a version of Krylov's weighted Sobolev space theory introduced in the article

- Krylov, 1999 : Weighted Sobolev spaces and Laplace's equation and the heat equations in a half space

with $\Delta^{\alpha/2}$, in place of Δ, on $C^{1,1}$ open sets.

Krylov's result: Let

$$\rho(x) = \mathrm{dist}(x, \mathbb{R}^d_+), \quad L_{p,\theta} = L_p(\mathbb{R}^d_+, \rho^{\theta-d}dx), \quad \mathbb{L}_{p,\theta}(T) = L_p([0,T]; L_{p,\theta}).$$

Krylov, for instance, proved sharp $L_{p,\theta}$ and $\mathbb{L}_{p,\theta}(T)$-regularity of

$$\rho^{-1}u, \quad Du, \quad \rho D^2u, \cdots, \rho^n D^{n+1}u, \cdots$$

for the elliptic and parabolic equations respectively.
In particular, for the parabolic equation

$$u_t = \Delta u + f, \quad x \in \mathbb{R}^d_+, \ t > 0; \quad u(0,\cdot) = 0,$$

Krylov proved

$$\|\rho^{-1}u\|_{\mathbb{L}_{p,\theta}(T)} + \|Du\|_{\mathbb{L}_{p,\theta}(T)} + \|\rho D^2u\|_{\mathbb{L}_{p,\theta}(T)} \leq C\|\rho f\|_{\mathbb{L}_{p,\theta}(T)},$$

provided that

$$d-1 < \theta < d-1+p.$$

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Our results ?

(i) Elliptic equation : For $\lambda \geq 0$, $\alpha \in (0,2)$,

$$\begin{cases} \Delta^{\alpha/2} u - \lambda u = f, & x \in D, \\ u = 0, & x \in D^c \end{cases}$$

(ii) (Deterministic) Parabolic equation :

$$\begin{cases} \partial_t u = \Delta^{\alpha/2} u + f, & (t,x) \in (0,T) \times D, \\ u(0,x) = u_0(x), & x \in D, \\ u(t,x) = 0, & (t,x) \in [0,T] \times D^c \end{cases}$$

(iii) Parabolic SPDE

Remark. Problems are considered on open sets, not only on domains.

Here, for $f \in C_c^\infty(\mathbb{R}^d)$, the fractional Laplacian $\Delta^{\alpha/2} := -(-\Delta)^{\alpha/2}$ is defined by

$$\Delta^{\alpha/2} f(x) = c(d,\alpha) \int_{\mathbb{R}^d} \frac{f(x+y) - f(x) - \nabla f(x) \cdot y 1_{|y| \le 1}}{|y|^{-d-\alpha}} dy.$$

Equivalently,

$$\begin{aligned} \Delta^{\alpha/2} f &= -\mathcal{F}^{-1}(|\xi|^\alpha \mathcal{F}(f)) \\ &= \lim_{t \to 0} \frac{\mathbb{E} f(x + X_t) - f(x)}{t}. \end{aligned}$$

Here X_t is a rotationally symmetric α-stable process, that is $\mathbb{E} e^{iX_t \cdot \xi} = e^{-t|\xi|^\alpha}$.

1. Bogdan et al. (2009). Define

$$u(x) = \begin{cases} (1-|x|^2)^{\alpha/2-1} & : x \in B_1(0) \\ 0 & : |x| > 1 \end{cases}$$

Then u satisfies

$$\begin{cases} \Delta^{\alpha/2} u = 0, & x \in B_1(0), \\ u = 0, & |x| > 1 \end{cases}$$

Remark. 1. Note $\lim_{|x|\uparrow 1} u(x) = \infty$.

2. Some control on the behavior of solution near ∂B or on $\mathbb{R}^d$ is needed for the uniqueness of the problem.

2. For some $c > 0$,

$$u(x) := c(1 - |x|^2)^{\alpha/2} \text{ in } B_1,$$

satisfies

$$\begin{cases} \Delta^{\alpha/2} u(x) = 1, & x \in D, \\ u(x) = 0, & x \in D^c. \end{cases}$$

Remark. u is in $C^{\alpha/2}$, but not in C^{γ} for any $\gamma > \alpha/2$.

3. Ros-Ston and Serra (2014):
If D is bounded, $f \in L_\infty(D)$, and $u \in H_2^{\alpha/2}(\mathbb{R}^d)$ satisfies

$$\begin{cases} \Delta^{\alpha/2} u = f, & x \in D, \\ u = 0, & x \in D^c, \end{cases}$$

then

$$\|u\|_{C^{\alpha/2}(\mathbb{R}^d)} \le C\|f\|_{L_\infty(D)}.$$

Remark. 1. If $\alpha = 2$, then we only have $u \in C^{2-\varepsilon}$ for any $\varepsilon > 0$.

2. Higher order interior H"older estimate is also proved

$$|u|^{(-\alpha/2)}_{\beta+\alpha;D} \le C|u|_{C^{\alpha/2}(\mathbb{R}^d)} + C|f|^{(\alpha/2)}_{\beta;D}, \quad \beta > 0.$$

Definition : We say $u(t,x)$ is a weak solution to the problem

$$\begin{cases} \partial_t u = \Delta^{\alpha/2} u + f, & (t,x) \in (0,T) \times D, \\ u(0,x) = u_0(x), & x \in D, \\ u(t,x) = 0, & (t,x) \in [0,T] \times D^c, \end{cases}$$

if (a) $u = 0$ *a.e.* in $[0,T] \times D^c$,
(b) $(u(t,\cdot),\phi)_{\mathbb{R}^d}$ and $(u(t,\cdot),\Delta^{\alpha/2}\phi)_{\mathbb{R}^d}$ exist for any $t \leq T$ and $\phi \in C_c^\infty(D)$,
(c) for any $\phi \in C_c^\infty(D)$ the equality

$$(u(t,\cdot),\phi)_{\mathbb{R}^d} = (u_0,\phi)_D + \int_0^t (u(s,\cdot),\Delta^{\alpha/2}\phi)_{\mathbb{R}^d} ds + \int_0^t (f(s,\cdot),\phi)_D ds$$

holds for all $t \leq T$.

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holds for all $t \leq T$.

Remark. If $\phi \in C_c^\infty(D)$, then

$$\|\Delta^{\alpha/2}\phi\|_{H^\gamma_{p,\theta+\alpha p/2}(D)} \leq C\|\phi\|_{H^{\gamma+\alpha}_{p,\theta-\alpha p/2}(D)}$$

provided that $d-1-\alpha p/2 < \theta < d-1+p+\alpha p/2$. Consequently, $(u,\Delta^{\alpha/2}\phi)_D < \infty$ if $u \in L_p(D,\rho^{\theta-d-\alpha p/2}dx)$.

Let X_t be a rotationally symmetric α-stable process, and

$$\tau_D = \tau_D^x := \inf\{t \geq 0 : x + X_t \notin D\}, \quad \text{first exit time.}$$

We add an element, called a cemetery point, $\partial \notin \mathbb{R}^d$ to $\mathbb{R}^d$, and define the killed process of X upon D by

$$X_t^D = X_t^{D,x} := \begin{cases} x + X_t & t < \tau_D, \\ \partial & t \geq \tau_D. \end{cases}$$

The process is forced to be killed or ignored once it leaves D.
Let $p^D(t,x,y)$ be the transition density of X^D, i.e., for any Borel set $B \subset \mathbb{R}^d$,

$$\mathbb{P}_x(X_t^D \in B) = \int_B p^D(t,x,y)dy.$$

For functions f on $\mathbb{R}^d$, we extend it by $f(\partial) := 0$. Then,

$$\mathbb{E}_x f(X_t^D) = \mathbb{E}_x f(X_t; t < \tau_D) = \int_D p^D(t,x,y)f(y)dy.$$

Denote

$$L_{p,\theta}(D) = L_p(D, \rho^{\theta-d}dx), \quad \mathbb{L}_{p,\theta}(D,T) = L_p([0,T]; L_{p,\theta}(D)).$$

Existence and Uniqueness

Let $p > 1, \theta \in (d-1, d-1+p), f \in \mathbb{L}_{p,\theta+\alpha p/2}(D,T),\ u_0 \in L_{p,\theta-\alpha p/2+\alpha}(D)$.
(i) The function

$$u(t,x) := \int_D p^D(t,x,y)u_0(y)dy + \int_0^t \int_D p^D(t-s,x,y)f(s,y)dyds$$

belongs to $\mathbb{L}_{p,\theta-\alpha p/2}(D,T) \cap \{u = 0 \text{ on } [0,T]\times D^c\}$.
(ii) It is the unique weak solution in this function space.
(iii) We have

$$\|\rho^{-\alpha/2}u\|_{\mathbb{L}_{p,\theta}(D,T)} \leq C(\|\rho^{\alpha/2}f\|_{\mathbb{L}_{p,\theta}(D,T)} + \|\rho^{-\alpha/2+\alpha/p}u_0\|_{L_{p,\theta}(D)}),$$

where C is independent of u and T.

Remark. 1. $u(t,x) = \mathbb{E}_x u_0(X_t^D) + \int_0^t \mathbb{E}_x f(s, X_{t-s}^D)ds$.

2. $d-1 < \theta < d-1+p$ is sharp

For $n = 0, 1, 2, \cdot$, let

$$H^n_{p,\theta}(D) := \{u : u, \rho Du, \cdots, \rho^n D^n u \in L_{p,\theta}(D)\}.$$

Actually, the space $H^\gamma_{p,\theta}(D)$ can be defined for any $\gamma \in \mathbb{R}$. By $B^\gamma_{p,\theta}(D)$ we denote the corresponding weighted Besov space.

Regularity of arbitrary order derivative

Let ψ be a smooth function such that $\psi \sim \rho$. For any $\gamma \in \mathbb{R}$ and $d-1 < \theta < d-1+p$,

$$\|\psi^{-\alpha/2} u\|_{\mathbb{H}^{\gamma+\alpha}_{p,\theta}(D,T)} \leq C \left(\|\psi^{\alpha/2} f\|_{\mathbb{H}^{\gamma}_{p,\theta}(D,T)} + \|\psi^{-\alpha/2+\alpha/p} u_0\|_{B^{\gamma+\alpha-\alpha/p}_{p,\theta}(D)} \right),$$

where $\mathbb{H}^\nu_{p,\theta}(D,T) = L_p([0,T]; H^\nu_{p,\theta}(D))$.

In particular, if $\gamma = 0$,

$$\|\rho^{-\alpha/2} u\|_{\mathbb{L}_{p,\theta}(D,T)} + \|\rho^{\alpha/2}\Delta^{\alpha/2} u\|_{\mathbb{L}_{p,\theta}(D,T)} \leq C \left(\|\rho^{\alpha/2} f\|_{\mathbb{L}_{p,\theta}(D,T)} + \|\rho^{-\alpha/2+\alpha/p} u_0\|_{B^{\alpha-\alpha/p}_{p,\theta}(D)} \right),$$

where $\mathbb{L}_{p,\theta}(D,T) := L_p([0,T]; L_{p,\theta}(D))$.

Let $u_t = \Delta u + f, \quad t > 0,\ x \in D; \quad u(0,x) = 0.$

1. Localization argument: If $\zeta \in C_c^\infty(D)$, then $u\zeta$ can be viewed as a function on $\mathbb{R}^d$, and

$$
\begin{aligned}
(u\zeta)_t &= \zeta \Delta u + \zeta f \\
&= \Delta(u\zeta) + f\zeta + [\zeta\Delta u - \Delta(u\zeta)] \\
&= \Delta(u\zeta) + f\zeta - [\nabla\zeta \cdot \nabla u + u\Delta\zeta] \quad \text{: cancellation of highest derivative}
\end{aligned}
$$

Using results on $\mathbb{R}^d$, we can control $\Delta(u\zeta)$ in terms of norm of lower order term, that is, $[\nabla\zeta \cdot \nabla u + u\Delta\zeta]$.

- Krylov used this argumet to prove, for instance,

$$\|\rho D^2 u\|_{\mathbb{L}_{p,\theta}(T)} \leq C\|Du\|_{\mathbb{L}_{p,\theta}(T)} + C\|\rho^{-1}u\|_{\mathbb{L}_{p,\theta}(T)} + C\|\rho f\|_{\mathbb{L}_{p,\theta}(T)}.$$

- With fractional Laplacian, one has to detal with

$$\zeta\Delta^{\alpha/2}u - \Delta^{\alpha/2}(u\zeta).$$

2. Integration by parts : by chain rule $\partial_t |u|^p = p|u|^{p-2} u u_t$

$$0 \leq |u(T,x)|^p = p \int_0^T |u|^{p-2} u(\Delta u + f) ds.$$

Appropriate integration by parts yields

$$\|\rho^{-1} u\|_{\mathbb{L}_{p,\theta}(T)} \leq C \|\rho f\|_{\mathbb{L}_{p,\theta}(T)}.$$

- Difficult to apply for $\Delta^{\alpha/2}$.

Key steps

Step 1. (Localization argument) If $u \in \mathbb{L}_{p,\theta-\alpha p/2}(D,T) \cap \{u = 0 : x \in D^c\}$ is a solution, then for any $\gamma \in \mathbb{R}$,

$$\begin{aligned} \|\psi^{-\alpha/2}u\|_{\mathbb{H}^{\gamma+\alpha}_{p,\theta}(D,T)} &\leq C\|\psi^{-\alpha/2}u\|_{\mathbb{L}_{p,\theta}(D,T)} \\ &+ C\|\psi^{\alpha/2}f\|_{\mathbb{H}^{\gamma}_{p,\theta}(D,T)} + C\|\psi^{-\alpha/2+\alpha/p}u_0\|_{B^{\gamma+\alpha-\alpha/2}_{p,\theta}(D)} \end{aligned}$$

Step 2. For $u := \mathbb{E}_x u_0(X^D_t) + \int_0^t \mathbb{E}_x f(s, X^D_{t-s})ds$, we have

$$\|\rho^{-\alpha/2}u\|_{\mathbb{L}_{p,\theta}(D,T)} \leq C\left(\|\rho^{\alpha/2}f\|_{\mathbb{L}_{p,\theta}(D,T)} + \|\rho^{-\alpha/2+\alpha/p}u_0\|_{L_{p,\theta}(D)}\right).$$

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Remark : Uniquenss ? : For $v \in C^\infty_c((0,T)\times D)$, we have (X. Zhang, 2012)

$$v(t,x) = \int_0^t \mathbb{E}_x f(s, X^D_{t-s})ds, \quad f := \partial_t v - \Delta^{\alpha/2}v.$$

Denote $d_x = \text{dist}(x, \partial D)$ if $x \in D$, and $d_x = 0$ otherwise. Denote

$$R_{t,x} = \frac{d_x^{\alpha/2}}{\sqrt{t} + d_x^{\alpha/2}}.$$

If D is $C^{1,1}$, then (Bogdan et al, 2014)

$$p^D(t,x,y) \leq C R_{t,x} R_{t,y} p(t, x-y),$$

where $p(t, x-y)$ is the transition density of X_t. By Hölder inequality,

$$\begin{aligned}\left|\int_D p^D(t, x-y) u_0(y) dy\right| \leq & N \left(\int_D p(t,x-y) d_y^{-\alpha\beta_1 p'/2} R_{t,y}^{(1-\beta_2)p'} dy\right)^{1/p'} \\ & \times \left(\int_D p(t,x-y) d_y^{\alpha\beta_1 p/2} R_{t,x}^p R_{t,y}^{\beta_2 p} |u_0(y)|^p dy\right)^{1/p}\end{aligned}$$

for any $\beta_1, \beta_2 \in \mathbb{R}$. We used $1 = d_y^{-1} d_y$ and $1 = R_{t,y}^{-1} R_{t,y}$. Then, we make good choice of β_1, β_2, multiply by $d_x^{\theta - d - \alpha p/2}$ to both sides, then integrate over $D \times [0, T]$.

Idea of Step 2: continued

In this ways, following computations by Nazarov and Kozlov, we prove

$$\left\|\rho^{-\alpha/2}(x)\int_D p^D(t,x-y)u_0(y)dy\right\|_{\mathbb{L}_{p,\theta}(D,T)} \leq C\|\rho^{-\alpha/2+\alpha/p}u_0\|_{L_{p,\theta}(D)}.$$

Similarly, we can prove

$$\left\|\rho^{-\alpha/2}(x)\int_0^t\int_D p^D(t-s,x,y)f(s,y)dyds\right\|_{\mathbb{L}_{p,\theta}(D,T)} \leq C\|\rho^{\alpha/2}f\|_{\mathbb{L}_{p,\theta}(D,T)}.$$

Following the proof of Krylov's embedding theorem, we can prove the following: Let $\gamma \geq 0$, $1/p < \mu < \nu \leq 1$, and

$$\gamma + \alpha - \nu\alpha - \frac{d}{p} \geq n + \delta, \quad n \in \mathbb{N}_+, \ \delta \in (0,1).$$

Then, for any $k \leq n$,

$$\begin{aligned} & \sum_{k=0}^{n} |\psi^{k+\frac{\theta}{p}+\alpha\left(\mu-\frac{1}{2}\right)} D_x^k (u - u_0)|_{C^{\mu-1/p}([0,T];C(D))} \\ & + \sup_{t,s\in[0,T]} \frac{[\psi^{n+\delta+\frac{\theta}{p}+\alpha\left(\mu-\frac{1}{2}\right)} D_x^n (u(t,\cdot) - u(s,\cdot))]_{C^{\delta}(D)}}{|t-s|^{\mu-1/p}} \\ \leq \quad & C\|\psi f\|_{\mathbb{H}^{\gamma}_{p,\theta}(D,T)} + C\|\psi^{-\alpha/2+\alpha/p} u_0\|_{B^{\gamma+\alpha-\alpha/p}_{p,\theta}(D)}. \end{aligned}$$

For instance, let $u_0 = 0$, and assume

$$\|\psi^{\alpha/2} f\|_{L_\infty([0,T]\times D)} < \infty.$$

Taking $\gamma = 0$, $\mu, \nu \uparrow 1$ and $p \uparrow \infty$,

$$\sup_{x\in D} |\psi^{\alpha/2-\delta}(x) u(\cdot, x)|_{C^{1-\varepsilon}([0,T])} < \infty$$

for any small $\delta, \varepsilon > 0$. This gives maximal regularity with respect to time. Next, taking p sufficiently large and μ, ν sufficiently close to $1/p$,

$$\sup_{x\in D} |\psi^{-\alpha/2+\delta'}(x) u(\cdot, x)|_{C^{\varepsilon'}([0,T])} + \sup_{t\in[0,T]} |\psi^{\alpha/2-\delta'} u(t, \cdot)|_{C^{\alpha-\varepsilon'}(D)} < \infty$$

for any small $\delta', \varepsilon' > 0$.

We have the similar results for the elliptic equation

$$\begin{cases} \Delta^{\alpha/2}u(x) - \lambda u(x) = f(x), & x \in D, \\ u(x) = 0, & x \in D^c, \end{cases} \tag{1}$$

In particular, we have

Existence and Uniqueness

Let $p > 1, \theta \in (d-1, d-1+p)$ and $f \in L_{p,\theta+\alpha p/2}(D)$.
(i) Let $\lambda > 0$ or D be bounded. Then, the function

$$u(x) = u^{(\lambda)}(x) := \int_D \left(\int_0^\infty e^{-\lambda t} p^D(t,x,y)dt \right) f(y)dy$$

is the unique weak solution in $L_{p,\theta-\alpha p/2}(D) \cap \{u = 0 \, \text{on} \; D^c\}$.

(ii) Let $\lambda = 0$ and $D = \mathbb{R}^d_+$. Then $u^{(1/n)}$ converges weakly, and the limit is the unique weak solution.

(iii) For the solution u, we have

$$\lambda \|\rho^{\alpha/2}u\|_{L_{p,\theta}(D)} + \|\rho^{-\alpha/2}u\|_{L_{p,\theta}(D)} + \|\rho^{\alpha/2}\Delta^{\alpha/2}u\|_{L_{p,\theta}(D)} \le C\|\rho^{\alpha/2}f\|_{L_{p,\theta}(D)}.$$

Remark. The above estimate as well as higher order regularity are proved as in the parabolic case.

Example: sharpness of $\theta \in (d-1, d-1+p)$

Put $D = B_1(0)$ and $G^D(x,y) = \int_0^\infty p^D(t,x,y)dt$. Let $f \in C_c^\infty(B_r(0))$ be non-negative, $r < 1$. Then (by Chen et al, 2010),

$$|y| < r,\ \frac{1+r}{2} < |x| < 1 \quad \Rightarrow G^D(x,y) \approx \rho^{\alpha/2}(x).$$

Thus, if $|x| > \frac{1+r}{2}$, then

$$u(x) := \int_D G^D(x,y)f(y)dy \approx \rho^{\alpha/2}(x) \int_D f(y)dy \approx (1-|x|)^{\alpha/2}.$$

Thus

$$\|\rho^{-\alpha/2}u\|_{L_{p,\theta}} \geq C \int_{(1+r)/2}^1 (1-s)^{\theta-d} s^{d-1} ds,$$

and the right-hand side is finite only if $\theta - d > -1$.

The necessity of $\theta < d-1+p$ can be checked using a duality argument.

SPDE :

$$\begin{cases} du = (\Delta^{\alpha/2} u + f)dt + \sum_k g^k dw_t^k, & (t,x) \in (0,T) \times D, \\ u(0,x) = u_0(x), & x \in D, \\ u(t,x) = 0, & (t,x) \in [0,T] \times D^c \end{cases}$$

Due to PDE result, without loss of generality, we can assume $f = 0$ and $u_0 = 0$. As in Deterministic cases, (arbitrary) higher order derivatives can be controlled by zero-th order derivative.

If $d - 1 < \theta < d - 1 + p$ we can prove

$$\mathbb{E}\|\rho^{-\alpha/2} u\|^p_{\mathbb{L}_{p,\theta}(D,T)} \leq C\mathbb{E}\||g|_{l_2}\|^p_{\mathbb{L}_{p,\theta}(D,T)}.$$

Idea: Apply B-D-G inequality to kill randomness for $\mathbb{E}|u(t,x)|^p$, then use direct computation arguments used for deterministic PDE.

감사합니다
Thanks for listening