Fractal Laplacian and Krein Strings

Desmond Li
Supervised by Dr. Thomas Jordan and Dr. Asma Hassannezhad

University of BRISTOL

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Research Background

My research is on the study of Kigami's fractal Laplacian and problems I am interested in are

- Eigenfunction analysis
- Spectral asymptotics
- Inverse spectral problem

Why do we even study fractal Laplacians?

The classical spectral theory of Laplacian focuses mainly on smooth structures e.g. smooth surface, smooth manifolds.

In first sight, the notion of putting a differential operator onto fractals is absurd.

Spectral theory problems on a smooth domain but with a fractal boundary, e.g. a filled Julia set have a smooth interior but fractal Julia set boundary.

A theory of Laplacian on fractals will shed light on these problems which is hard or impossible to solve otherwise.

Energy form

We will focus on the simplest case, the unit interval. We will begin by introducing the energy form.

Let $u, v : [0,1] \to \mathbb{C}$ be continuous functions, we define the bilinear form $\mathcal{E}(u,v)$ by the following limit

$$\lim_{m\to\infty} 2^m \sum_{k=0}^{2^m-1} \left(u\left(\frac{k+1}{2^m}\right) - u\left(\frac{k}{2^m}\right)\right) \left(\overline{v\left(\frac{k+1}{2^m}\right) - v\left(\frac{k}{2^m}\right)}\right).$$

We say that a given continuous function $w : [0,1] \to \mathbb{C}$ is of finite energy if $\mathcal{E}(w,w) < \infty$ and the collection of all finite energy functions is called the domain of energy, denoted by $\mathrm{dom}\mathcal{E}$.

Kigami's Laplacian

Definition

Let μ be a strictly positive and regular Borel probability measure on $[0, 1]$ and let $u \in \text{dom}\mathcal{E}$, we say $u \in \text{dom}\Delta_\mu$ if there exists some function $f \in L^2([0, 1], \mu)$ such that for every function $v \in \text{dom}\mathcal{E}$ satisfying $v(0) = v(1) = 0$, we have that

$$\mathcal{E}(u, v) \quad = \quad -\int_{[0,1]} f\overline{v}\mathrm{d}\mu.$$

We then say that f is the Laplacian of u, i.e.

$$\Delta_\mu u \quad = \quad f.$$

Observe that this resembles the integration by parts in usual calculus, which is the case when μ is the Lebesgue measure.

Krein strings

First studied by Krein (1951).

Motivated from studying vibrations of strings with non-uniform mass.

Closely relates to the classical Sturm-Liouville theory.

Defined only on intervals, there have been generalisations to other domains like star-graphs (Eckhardt 2013). But ultimately they are still finite unions of 1-dimensional lines.

Krein strings

When studying vibrations, we always make use of the wave equation. For “good” mass distributions, we can simplify the problem into the following differential equation

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}w(x) \quad = \quad -\lambda\rho(x)w(x).$$

Where w is the position of string relative to equilibrium position, λ is the frequency and $\rho(x)$ is the density function.

For general mass distributions, the density function ρ might not exist!

Krein strings

To fix this, we first consider only the case when the density function does exist. Then we can transform the equation

$$\frac{\mathsf{d}^2}{\mathsf{d}x^2} w(x) \quad = \quad -\lambda \rho(x) w(x)$$

into

$$\frac{\mathsf{d}}{\mathsf{d}M(x)} \frac{\mathsf{d}}{\mathsf{d}x} w(x) \quad = \quad -\lambda w(x).$$

Where M is the cumulative mass distribution function of the string with density function ρ. In general, we can think of M simply to be the mass measure and $\frac{\mathsf{d}}{\mathsf{d}M} \frac{\mathsf{d}}{\mathsf{d}x}$ to be an operator, reducing the problem to an eigenvalue problem.

Quasi-derivative

To define the Krein operator properly, we must first make sense of the operator $\frac{\mathsf{d}}{\mathsf{d}M}$ for general cumulative mass distribution M. We do this by using the notion of quasi-derivative.

Let μ be a strictly positive and regular Borel probability measure on the unit interval and $g : [0,1] \to \mathbb{C}$ a continuous function. g is said to be of class $\mathsf{AC}_{L^2}([0,1],\mu)$ if there is some function $h \in L^2([0,1],\mu)$ such that

$$g(x) \quad = \quad g(c) + \int_{[c,x]} h \mathsf{d}\mu$$

for every $c, x \in [0,1]$. The function h is then said to be the quasi-derivative of g, i.e.

$$\frac{\mathsf{d}g}{\mathsf{d}M} \quad = \quad \frac{\mathsf{d}g}{\mathsf{d}\mu} \quad = \quad h.$$

Krein operator

Definition

Let μ be a strictly positive and regular Borel probability measure on the unit interval and $w \in \mathcal{C}^1$. We say $w \in \mathrm{dom}\kappa$ if $w' \in \mathrm{AC}_{L^2}([0,1], \mu)$ and

$$\kappa w \quad := \quad \frac{\mathrm{d}}{\mathrm{d}M} w'.$$

The quasi-derivative of w' with respect to the measure μ.

Despite Kigami's Laplacian and Krein's operator are constructed quite differently, it turns out that the two operators are in fact identical.

Kigami's Laplacian and Krein's operator

Theorem

Let μ be a strictly positive and regular Borel probability measure. We have that the Kigami's Laplacian and Krein's operator are equivalent, in the sense that their domain sets are equal

$$\mathrm{dom}\Delta_\mu \quad = \quad \mathrm{dom}\kappa$$

and for every function $w \in \mathrm{dom}\Delta_\mu$, we have that

$$\Delta_\mu w \quad = \quad \kappa w.$$

Proof outline

1. $\text{ker}\Delta_\mu = \text{ker}\kappa = \{ax + b | a, b \in \mathbb{C}\}$.

2. $\mathcal{C}^1([0,1]) \subseteq \text{dom}\mathcal{E}$.

3. $\text{dom}\mathcal{E} \subseteq \text{BV}([0,1])$.

4. $\text{AC}_{L^2}([0,1],\mu) \subseteq \text{BV}([0,1])$.

5. Using integration by parts on a Riemann-Stieljes integral and Radon-Nikodym theorem we have the inclusion $\text{dom}\kappa \subseteq \text{dom}\Delta_\mu$.

6. $\text{dom}\Delta_\mu \subseteq \text{dom}\kappa$.

Proof overview of dom$\kappa \subseteq$ domΔ_μ

Proof.

Let $u \in$ domκ, from definition we have that $u \in \mathcal{C}^1$ and by 2 we have that $u \in$ dom$\mathcal{E}$. We will also let $v \in$ dom$\mathcal{E}$ satisfying $v(0) = v(1) = 0$. We have that the energy can be expressed into a Riemann-Stieljes integral as v is of bounded variation by 3.

$$\mathcal{E}(u, v) = \int_{[0,1]} u' \mathrm{d}\overline{v}.$$

Applying integration by parts, we obtain

$$= -\int_{[0,1]} \overline{v} \mathrm{d}u'.$$

Noting that the boundary terms vanishes. From 4, we have that w' is of bounded variation. Therefore, we can rewrite the integral as

$$= -\int_{[0,1]} \overline{v} \mathrm{d}\mu_{u'}$$

Proof overview of dom$\kappa \subseteq$ domΔ_μ

Proof.

Where $\mu_{u'}$ is the (signed) Lebesgue-Stieljes measure of u', which satisfies

$$\mu_{u'}(A) \ = \ \int_A \kappa u \mathsf{d}\mu$$

for any Borel subset A of $[0,1]$. Thus, by Radon-Nikodym theorem, we have that

$$\mathcal{E}(u,v) \ = \ -\int_{[0,1]} (\kappa u)\overline{v} \mathsf{d}\mu.$$

As v is arbitrary, we must have that $u \in$ domΔ_μ and

$$\Delta_\mu u = \kappa u.$$

□

Quick summary

Kigami's Laplacian	Krein's Operator
Motivated by problems involving Laplacians on smooth domain but fractal boundary.	Motivated from studying strings of non-uniform mass and Sturm-Liouville operators.
Implicitly constructed via the weak formulation.	Explicitly defined using quasi-derivative.
Most results are only applicable to self-similar probability measures on post-critically finite sets.	Results are applicable to any Borel measure but only on intervals or sets composed of finite union of lines.

Thank You For Your Attention!

Q&A