$\boxed{1}$ 改

with Mimura, Munemasa, Seki, Yoshino ; arXiv 2012.15669

in progress with Binda, H. Miyazaki, R. Sugiyama 2101.00839

2209.11816

<u>Green-Tao 2008</u>

The primes contain arbitrarily long arithmetic progressions.

$5,\ 11,\ 17,\ 23,\ 29,\ \cancel{35}$

$7,\ 37,\ 67,\ 97,\ 127,\ 157,\ \underset{11\times 17}{\underset{\shortparallel}{\cancel{187}}},\ \cancel{217}$

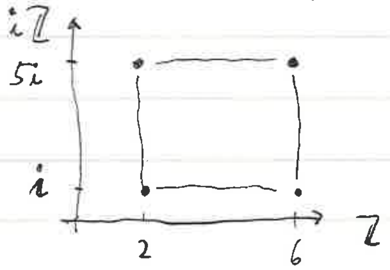
<u>Tao 2006</u>

$\forall S \subseteq \mathbb{Z}[i]$ finite set $\exists x \in \mathbb{Z}\setminus\{0\}$ $\exists y \in \mathbb{Z}[i]$ such that

$x\cdot S + y \subseteq \{\text{prime elements}\}$.

時間
なさそう

$S =$ ⸬



$5 = (2+i)(2-i)$

$29 = (2+5i)(2-5i)$

$37 = 36 + 1$

$61 = 36 + 25$

~~predicted~~ "likely to extend" at least if

- $\mathcal{O}_K$ UFD
- $\mathcal{O}_K^\times$ finite
- $K/\mathbb{Q}$ Galois

$\to$ 8 cases left

<u>KMMSY 2020</u>

The same holds for all number fields.

($\exists$ analogue for ~~integral~~ affine integral curves / $\mathbb{F}_q$)

(sketch of proof)

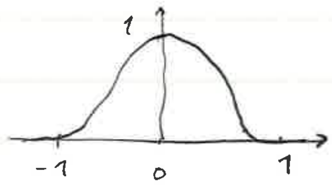
Feed 2 things to a combinatorics machine:

$\boxed{\text{A pseudorandom } \underline{\text{weight}} \text{ function } \nu : \mathcal{O}_K \to \mathbb{R}_{\geq 0}}$ + $\boxed{\text{A subset } \subseteq \mathcal{O}_K \text{ of positive } \underline{\text{weighted}} \text{ density}}$ $\xrightarrow[\text{thm of Green–Tao}]{\text{relative Szemerédi}}$ $\boxed{\text{constellations in your set}}$

prime elements

Our main contribution

Key def.



$\chi : \mathbb{R} \to [0, 1]$ any C^∞ such that $R > 1$.

Then

$$\Lambda_{R,\chi} : \{\text{non-zero ideals} \subseteq \mathcal{O}_K\} \longrightarrow \mathbb{R}$$

$$\mathfrak{a} \longmapsto \log R \sum_{\mathfrak{b} \mid \mathfrak{a}} \mu(\mathfrak{b})\, \chi\left(\frac{\log N(\mathfrak{b})}{\log R}\right)$$

$\mathcal{O}_K \setminus \{0\}$

all ideals

$\square$

§ Beyond (in progress)

Inspired by Serre,

Colliot-Thélène / Swinnerton-Dyer 1994

Assume $\boxed{\text{Schinzel's Hypothesis.}}$ ← a twin-prime type conjecture

X / K geometrically integral, smooth with $X \to \mathbb{P}^1$ whose generic fiber is Severi–Brauer. Then Hasse principle for K-rational points holds for X. (Brauer–Manin obstruction)

International Centre for Mathematical Sciences
Edinburgh, Scotland, UK
t: +44 (0)131 650 9831 e: info@icms.org.uk
w: www.icms.org.uk

こっち後

$\mathbb{Q}$	Green-Tao 2008	$\leq$ Green-Tao-Ziegler 2012	$\rightsquigarrow$	Hasse principle 2014 (fibers Severi-Brauer $X \to \mathbb{P}^1_{\mathbb{Q}}$)
# fields	KMMSY 2020	$\boxed{}$ 6	$\rightsquigarrow$ ✓ (kind of) 2020	$\boxed{}$

Quasi Theorem (hopefully 2023)

Green-Tao-Ziegler also holds for number fields.

こっち先

Harpaz-Skorobogatov-Wittenberg 2014

Over $\mathbb{Q}$, removed Schinzel's Hyp. using: adding minor conditions on bad fibers

Green-Tao-Ziegler 2012

$\psi_1, \dots, \psi_t : \mathbb{Z}^d \to \mathbb{Z}$ affine-linear maps.

They ~~assume~~ attain simultaneous prime values if

- $\forall p$ prime $\exists x_p \in \mathbb{Z}^d$ such that $\psi_i(x_p) \neq 0$ in $\mathbb{Z}/p\mathbb{Z}$ for all i.
- linear parts ~~of~~ $\dot{\psi}_i$ are pairwise linearly independent $/\mathbb{Q}$

$\boxed{4}$ 2

(Strategy)

$\mathbb{Q}$: Methods of Green-Tao, streamlined by Tao-Teräväinen
2010 ... 2021

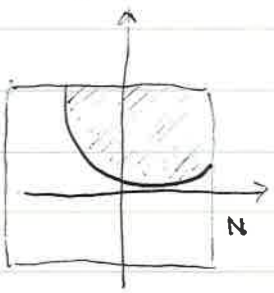
powered by a combinatorics "Gowers uniformity norm" of functions $\mathbb{Z} \to \mathbb{C}$

K number field : upgrade ↑ to number fields.

- Rework some combinatorics for $\mathbb{Z}^n$ not $\mathbb{Z}$
  (squares ≠ circles any more !)
- "uniform" choice of bases ~~[illegible]~~ $\mathfrak{a} \cong \mathbb{Z}^n$ for all ideals $\mathfrak{a}$
- Our previous pseudorandom weights play a role.
- Prime Number Theorem in arithmetic progressions $aX+b$

⇝ Mitsui's Prime Number Theorem (refinement by K.)
1956 ... 2022

$\mathcal{O}_K$



主張を文字で書く時間なさそう

Fix $\mathcal{O}_K \cong \mathbb{Z}^n$

$C \subseteq [-N, N]^n \subseteq \mathcal{O}_K \otimes_{\mathbb{Z}} \mathbb{R}$ convex open

$\mathfrak{q} \subseteq \mathcal{O}_K$ non-zero ideal, $\alpha \in (\mathcal{O}_K/\mathfrak{q})^*$

$$\sum_{\substack{x \in C \\ x \equiv \alpha \bmod \mathfrak{q}}} \Lambda_K(x) = \frac{1}{\operatorname{res}_{s=1}(\zeta_K(s))} \sum_{\substack{x \in C \\ x \equiv \alpha}} 1 + O\left(N^n \cdot e^{-\sqrt{\log N}/O_K(1)}\right)$$

□