Stochastic parabolic equation and Dirichlet boundary condition

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Harmonic Analysis, Stochastics and PDEs in Honour of the 80th Birthday of Nicolai Krylov

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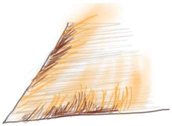
In this talk we are interested in :

Stochastic heat diffusion near the boundaries
of infinite wedges in $\mathbb{R}^2$
under Dirichlet boundary control of the heat density(temperature).

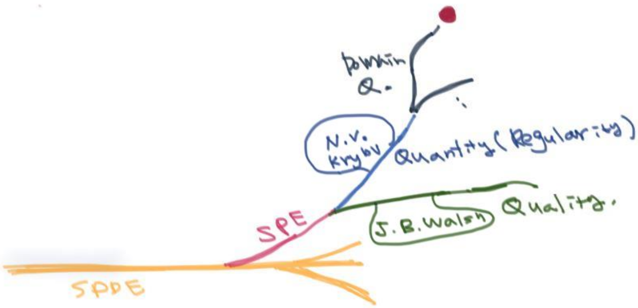


* The second half of this talk is based on the co-work with K.H. Kim, Jinsol Seo, P.A. Cioica-Licht, and F. Lindner.

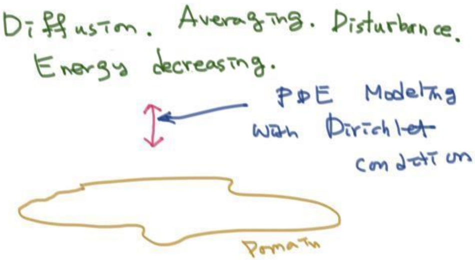
<<Related stories>>

<The landscape>

☐ We discuss



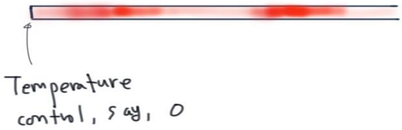
☐ Understanding



- Better PDE modeling, more reliable solution, better understanding of diffusion and the domain.
- The quantitative information including energy near the boundary really matters.

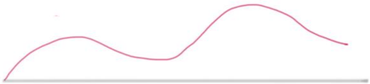
<PDE modeling. with $d = 1$ >

☐ (1 dim) Heat diffusion (averaging) on a half line



A very long thin insulated rod

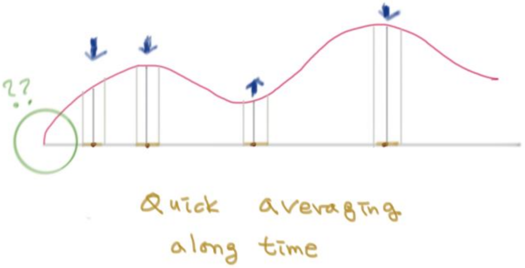
- The initial temperature $\phi(x)$. At most points, ϕ is unbalanced.



- Q. How does the temperature, the density $u(t,x)$ of heat, change along time?

- Nature quickly resolves the unbalances through the heat flux, say, by Fourier's law.
- (Heat-type averaging) The heat equation : with $k > 0$

$$u_t(t,x) = k u_{xx}(t,x), \quad (0,T)\times(0,\infty) \quad ; \quad u(t,0) = 0, \quad t \geq 0.$$



☐ With disturbance

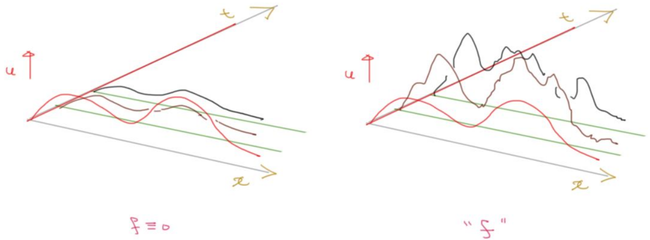
- $f(t,y)dy$: the changing rate of the occurring heat near y at time t.
- (Diffusion vs. disturbance) The model equation

$$u_t(t,x) = ku_{xx}(t,x) + f(t,x).$$

or, for any $t_2 > t_1 \geq 0$

$$u(t_2,x) - u(t_1,x) = k\int_{t_1}^{t_2} u_{xx}(s,x)ds + \int_{t_1}^{t_2} f(s,x)ds$$

$$; \quad du(t,x) = ku_{xx}(t,x)dt + f(t,x)dt.$$



☐ Toward a better model. Stochastic consideration to $f\,dt$

- Naturally, $f\,dt$ may have very little randomness caused by the sum of micro uncertainties:

$$u(\omega, t+dt, x) - u(\omega, t, x) = ku_{xx}(\omega, t, x)dt + f(t,x)dt + g(t,x)(F_{t+dt}(\omega) - F_t(\omega)),$$

$$; \quad du(t,x) = ku_{xx}(t,x)dt + f(t,x)dt + g(t,x)dF_t.$$

  where $F_t(\omega)$ is a stochastic process and $g(t,x)(F_{t+dt} - F_t)$ models the uncertainty to $f\,dt$.

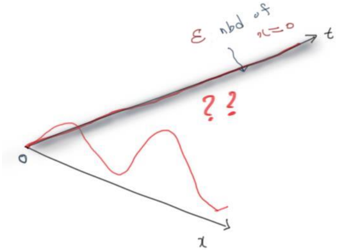
- For instance, we may design F in a way that
  - $F_{t+dt} - F_t \sim \mathcal{N}(0, dt)$.
  - The increments $\{F_{t+dt} - F_t\}$ are independent.
- Then any Brownian motion $W_\cdot$ will do the job.
- Any Brownian motion in any interval $[t_1, t_2]$ is of unbounded variation a.s..

☐ Initial boundary value problem(IBVP) of stochastic heat equation on half line

$$
\begin{aligned}
du(\omega,t,x) &= ku_{xx}(\omega,t,x)dt + f(t,x)dt + g(t,x)dW_t(\omega), \\
&\qquad\qquad \omega \in \Omega, \quad 0 < t < T, \quad 0 < x, \\
u(0,x) &= \phi(x), \quad 0 \le x, \\
u(t,0) &= 0, \quad 0 \le t \le T.
\end{aligned}
$$

* The equation part can be illegally expressed by

$$u_t = ku_{xx} + f + \dot{W}_t g.$$



<Dirichlet condition with $d = 1$>

☐ (N.V. Krylov) Close look; the temperature behavior near the boundary point

- (Heuristic) The IBVP leads us to

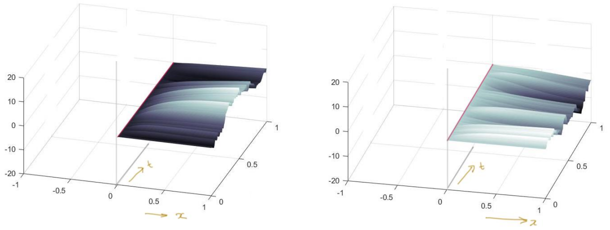
$$0 \approx u(t_2, \varepsilon) - u(t_1, \varepsilon) = k \int_{t_1}^{t_2} u_{xx}(s, \varepsilon) ds + \int_{t_1}^{t_2} f(s, \varepsilon) ds + \int_{t_1}^{t_2} g(s, \varepsilon) dW_s$$

  for any $t_1 < t_2$.

- If $g(\cdot, \varepsilon) \approx 0$, when f is very nice, the second derivative $u_{xx}(\cdot, \varepsilon)$ does not suffer much although it still suffers little bit by the Dirichlet condition.

- If $g(\cdot, \varepsilon) \approx 1$,
  - Even if f is really good, $u_{xx}(\cdot, \varepsilon)$ on any interval $[t_1, t_2]$ can not be bounded as $W_\cdot$ is of unbounded variation on any interval (a.s.).
  - For almost sure events, the second derivatives near the boundary can be endangered during time in $[0, T]$.

– (Simulation with $T = 1$) The (random event-wise) solutions are much worse than the following pictures



<Two random events of u with zero initial condition>

and the second derivative and even the first one near boundary suffers a lot.

– Need weight? In view of summability (integration) we note

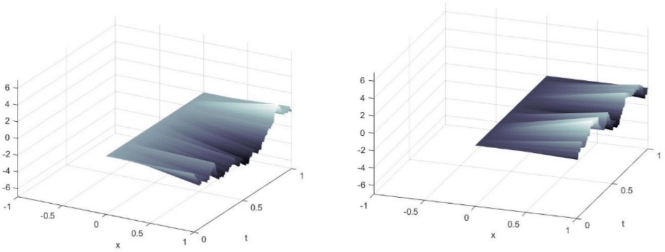
$$(x^{0.5})' \sim x^{-0.5}$$
$$x(x^{0.5})'' \sim x^{-0.5}$$
$$\frac{1}{x} x^{0.5} \sim \underbrace{x^{-0.5}}_{\text{o.k!}}$$

- If g vanishes at the boundary, then we have a better situation.

  For an extreme example, the (random) function $u(t,x) = xW_t$. satisfies SPE (stochastic parabolic equation)

$$u_t = u_{xx} + x\dot{W}_t$$

with $u_{xx} \equiv 0$. * A reference stochastic heat equation on half line.

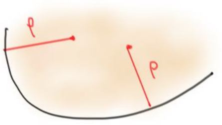


$<$Two events of $u = xW_t$ $>$

- Looking at $xu_t = xu_{xx} + xf + xg\dot{W}_t$, we have

$$0 \approx \varepsilon(u(t_2,\varepsilon) - u(t_1,\varepsilon)) = k\int_{t_1}^{t_2} \varepsilon u_{xx}(s,\varepsilon)ds + \int_{t_1}^{t_2} \varepsilon f(s,\varepsilon)ds + \int_{t_1}^{t_2} \varepsilon g(s,\varepsilon)dW_s.$$

- Considering xu_{xx} instead of u_{xx} is appropriate.
- We also consider u_x, the information of energy, and $\frac{1}{x}u$ together with xu_{xx} in one package.
- For higher dimensional domains D, we use $\rho(x) = \mathrm{dist}(x, \partial D)$ instead of x.



<Our quantitative information>

☐ (N.V. Krylov) Caring the derivatives near the boundary

We adapt the following weighted norms for the solution u :

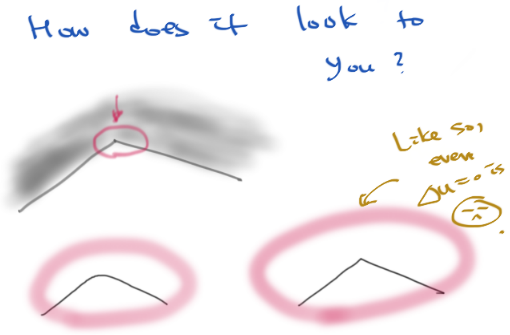
$$\|u\|_{\mathbb{H}^2_{p,\theta}(T)} = \left(\sum_{k=0}^{2} \mathbb{E} \int_0^T \int_{\mathbb{R}_+} |x^{k-1} D^k u(x)|^p x^{\theta-1} \, dx \, dt \right)^{1/p}.$$

- $x^{\theta-1}$ (1 is the dimension for now) delicately adjusts the norm in accordance with the smootheness of the boundary.
- It turns out that $-1 < \theta - 1 < -1 + p$ with our half-line domain.
- $-1 < \theta - 1$ is crucial. For our favored function $u = xW_t$ we have

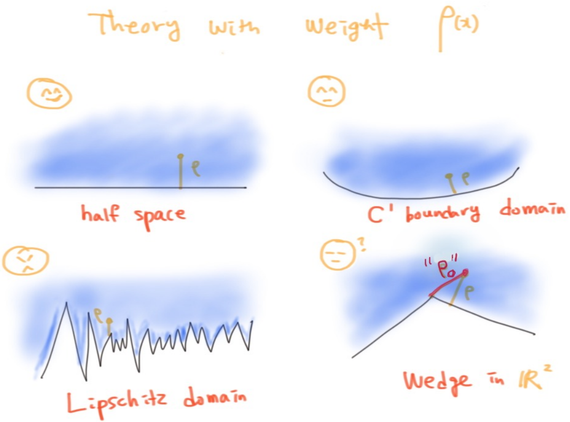
  $$\mathbb{E} \int_0^T \int_0^{\varepsilon} |x^{-1} u|^p x^{\theta-1} dxdt < \infty \iff -1 < \theta - 1$$

  as all the moments of a Gaussian distribution is finite.
- For higher dimensional domain D, we use $\rho(x)$ and d instead of x and 1. The range of $\theta - d$ is then $(-1, -1 + p)$. If the boundary is bad, it may shrink?

- ☐ The way we choose to look matters.



☐ Domains with dimension ≥ 2; question of the range of θ



☐ Some mathematical notes

- (N.V. Krylov, 1994) By Burkholder-Davis-Gundy inequality and a help of harmonic analysis $\sim$ parabolic sharp function estimate, Marcinkiewicz interpolation theorem, Littlewood-Paley type of estimate, we find that $p \geq 2$ is required for L_p-regularity theory of stochastic parabolic equation.
- (N.V. Krylov, 1999) Working with half space, to measure $\|u\|_{\mathbb{H}^2_{p,\theta}(T)}$, using equation itself is much more effective than using solution representation. The main ingredients are
  - Itô formula to $|u|^p$, hence the structure of the equation.
  - Hardy's inequality
  - (beautiful) a package of zooming in/out of a function near the boundary with care
  - C^1 boundary domains with KH Kim

- (KH Kim) The result for Lipschitz domains using perturbation method and Hardy's inequality. The range of θ we can be sure is significantly shrunken:

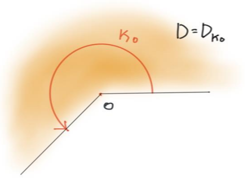
$$\theta \in (d-2+p-\varepsilon, d-2+p+\varepsilon)$$

as we can compare it with

$$(d-1, d-1+p).$$

☐ We discussed.

- Can we pose a theory between smooth boundary domain theory and Lipschitz domain theory?
- Say, take wedges in $\mathbb{R}^2$.



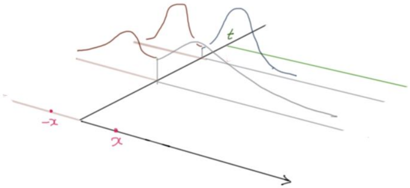
- Angle κ_0 will surely matter.
- Using the equation itself will then be shortcoming.
- What do we need more?

☐ Works of V.A. Solonnikov, V.A. Kozlov, A.L. Nazarov for deterministic parabolic equations, using solution representation using Green's function, came along in the search.

<The tool we have overlooked before>

☐ Solution representation with the simplest domain having boundary, $D = \mathbb{R}_+$

- Green's function for (1) $\mathcal{L} = \partial_t - \Delta_x$, (2) half line, and (3) Dirichlet condition.



$$G(t,s,x,y) = \begin{cases} \frac{1}{\sqrt{4\pi(t-s)}} e^{-\frac{|y-x|^2}{4(t-s)}} - \frac{1}{\sqrt{4\pi(t-s)}} e^{-\frac{|y+x|^2}{4(t-s)}}, & s < t \\ 0, & s \geq t \end{cases}$$

- For any fixed $t \in \mathbb{R}$, $x \in \mathbb{R}_+$ and for all $s \in \mathbb{R}$, $y \in \mathbb{R}_+$

$$(-\partial_s - \Delta_y)G(t,s,x,y) = \delta_{t,x}.$$

  and

$$u(t,x) := \int_{-\infty}^{t} \int_{\mathbb{R}_+} G(t,s,x,y) f(s,y) dyds$$

  solves $(\partial_t - \Delta_x)u = f$.

- $u(t,0) = 0$ as $G(t,s,0,y) = 0$ for any s and $y \in \mathbb{R}_+$.

- With Laplace operator Δ, $G(t,s,x,y)$ is a function of $t-s$, x, y and we can write

$$G(t-s,x,y)$$

  instead of $G(t,s,x,y)$.

☐ In general, given

(1) Parabolic operator $\mathcal{L} := \partial_t - L_x$,

(2) Domain D,

(3) Dirichlet condition,

we have Green's function $G(t, s, x, y)$ satisfying

- For fixed (t, x), the function $v = G(t, \cdot, x, \cdot)$ gives
$$\mathcal{L}^* v = \delta_{t,x},$$
where $\mathcal{L}^* = -\partial_s - L_y$.
- (Looks green) For fixed (s, y),
$$\mathcal{L} G(\cdot, s, \cdot, y) = \delta_{s,y},$$
- $$G(t, s, x, y) = 0, \quad s \geq t.$$
- $$u(t, x) := \int_{-\infty}^{t} \int_D G(t, s, x, y) f(s, y) dy ds$$
solves
$$\mathcal{L} u = f; \quad u(t, x) = 0 \text{ for } x \in \partial D.$$

☐ The estimation of G is a main topic of potential analysis.

☐ We are drawn to find a special estimate of G for (1) the operator $\mathcal{L} := \partial_t - \Delta_x$, (2) a wedge domain in $\mathbb{R}^2$, and (3) Dirichlet condition.

- $*$ Special in the sense that, through the estimate, we can obtain a nice regularity result for stochastic heat equation near the vertex.

$$|G(t, x, y)| \le \quad ?$$

$\wr$

The prob. that a process starting at x be near y at time t, before hitting ∂

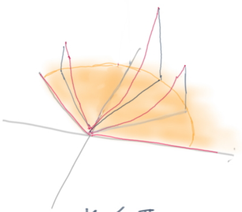
<< Stochastic heat-type diffusion on wedges in $\mathbb{R}^2$ >>

☐ Evidences : The solution is prone to be radial near the vertex.

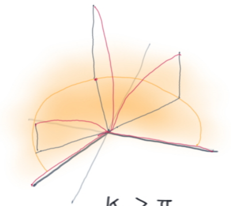
- (The equilibrium, when diffusion ends)

$$0 = \Delta u \quad \text{in } D_{\kappa_0} \quad ; \quad u = 0 \quad \text{on } \partial D_{\kappa_0}.$$

  - Using $Z^\alpha = r^\alpha e^{i\alpha\eta}$ with $\alpha = \pi/\kappa_0$, we find a solution $u = r^\alpha \sin(\alpha\eta)$.
  - The distance to the "point" matters more than the distance to the boundary.
  - κ_0 vs. π also matters.



$\kappa_0 < \pi$

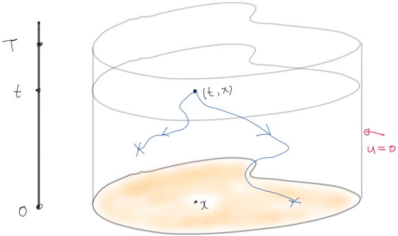


$\kappa_0 > \pi$

- (To the equilibrium)

$$u_t = \Delta u \text{ in } (0,T) \times D_{\kappa_0} \ ; \ u = 0 \text{ on } (0,T) \times \partial D_{\kappa_0} \ ; \ u(0,\cdot) = \phi \text{ on } D_{\kappa_0}.$$

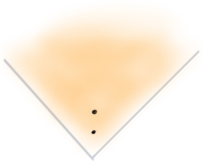
  – (A probabilistic view) For each (t,x), we let the time-space process $(t-s, x+\sqrt{2}B_s)$ flow, where B is a two-dimensional Brownian motion. It will hit the parabolic boundary at (stopping) time τ.

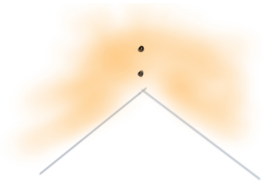


Itô's formula gives

$$u(t,x) = \mathbb{E}u(t-\tau, x+\sqrt{2}B_\tau) = \mathbb{E}\left[I_{\tau=t}\, \phi(x+\sqrt{2}B_t)\right] \left(= \int_D G(t,x,y)\phi(y)dy \right).$$

– Two situations. Again, angle matters.





This observation resembles the case of the equilibrium.

- (A reference computation, SPE) The stochastic diffusion $u = W_t \cdot r^{\pi/\kappa_0} \sin\left(\frac{\pi}{\kappa_0}\eta\right)$, where $W.$ is a 1-dimensional BM, solves

$$u_t = \Delta u + \dot{W}_t g \ \text{ in } (0,T)\times D_{\kappa_0} \quad ; \quad u = 0 \text{ on } (0,T)\times \partial D_{\kappa_0}$$

for all ω with $g = r^{\pi/\kappa_0}\sin\left(\frac{\pi}{\kappa_0}\eta\right)$. Then near the vertex point, introducing $\rho_0(x) = |x|$,

$$\begin{aligned} &\mathbb{E}\int_0^T \int_{\text{near }(0,0)} |\rho_0{}^{-1}u|^p \rho_0{}^{\Theta-2}dxdt \\ =\ &\mathbb{E}\int_0^T\int_0^{\kappa_0}\int_0^{\varepsilon} |W_t|^p \left| r^{-1} r^{\pi/\kappa_0}\sin\left(\frac{\pi}{\kappa_0}\eta\right)\right|^p r^{\Theta-2} r dr\, d\eta\, dt \\ =\ &\int_0^{\varepsilon} r^{(-1+\pi/\kappa_0)p+\Theta-1}dr \cdot \int_0^{\kappa_0}\left|\sin\left(\frac{\pi}{\kappa_0}\eta\right)\right|^p d\eta \cdot \mathbb{E}\int_0^T |W_t|^p dt. \quad (*) \end{aligned}$$

We note

$$(*) < \infty \quad \Leftrightarrow \quad p\left(1-\frac{\pi}{\kappa_0}\right) < \Theta.$$

□ (Our regularity relation) Let u, f, g be related by

$$u_t = \Delta u + f + \dot{W}_t g \text{ in } (0,T)\times D_{\kappa_0} \quad ; \quad u = 0 \text{ on } (0,T)\times \partial D_{\kappa_0},$$
$$u(0,\cdot) = 0 \text{ in } D_{\kappa_0}$$

for all ω. Then we have a result mirror to the case of smooth domain.

Theorem (Cioica, Kim, Lee, Lindner, 2018)

For any

$$p\left(1-\frac{\pi}{\kappa_0}\right) < \Theta < p\left(1+\frac{\pi}{\kappa_0}\right),$$

we have

$$\mathbb{E}\int_0^T\int_{D_{\kappa_0}} |{\rho_0}^{-1}u|^p {\rho_0}^{\Theta-2}dxdt$$
$$\leq N\left(\mathbb{E}\int_0^T\int_{D_{\kappa_0}} |\rho_0 f|^p {\rho_0}^{\Theta-2}dxdt + \mathbb{E}\int_0^T\int_{D_{\kappa_0}} |g|^p {\rho_0}^{\Theta-2}dxdt\right).$$

- As κ_0 gets bigger, we give up to ask u to be good.

☐ Some notes on the result

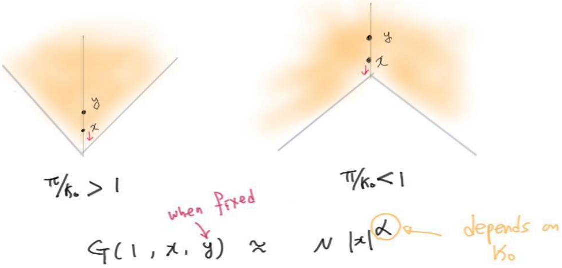
- Solution representation

$$
\begin{aligned}
u(t,x) &= \int_{D_{\kappa_0}} G(t,x,y)\phi(y)dy \\
&+ \int_0^T \int_{D_{\kappa_0}} G(t-s,x,y)f(s,y)dyds \\
&+ \int_0^T \int_{D_{\kappa_0}} G(t-s,x,y)g(s,y)dy\frac{dW}{ds}ds
\end{aligned}
$$

; the initial condition part + the deterministic part + the stochastic part.

- The result for the infinite wedge domains in $\mathbb{R}^2$ heavily relies on the Green's function $G(t,x,y)$ of heat diffusion with wedge domains.

- (Guessing the shape of G, a probabilistic view) $G(t,x,y)dy$: the probability that
  - (i) $x+\sqrt{2}B_s$ stays in D for $s<t$ and
  - (ii) $x+\sqrt{2}B_t$ is in the neighborhood of y with the infinitesimal area dy.
  - If x is close to the boundary, then $G(t,x,y)\approx 0$ as $t>>1$.
  - If x and y are far away from boundary, then $G(t,x,y)dy \approx \Phi(t,x,y)dy$, where Φ is the heat density.
  - Also, we observe



- (V.A Kozlov) It turns out that for any $0 < \lambda_1, \lambda_2 < \pi/\kappa_0$, there exist constants σ, N such that

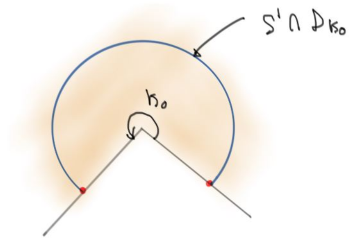
$$|G(t,x,y)| \leq N\frac{1}{t}e^{-\sigma\frac{|x-y|^2}{t}}\left(\frac{|x|}{|x|+\sqrt{t}}\right)^{\lambda_1}\left(\frac{|y|}{|y|+\sqrt{t}}\right)^{\lambda_2}.$$

  Back then, this became what we were looking for.

- (A view) $(\pi/\kappa_0)^2$ is the first eigenvalue of the eigenvalue/function problem of

$$\Delta_{LB}u = \lambda u \quad \text{on } S^1 \cap D_{\kappa_0}$$

  with Dirichlet condition on the boundary points for the manifold $S^1 \cap D_{\kappa_0}$; the information of the lowest energy state that the domain can have.



- For our regularity result, we desperately use the solution representation.
- Need a long sequence of elementary, subtle, russian-style arguments to get it done.
- Especially, the stochastic part of the solution needs more care as we lose some quantitative information at the very beginning as we apply Burkholder-Davis -Gundy inequality.

☐ We then look around and see what we have.

- Proceeding with the result, it is quite suffering to obtain a result for the second order derivatives.
- Except only one point, the vertex, a wedge domain has smooth boundary, doesn't it?
- The estimate

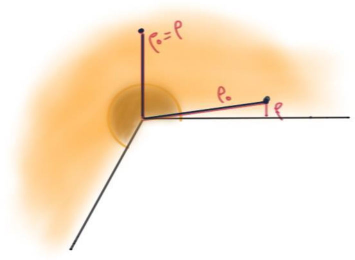
$$|G(t,x,y)| \leq N \frac{1}{t} e^{-\sigma \frac{|x-y|^2}{t}} \left(\frac{|x|}{|x| + \sqrt{t}} \right)^{\lambda_1} \left(\frac{|y|}{|y| + \sqrt{t}} \right)^{\lambda_2}$$

  is quite loose for the boundary points which are not the vertex.

- We need a refined estimate to build a more satisfactory (= less suffering) regularity theory.

☐ Next step

Incorporates the distance to the point and the distance to the boundary.



- $\rho \leq \rho_0$.
- $\theta - 2 \in (-1, -1 + p)$ for $\rho = \rho(x)$ should come into play in the design of weighted Sobolev spaces for solutions u and inputs f, g.

☐ A refined Green's function estimate. Wedge domains in $\mathbb{R}^2$.

Theorem (Kim, Lee, Seo, 2022)

For any $0<\lambda_1,\lambda_2<\pi/\kappa_0$, *there exist constants* σ, $N(\lambda_1,\lambda_2)$ *such that*

$$G(t,x,y) \le N\frac{1}{t^{2/2}}e^{-\sigma\frac{|x-y|^2}{t}}\left(\frac{\rho_0(x)}{\rho_0(x)+\sqrt{t}}\right)^{\lambda_1-1}\left(\frac{\rho_0(y)}{\rho_0(y)+\sqrt{t}}\right)^{\lambda_2-1}$$
$$\times\frac{\rho(x)}{\rho(x)+\sqrt{t}}\,\frac{\rho(y)}{\rho(y)+\sqrt{t}}.$$

∗ In fact, higher dimensional version is also available.

- Using it, we manage to measure the weighted norm designed by

$$\mathbb{E}\int_0^T\int_{D_{\kappa_0}}\left(|\rho^{-1}u|^p+|u_x|^p+|\rho u_{xx}|^p\right)\rho_\circ^{\Theta-\theta}\rho^{\theta-2}\,dx\,dt$$

with ranges

$$p\left(1-\frac{\pi}{\kappa_0}\right)<\Theta<p\left(2-1+\frac{\pi}{\kappa_0}\right)\quad\text{and}\quad 2-1<\theta<2-1+p.$$

- Away from the vertex, $\rho_\circ^{\Theta-\theta}$ is weak in play.

- $$\mathbb{E}\int_0^T\int_{D_{\kappa_0}}\left(|\rho^{-1}u|^p+|u_x|^p+|\rho u_{xx}|^p\right)\rho_\circ^{\Theta-\theta}\rho^{\theta-2}\,dx\,dt.$$

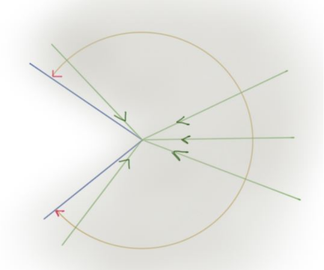
  Also, when away from the sides, as

  $$\rho_\circ^{\Theta-\theta}\rho^{\theta-2}=\rho_0^{\Theta-2}\left(\frac{\rho}{\rho_0}\right)^{\theta-2},$$

  and the term

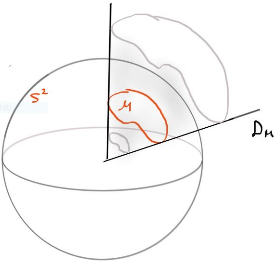
  $$\left(\frac{\rho}{\rho_0}\right)^{\theta-2}$$

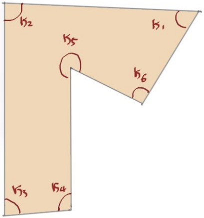
  is not crucial and ρ, ρ_0 are comparable.



- Two types of estimate are solidly in one package now. Good start for better theories to be built.

* In fact, the refined Green's function estimate for general parabolic operators and the conic domains in $\mathbb{R}^d$ $(d \geq 2)$





and the deterministic version (hence $p > 1$) of the corresponding regularity theory for polygonal domains in $\mathbb{R}^2$ are built and now available; [Kim, Lee, Seo, 2021].

All the wishes
for
our Teacher