Fully Scalable Solver for Frequency-Domain Visco-Elastic Wave Equations in 3D Heterogeneous Media

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Application

- 2D/3D Modelling of acoustic, elastic, or electromagnetic wave fields in heterogeneous media
- Exploring (unknown) heterogeneous media with full waveform inversions (FWI)

High-frequency exploration or simulation in the time domain is a major challenge:

- high resolution and fine grid
- CFL restriction of (explicit) numerical time integration such as Newmark or Runge-Kutta methods

$\Rightarrow$ We instead consider the frequency-domain wave equations.

J. Virieux and S. Operto *"An overview of full waveform inversion in exploration geophysics"*, Geophysics, 2010.

P. Amestoy, R. Brossier, A. Buttari, J.-Y. L'Excellent, T. Mary, L. Métivier, A. Miniussi, and S. Operto, *"Fast 3D frequency-domain full-waveform inversion with a parallel block low-rank multifrontal direct solver: Application to OBC data from the North Sea"*, Geophysics, 2016.

Y. Li, R. Brossier, L. Métivier, *"3D frequency-domain elastic wave modeling with the spectral element method using a massively parallel direct solver"*, Geophysics, 2020.

Classical Discretization Method: FD, P^r-FEM/-SEM, P^r-DG, BEM, FV, RT etc.

$$\boxed{\Rightarrow \qquad \mathbf{A}_h \mathbf{u}_h = \mathbf{f}_h \qquad \text{(linear system)}}$$

- $\mathbf{A}_h \in \mathbb{C}^{N\times N}$ is symmetric, but not Hermitian, indefinite and ill-conditioned
- number of grid points $N \sim h^{-d} \sim k^{d(r+1)/r}$, r polynomial order, k wave number, h mesh size, d spatial dimension

Standard Numerical Methods:

- Direct Solvers (LU-Decomposition)
- Iterative Methods (GMRES)
- Domain Decomposition Methods
- Multigrid Methods

C. D. Riyanti, A. Kononov, Y. A. Erlangga, C. Vuik, C. Oosterlee, R. E. Plessix, and W. A. Mulder, *"A parallel multigrid-based preconditioner for the 3D heterogeneous high-frequency Helmholtz equation"*, J. Comput. Phys., 2007.

O. G. Ernst and M. J. Gander, *"Why it is Difficult to Solve Helmholtz Problems with Classical Iterative Methods"*, Lect. Notes Comput. Sci., 2012

P. Tsuji, J. Poulson, B. Engquist, and L. Ying, *"Sweeping preconditioners for elastic wave propagation with spectral element methods"*, ESAIM-M2AN, 2014

Y. Li, R. Brossier, L. Métivier, *"3D frequency-domain elastic wave modeling with the spectral element method using a massively parallel direct solver"*, Geophysics, 2020.

Outline

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Controllability Methods

Numerical Experiments

Other applications

Conclusion

SEISCOPE

Acoustic wave field: u

$$-\frac{\omega^2}{c^2}u - \Delta u = f$$

Parameter

- $\omega > 0$ angular frequency
- $c(x) \geq c_0 > 0$ wave speed

Elastic wave fields: $\mathbf{u}$

$$-\omega^2 \rho \mathbf{u} - \nabla \cdot \sigma(\mathbf{u}) = \mathbf{f}$$

Parameter

- $\omega > 0$ angular frequency
- $\rho(x) \geq \rho_0 > 0$ density
- $v_P(x)$ and $v_S(x)$ velocities
- $\sigma(\mathbf{u}) = \mathbf{C} : \varepsilon(\mathbf{u})$ (stress), $\mathbf{C}$ elastic modulus tensor
- $\varepsilon(\mathbf{u}) = \frac{1}{2}(\nabla \mathbf{u} + \nabla \mathbf{u}^\top)$ (strain)

Electromagnetic wave fields: (e, h)

$$\begin{aligned} -i\omega\varepsilon \mathbf{e} - \nabla \times \mathbf{h} &= \mathbf{j} \\ -i\omega\mu \mathbf{h} + \nabla \times \mathbf{e} &= 0 \end{aligned}$$

Parameter

- $\omega > 0$ angular frequency
- $\varepsilon(x) \geq \varepsilon_0 > 0$ permittivity
- $\mu(x) \geq \mu_0 > 0$ permeability

Elastic wave equations:

$$-\omega^2 \rho \mathbf{u} - \nabla \cdot \sigma(\mathbf{u}) \;=\; \mathbf{f} \text{ in } \Omega$$

Spongelayer $+$ First-order ABC on $\Gamma_S \subset \partial\Omega$ $+$ Free-surface on $\Gamma_N \cup \Gamma_D \subset \partial\Omega$

- S. Mönkölä, *"Time-harmonic solution for acousto-elastic interaction with controllability and spectral elements"*, J. Comput. Appl. Math., 2010.
- R. Brossier, L. Métivier, J. H. Tang, *"Solving frequency-domain elastic wave equations via parallel controllability methods"*, IMAGE Abstract, 2021.
- R. Brossier, L. Métivier, J. H. Tang, *"Fully scalable solver for frequency-domain visco-elastic wave equations in 3D heterogeneous media: a controllability approach"*, submitted, 2021.

Acoustic wave equation:

- M. O. Bristeau, R. Glowinski, J. Périaux, *"Controllability Methods for the Computation of Time-Periodic Solutions; Application to Scattering"*, J. Comput. Phys., 1998.
- R. Glowinski, T. Rossi *"A mixed formulation and exact controllability approach for the computation of the periodic solutions of the scalar wave equation. (I): Controllability problem formulation and related iterative solution"*, J. Comput. Phys., 2006.
- E. Heikkola, S. Mönkölä, A. Pennanen, T. Rossi, *"Controllability method for acoustic scattering with spectral elements"*, J. Comp. Appl. Math., 2007.
- M. J. Grote, J. H. Tang, *"On controllability methods for the Helmholtz equation"*, J. Comp. Appl. Math., 2019.
- M. J. Grote, F. Nataf, J. H. Tang, P.-H. Tournier, *"Parallel controllability methods for the Helmholtz equation"*, Comput. Method App. M., 2020.

Electromagnetic:

- M.-O. Bristeau, R. Glowinski, J. Périaux, and T. Rossi, *"3D Harmonic Maxwell Solutions on Vector and Parallel Computers using Controllability and Finite Element Methods"*, technical report, 1999.
- T. Chaumont-Frelet, M. J. Grote, S. Lanteri, J. H. Tang, *"A controllability method for Maxwell's equations"*, submitted, 2021.

Instead of solving the frequency-domain elastic equation directly, we go back to the time-domain:

Frequency-domain: $\mathbf{u}(x)$

$$
\begin{aligned}
-\omega^2 \rho \mathbf{u} - \nabla \cdot \sigma(\mathbf{u}) &= \mathbf{f} \text{ in } \Omega \\
-i\omega\rho \mathbf{B}\mathbf{u} + \sigma(\mathbf{u})\mathbf{n} &= \mathbf{0} \text{ on } \Gamma_S \\
\sigma(\mathbf{u})\mathbf{n} &= \mathbf{0} \text{ on } \Gamma_N \\
\mathbf{u} &= \mathbf{0} \text{ on } \Gamma_D
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{B} &= (\mathbf{v}_P - \mathbf{v}_S)\mathbf{n}\mathbf{n}^\top - \mathbf{v}_S \mathbf{I}_d \\
\sigma(\mathbf{u}) &= \mathbf{C} : \varepsilon(\mathbf{u}), \ \mathbf{C} = C_{ijkl} \\
\varepsilon(\mathbf{u}) &= \frac{1}{2}\left(\nabla \mathbf{u} + \nabla \mathbf{u}^\top\right)
\end{aligned}
$$

$\Longleftrightarrow$

Time-domain: $\mathbf{U}(x,t) = \mathrm{Re}\{\mathbf{u}(x)\mathrm{e}^{-i\omega t}\}$

$$
\begin{aligned}
\rho \frac{\partial^2}{\partial^2 t}\mathbf{U} - \nabla \cdot \sigma(\mathbf{U}) &= \mathrm{Re}\left\{\mathbf{f}\mathrm{e}^{-i\omega t}\right\} && \text{in } \Omega,\ t > 0 \\
\rho \mathbf{B} \frac{\partial}{\partial t}\mathbf{U} + \sigma(\mathbf{U})\mathbf{n} &= \mathbf{0} && \text{on } \Gamma_S,\ t > 0 \\
\sigma(\mathbf{U})\mathbf{n} &= \mathbf{0} && \text{on } \Gamma_N,\ t > 0 \\
\mathbf{U} &= \mathbf{0} && \text{on } \Gamma_D,\ t > 0 \\
\mathbf{U}(0) &= \mathbf{U}_0 && \text{in } \Omega \\
\frac{\partial}{\partial t}\mathbf{U}(0) &= \mathbf{V}_0 && \text{in } \Omega
\end{aligned}
$$

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\sigma(\mathbf{u})\mathbf{n} &= \mathbf{0} \text{ on } \Gamma_N \\
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\rho \mathbf{B} \frac{\partial}{\partial t}\mathbf{U} + \sigma(\mathbf{U})\mathbf{n} &= \mathbf{0} && \text{on } \Gamma_S,\ t > 0 \\
\sigma(\mathbf{U})\mathbf{n} &= \mathbf{0} && \text{on } \Gamma_N,\ t > 0 \\
\mathbf{U} &= \mathbf{0} && \text{on } \Gamma_D,\ t > 0 \\
\mathbf{U}(0) &= \mathbf{U}_0 && \text{in } \Omega \text{ (unknown)} \\
\frac{\partial}{\partial t}\mathbf{U}(0) &= \mathbf{V}_0 && \text{in } \Omega \text{ (unknown)}
\end{aligned}
$$

$$
\Rightarrow \mathbf{u}(x) = \mathbf{U}_0(x) + \frac{i}{\omega}\mathbf{V}_0(x)
$$

Looking for a T-time-periodic solution ($T = \frac{2\pi}{\omega} = \frac{1}{freq}$):

Theorem: eigenmode decomposition

Let $\mathbf{u}$ be the unique frequency-domain solution and $\mathbf{U}(\mathbf{x}, \mathbf{t})$ be a T-time periodic solution of the elastic wave equation in the sense that

$$\begin{aligned} \mathbf{U}(x,T) &= \mathbf{U}(x,0) &= \mathbf{U}_0(x), \\ \dot{\mathbf{U}}(x,T) &= \dot{\mathbf{U}}(x,0) &= \mathbf{V}_0(x). \end{aligned}$$

Then there is a (Fourier-) series $\boldsymbol{\gamma}_\ell$, $\ell \neq 1$, of eigenmodes such that

$$\mathbf{U}(x,t) = \mathrm{Re}\left\{\mathbf{u}(x)\mathrm{e}^{-i\omega t}\right\} + \boldsymbol{\gamma}_0(x) + \sum_{\ell=2}^{\infty} \mathrm{Re}\left\{\boldsymbol{\gamma}_\ell(x)\mathrm{e}^{-i\omega \ell t}\right\},$$

where γ_ℓ solves the frequency-domain wave equation with $\omega\ell$ instead of ω and $\mathbf{f} = 0$.

Corollary

Let $|\Gamma_S| > 0$ and $|\Gamma_D| > 0$. Then $\boldsymbol{\gamma}_\ell = \boldsymbol{\gamma}_0 = 0$ a.e. in Ω and

$$\mathbf{U}(x,t) = \mathrm{Re}\left\{\mathbf{u}(x)\mathrm{e}^{-i\omega t}\right\} \quad \Rightarrow \quad \mathbf{u}(x) = \mathbf{U}_0(x) + \frac{i}{\omega}\mathbf{V}_0(x).$$

We formulate a PDE-constrained optimization problem using controllability technique to find the T-time-periodic solution:

$$\boxed{\min_{\mathbf{v}\in H} J(\mathbf{v})}$$

with the cost functional $J : H \to \mathbb{R}_{\geq 0}$, $H = (H^1(\Omega))^d \times (L^2(\Omega))^d$ Hilbert space,

$$J(\mathbf{U}_0, \mathbf{V}_0) = \frac{1}{2}\int_\Omega \mathbf{C} : \varepsilon(\mathbf{U}(x,T) - \mathbf{U}_0(x)) : \varepsilon(\mathbf{U}(x,T) - \mathbf{U}_0(x))\; dx + \frac{1}{2}\int_\Omega \rho(x)|\dot{\mathbf{U}}(x,T) - \mathbf{V}_0(x)|^2\; dx,$$

where the state variable $\mathbf{U}$ is the solution of the elastic wave equation with the initial value / control variable $\mathbf{v} = (\mathbf{U}_0, \mathbf{V}_0)$,

- $\mathbf{U}_0$ initial displacement,
- $\mathbf{V}_0$ initial speed.

The cost functional J is minimal if, and only if, $J(\mathbf{U}_0, \mathbf{V}_0) = 0$, that is

$$\varepsilon(\mathbf{U}(\mathbf{x},\mathbf{T}) - \mathbf{U}_0(\mathbf{x})) = \mathbf{0} \quad \text{and} \quad \dot{\mathbf{U}}(\mathbf{x},\mathbf{T}) - \dot{\mathbf{U}}_0(\mathbf{x}) = \mathbf{0} \quad \not\Rightarrow \quad \boldsymbol{\eta}(x) := \mathbf{U}(x,T) - \mathbf{U}_0(x) = \mathbf{0}$$

where $\boldsymbol{\eta} \in \mathcal{U} := \mathrm{kernel}(\nabla\cdot\sigma) = \{\psi_1\} \oplus \cdots \oplus \{\psi_6\} = \mathrm{Span}\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, (0,-x_3,x_2), (x_3,0,-x_2), (x_2,-x_1,0)\}$

$$\Rightarrow \underbrace{\mathbf{U}(x,t) - \boldsymbol{\eta}(x)\frac{t}{T}}_{T\text{-time periodic solution}} = \underbrace{\mathrm{Re}\left\{\mathbf{u}(x)\mathrm{e}^{-i\omega t}\right\} + \boldsymbol{\gamma}_0(x) + \sum_{\ell=2}^{\infty}\mathrm{Re}\left\{\boldsymbol{\gamma}_\ell(x)\mathrm{e}^{-i\omega\ell t}\right\}}_{\text{Theorem}}$$

Sound-soft scattering ($|\Gamma_D|, |\Gamma_S| > 0$)

$\boldsymbol{\eta} = \boldsymbol{\gamma}_0 = \boldsymbol{\gamma}_\ell = \mathbf{0}$, $\forall \ell \geq 2$

$\Rightarrow \mathbf{U}(x,t) = \mathrm{Re}\left\{\mathbf{u}(x)\mathrm{e}^{-i\omega t}\right\}$

$\Rightarrow \mathbf{u}(x) = \mathbf{U}_0(x) + \frac{i}{\omega}\mathbf{V}_0(x)$

S. Mönkölä, *"Time-harmonic solution for acousto-elastic interaction with controllability and spectral elements"*, J. Comput. Appl. Math., 2010

(Sound-hard) scattering ($|\Gamma_S| > 0$)

$\boldsymbol{\eta} = \boldsymbol{\gamma}_\ell = \mathbf{0}$, $\forall \ell \geq 2$, $\boldsymbol{\gamma}_0 \in \mathcal{U}$

- Orthogonal projection on $\mathbf{v} = \mathbf{U}_0 + \frac{i}{\omega}\mathbf{V}_0$:

$$\mathbf{u} = \mathbf{v} - \frac{1}{\omega^2}\sum_{\ell=1}^{6}(f + \omega^2\mathbf{v} + \nabla\cdot\sigma(\mathbf{v}), \psi_\ell)\psi_\ell$$

- Filtering post-procedure:

$$\mathbf{u}(x) = \frac{2}{T}\int_0^T \mathbf{U}(x,t)\mathrm{e}^{i\omega t}\,dt$$

Bounded domain ($\Gamma_S = \emptyset$)

Filtering + orthogonal Projection:

$$\mathbf{u}(x) = \hat{\mathbf{u}}(x) + \frac{i}{2\pi}\boldsymbol{\eta}(x)$$

where

$$\hat{\mathbf{u}}(x) = \frac{2}{T}\int_0^T \mathbf{U}(x,t)\mathrm{e}^{i\omega t}\,dt$$

$$\boldsymbol{\eta} = \frac{2i\pi}{\omega^2}\sum_{\ell=1}^{6}(\mathbf{f} + \omega^2\hat{\mathbf{y}} + \nabla\cdot\sigma(\hat{\mathbf{y}}), \psi_\ell)\psi_\ell$$

i. Initial Step

  a. Compute the gradient $\hat{g}^{(0)} = J'(v^{(0)}) \in H^*$
  (required the **forward and backward time-dependent wave fields**).

  b. Find the (Riesz) representative $g^{(0)}$ of $\hat{g}^{(0)}$ in H,

$$\Rightarrow (g, \varphi)_H = \langle \hat{g}, \varphi \rangle_{H^*}, \qquad \forall \varphi \in H, \qquad \text{(new elliptic problem } (\Rightarrow \text{(inner) CG method)),}$$

  with $g = g^{(0)}$ and $\hat{g} = \hat{g}^{(0)}$.

  c. Set $d^{(0)} = r^{(0)} = -g^{(0)}$.

ii. Iterative Step
For $k = 0, 1, \ldots$

  a. Compute the gradient $\hat{g}^{(k)} = J'(d^{(k)}) \in H^*$ and the representative $g^{(k)}$ of $\hat{g}^{(k)}$.

  b. Compute $\alpha_k = \frac{(r^{(k)}, r^{(k)})_H}{(g^{(k)}, d^{(k)})_H}$.

  c. Update $v^{(k+1)} = v^{(k)} + \alpha_k d^{(k)}$.

  d. Update $r^{(k+1)} = r^{(k)} - \alpha_k g^{(k)}$.

  e. Break when $|r^{(k+1)}| < \text{tol}\ |r^{(0)}|$.

  f. Compute $\beta_k = \frac{(r^{(k+1)}, r^{(k+1)})_H}{(r^{(k)}, r^{(k)})_H}$.

  g. Update $d^{(k+1)} = r^{(k)} + \beta_k d^{(k)}$.

Looking for the solution of the elliptic problem:

$$\begin{aligned} -\nabla \cdot \sigma(g) &= \hat{g} \text{ in } \Omega, \\ \sigma(g)\mathbf{n} &= 0 \text{ on } \Gamma_S \cup \Gamma_N, \\ g &= 0 \text{ on } \Gamma_D. \end{aligned}$$

Remark: In the absence of Γ_D the above elliptic problem is not well-posed. We need additional conditions:

- $\hat{g}$ is in range of $\mathcal{L} := -\nabla \cdot \sigma$.
- The mean of g is zero, $\int g \; dx = 0$.

Recall that $\mathrm{kernel}(\mathcal{L}) = \mathrm{Span}\{(1,0,0),(0,1,0),(0,0,1),(0,-z,y),(z,0,-y),(y,-x,0)\}$. Use the orthogonal projection:

$$\hat{g} \leftarrow \hat{g} - \frac{(\hat{g},\boldsymbol{\psi})}{(\boldsymbol{\psi},\boldsymbol{\psi})}\boldsymbol{\psi}, \qquad \forall \boldsymbol{\psi} \in \mathrm{kernel}(\mathcal{L}).$$

Looking for the solution of the elliptic problem:

$$-\nabla \cdot \sigma(g) \;=\; \hat{g} \text{ in } \Omega$$

From SE discretization we obtain

$$\mathbf{K}\mathbf{g} = \hat{\mathbf{g}},$$

where the system matrix $\mathbf{K}$ is symmetric positive definite.

To accelerate the convergence, we use the (diagonal) Jacobi preconditioner $\mathbf{P}^{-1}$ of $\mathbf{K}$.

- It requires only the solutions of **time-dependent wave equations** over a short time period (T) and the solution of a **linear system**.
  - Explicit methods for the time integration are inherently parallel (vector-vector and matrix-vector multiplications).
  - The linear system is independent of the frequency and the system matrix is symmetric positive definite ($\Rightarrow$ (Inner) CG method).
- The optimization problem is quadratic and strictly convex ($\Rightarrow$ (Outer) CG method).

Let N be the number of grid points in one dimension.

SEM $\Rightarrow$ each matrix-vector and vector-vector multiplication $= \mathcal{O}(N^3)$

- **3D time-domain modelling**
  - time-domain modeling $\Rightarrow \mathcal{O}(N_t N^3)$
    N_t = number of time steps controlled by CFL restriction
- **3D CMCG modelling**
  - a single period time-domain modeling $\Rightarrow \mathcal{O}(N_T N^3) = \mathcal{O}(N^3)$
    $N_T = \frac{T}{\Delta t}$ (number of time steps), $T = \frac{2\pi}{\omega}$ (single period), $\Delta t \sim \frac{1}{N} \sim \frac{1}{\omega}$ (time step)
    $\Rightarrow N_T$ is independent of N and ω (= constant)
  - CMCG method in N_{cg} CG-iterations $\Rightarrow \mathcal{O}(N_{cg} N^3)$
  - CMCG method for problems with N_ω frequencies $\Rightarrow \mathcal{O}(N_\omega N_{cg} N^3)$

$\Rightarrow$ improvement in complexity when $\underline{\underline{N_\omega N_{cg} \ll N_t}}$!

Outline

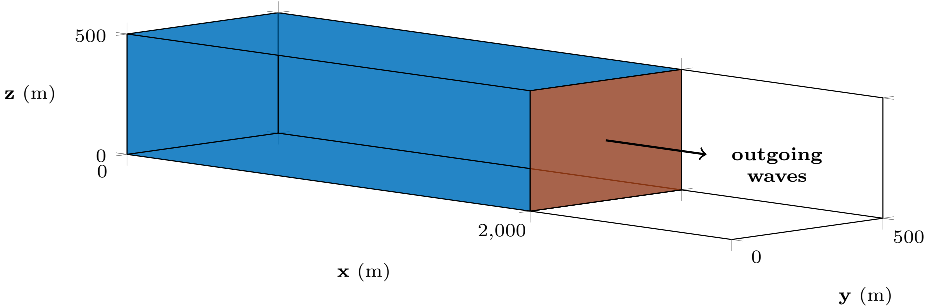
Outline

We consider the plane waves $\mathbf{u}$ in $\Omega = (0, 2000m) \times (0, 500m) \times (0, 500m)$, given by

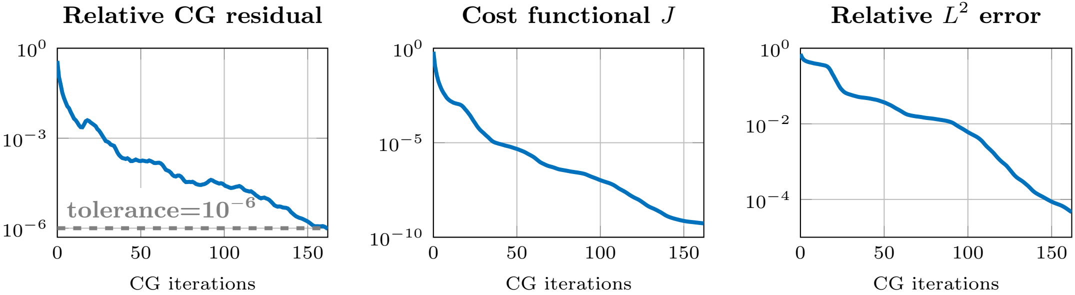
$$\mathbf{u}(\mathbf{x}) = \mathbf{A} \exp(i\mathbf{k} \cdot \mathbf{x}),$$

with ABCs on $\{x = 2000m\}$ and Dirichlet boundary conditions everywhere:

$\mathbf{A} \parallel \mathbf{k}$ and $\|\mathbf{k}\| = \frac{\omega}{v_P}$ with frequency $f = 10\text{Hz}$, $v_p = 5000\text{m/s}$, $v_s = 2500\text{m/s}$, $\rho = 1\ \text{g/m}^3$



We apply the CMCG method ($\mathcal{P}^5$-SEM + RK4 ($h = 50m$)) with CG tolerance$=10^{-6}$ to find the time-harmonic solution. It converges at 162 iterations ($\Rightarrow$ wave solver for 324 time periods).

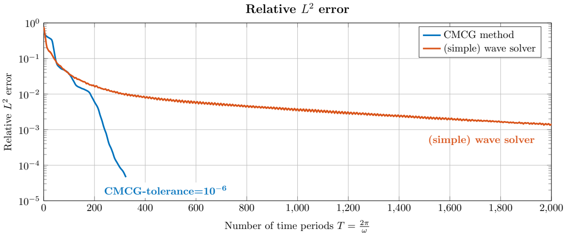


Questions:

- From the limit amplitude principle, a sufficiently long solution of the wave equation yields a time-harmonic solution. Do we then still need the CMCG method?
- Is it possible to accelerate the convergence with an initial estimate?

Planewaves(3)

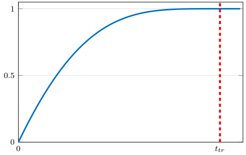
It was shown that the "simple" wave equation solver may not converge.



Planewaves(4)

An alternative wave solver has been proposed: a smooth (transient) function $\theta_{tr}(t)$ is multiplied with the external source terms. $t_{tr} \in T\mathbb{N}$ is the length of the transient phase and $T = 2\pi/\omega$ is a single period.

$\theta_{tr}(t),\ t \in (0, t_{tr})$



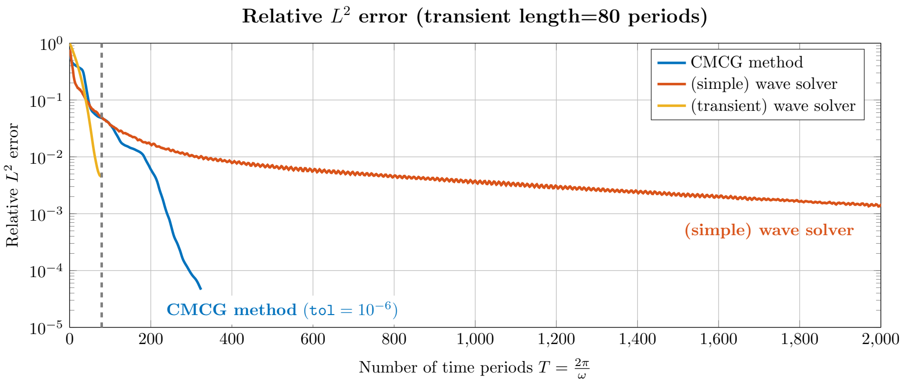
$$
\begin{aligned}
\rho \frac{\partial^2}{\partial^2 t}\mathbf{U}(t) - \nabla \cdot \sigma(\mathbf{U}(t)) &= \theta_{tr}(t)\,\mathrm{Re}\left\{\mathbf{f}\mathrm{e}^{-i\omega t}\right\} && \text{in } \Omega,\ t \in (0, t_{tr}),\\
\rho \mathbf{B} \frac{\partial}{\partial t}\mathbf{U}(t) + \sigma(\mathbf{U}(t))\mathbf{n} &= \theta_{tr}(t)\,\mathrm{Re}\left\{\mathbf{g}_S\mathrm{e}^{-i\omega t}\right\} && \text{on } \Gamma_S,\ t \in (0, t_{tr}),\\
\sigma(\mathbf{U}(t))\mathbf{n} &= \theta_{tr}(t)\,\mathrm{Re}\left\{\mathbf{g}_N\mathrm{e}^{-i\omega t}\right\} && \text{on } \Gamma_N,\ t \in (0, t_{tr}),\\
\mathbf{U}(t) &= \theta_{tr}(t)\,\mathrm{Re}\left\{\mathbf{g}_D\mathrm{e}^{-i\omega t}\right\} && \text{on } \Gamma_D,\ t \in (0, t_{tr})
\end{aligned}
$$

G. Mur, *"The finite-element modeling of three-dimensional electromagnetic fields using edge and nodal elements"*,IEEE Transactions on Antennas and Propagation, 1993

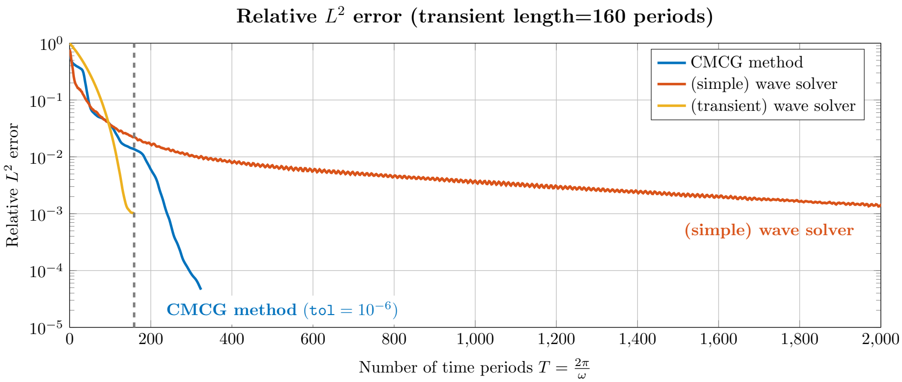
M.-O. Bristeau, R. Glowinski and J. Périaux, *"Controllability methods for the calculation of time-periodic solutions. Application to scattering"*, J. Comput. Phys., 1998

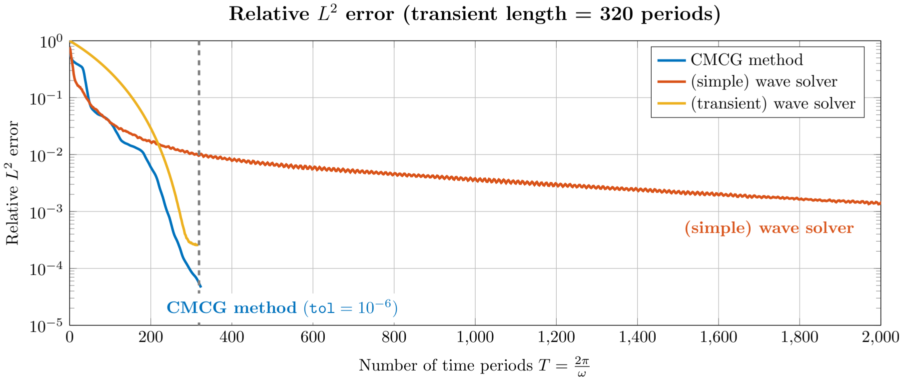
M. J. Grote, F. Nataf, J. H. Tang, P.-H. Tournier, *"Parallel controllability methods for the Helmholtz equation"*, Comput. Method App. M., 2020

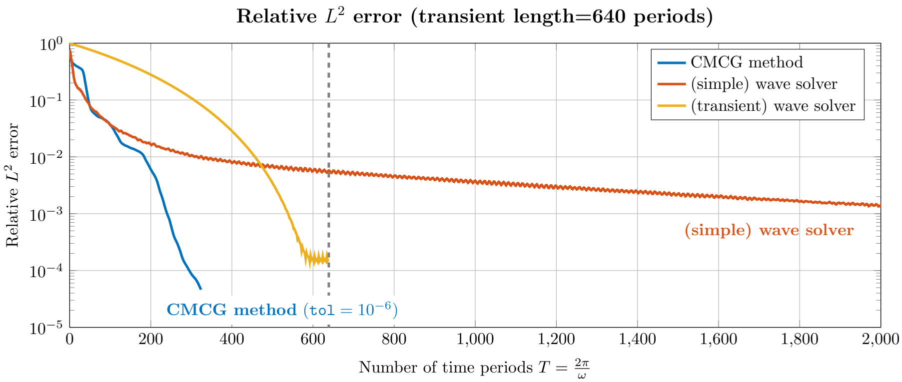
Planewaves(5)



Planewaves(5)



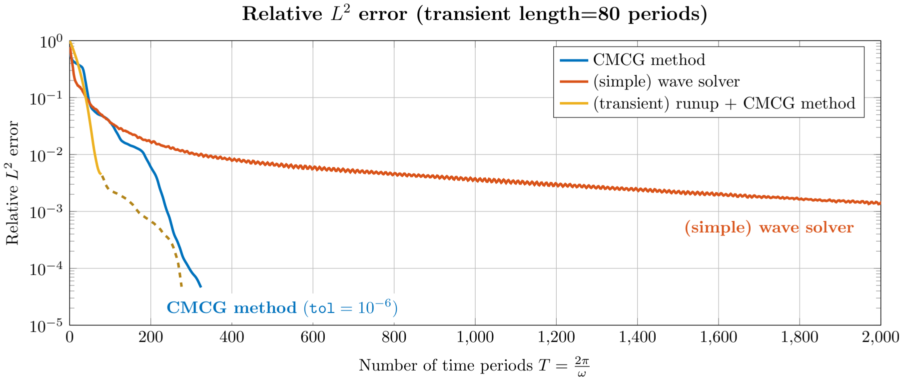

Relative L^2 error (transient length = 320 periods)
Relative L^2 error
10^0
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
0
200
400
600
800
1,000
1,200
1,400
1,600
1,800
2,000
Number of time periods $T = \frac{2\pi}{\omega}$
CMCG method
(simple) wave solver
(transient) wave solver
(simple) wave solver
CMCG method (tol $= 10^{-6}$)


Relative L^2 error (transient length=640 periods)
CMCG method
(simple) wave solver
(transient) wave solver
(simple) wave solver
CMCG method (tol $= 10^{-6}$)
Relative L^2 error
Number of time periods $T = \frac{2\pi}{\omega}$
10^0
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
0
200
400
600
800
1,000
1,200
1,400
1,600
1,800
2,000

Only the wave solver may converge slowly

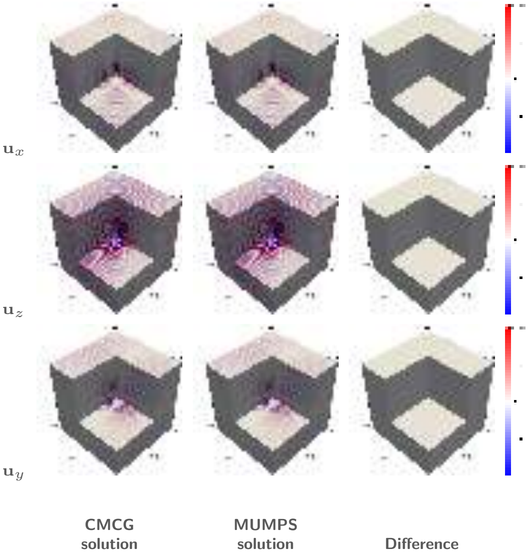
⇒ Use the wave solver for the initial estimate

⇒ Apply the initial estimate to the CMCG method as the initial guess



⇒ The length of the run-up phase is a free parameter; the CMCG method always converges.

Outline



Elastic Model ($\mathcal{P}^5$-SEM):

$$\begin{cases} -\omega^2\rho(x)\mathbf{u}(x) - \nabla\cdot\sigma(\mathbf{u}) = & \mathbf{f}(x) & \text{in } \Omega = (0, 2.5km)^3 \\ -i\omega\rho(x)\mathbf{B}(x)\mathbf{u}(x) + \sigma(\mathbf{u}(x))\mathbf{n} = & 0 & \text{on } \partial\Omega \end{cases}$$

Source: $(x_s, z_s, y_s) = (1235.3m, 1235.3m, 1235.3m)$

n_{sponge}	n_{DOFs}/λ	$V_P[\frac{\text{m}}{\text{s}}]$	$V_S[\frac{\text{m}}{\text{s}}]$	$\rho[\frac{\text{g}}{\text{m}^3}]$	f [Hz]
4	5	5000	2500	1	20

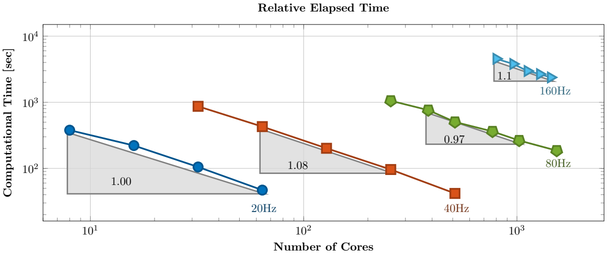
Comparison: CMCG method vs. MUMPS direct solver

	#Cores	Elapsed Timing	Max alloc. Mem.	rel. L^2 error u_x	rel. L^2 error u_z	rel. L^2 error u_y
MUMPS	256	855s	1423 GB	4.6%	8.7%	4.6%
CMCG	8	378s	**0.8 GB**	3.6%	7.3%	3.6%

Y. Li, R. Brossier, L. Métivier, *"3D frequency-domain elastic wave modeling with the spectral element method using a massively parallel direct solver"*, Geophysics, 2020.

Pointsource problems (20Hz-160Hz): CMCG

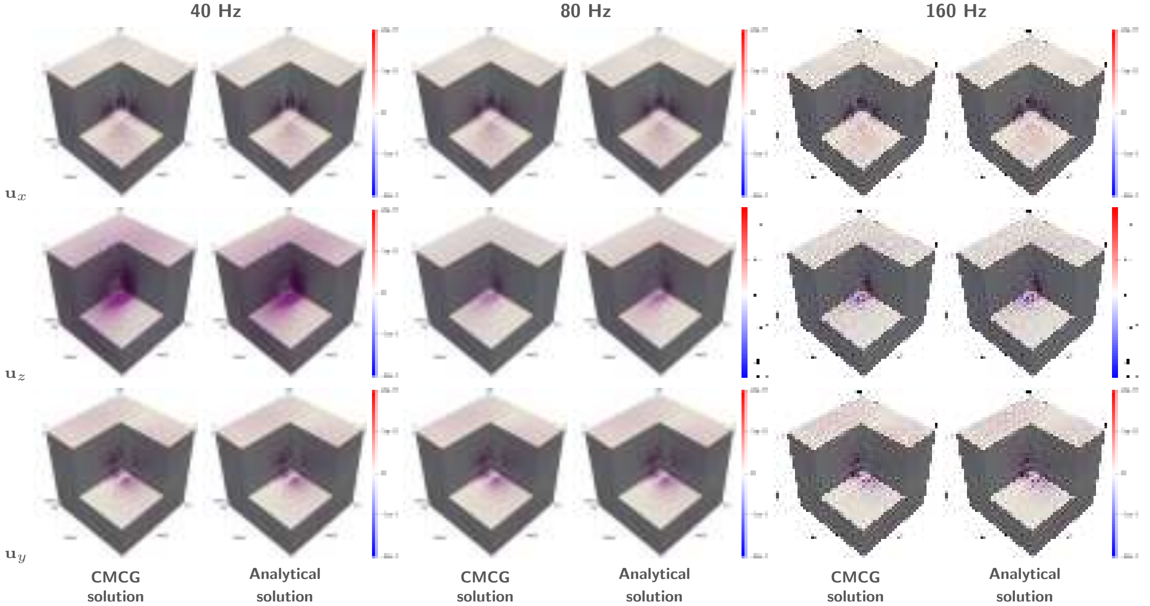
f	n_{DOFs}	#{Cores}	Estim. memory	CG iter.	Elapsed timing
20Hz	8M	8	0.8gb	17	378s
		16	0.8gb	17	222s
		32	0.8gb	17	105s
		64	0.8gb	17	47s
40Hz	42M	32	3.9gb	21	868s
		64	3.9gb	21	428s
		128	4.0gb	21	201s
		256	4.0gb	21	96s
		512	4.1gb	21	42s
80Hz	257M	256	23.9gb	23	1054s
		384	24.0gb	23	761s
		512	24.1gb	23	504s
		768	24.3gb	23	362s
		1024	24.5gb	23	264s
		1536	24.7gb	23	186s
160Hz	1784M	800	165.3gb	27	4547s
		960	165.7gb	27	3796s
		1120	166.9gb	27	2973s
		1280	165.7gb	27	2665s
		1440	166.1gb	27	2375s



Total elapsed timing spent in the CMCG method by increasing the number of cores (f = **20**, **40**, **80**, and **160** Hz)

⇒ **(Super) linear scaling**

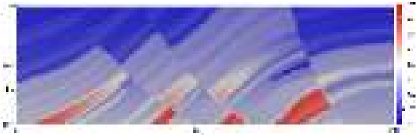
Pointsource problems: solutions at 40Hz, 80Hz, and 160Hz



Elastic Model ($\mathcal{P}^5$-SEM + RK4):

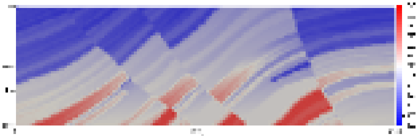
$$
\begin{aligned}
-\omega^2\rho(x)\mathbf{u}(x) - \nabla\cdot\sigma(\mathbf{u}) &= \mathbf{f}(x) && \text{in} \quad \Omega = (0, 39km)\times(0,12km)\times(0,10km),\\
\sigma(\mathbf{u})\mathbf{n} &= 0 && \text{on} \quad \{z=0\},\\
-i\omega\rho(x)\mathbf{B}(x)\mathbf{u}(x) + \sigma(x,\mathbf{u})\mathbf{n} &= 0 && \text{on} \quad \{x=0\}\cup\{x=39\}\cup\{z=12\}\cup\{y=0\}\cup\{y=10\},
\end{aligned}
$$

Source location: $(x_s, z_s, y_s) = (1950m, 100m, 500m)$



(a) velocity v_P

(b) velocity v_S

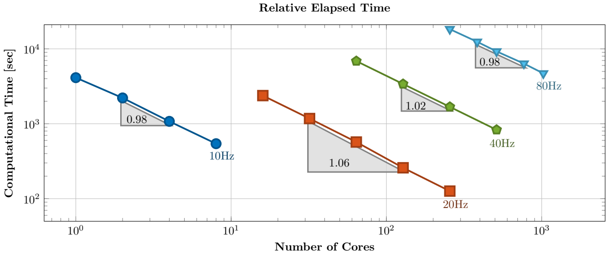


(c) density ρ

3D-Elastic Marmousi: total computational time

Methods: $\mathcal{P}^5$-SEM + 4 sponge elements + RK4 + transient/simple runup + cmcg + filtering procedure

f	n_{DOFs}	#{Cores}	Estim. memory	CG iter.	Elapsed timing
10Hz	5M	1	0.5gb	25	4126s
		2	0.5gb	25	2213s
		4	0.5gb	25	1077s
		8	0.5gb	25	542s
20Hz	26M	16	2.3gb	17	2378s
		32	2.4gb	17	1175s
		64	2.4gb	17	571s
		128	2.4gb	17	259s
		256	2.5gb	17	126s
40Hz	155M	64	14.2gb	29	6870s
		128	14.3gb	29	3397s
		256	14.6gb	29	1682s
		512	14.7gb	29	831s
80Hz	1061M	256	97.5gb	21	18323s
		384	97.7gb	21	12425s
		512	97.9gb	21	9253s
		768	98.3gb	21	6358s
		1024	98.8gb	21	4704s

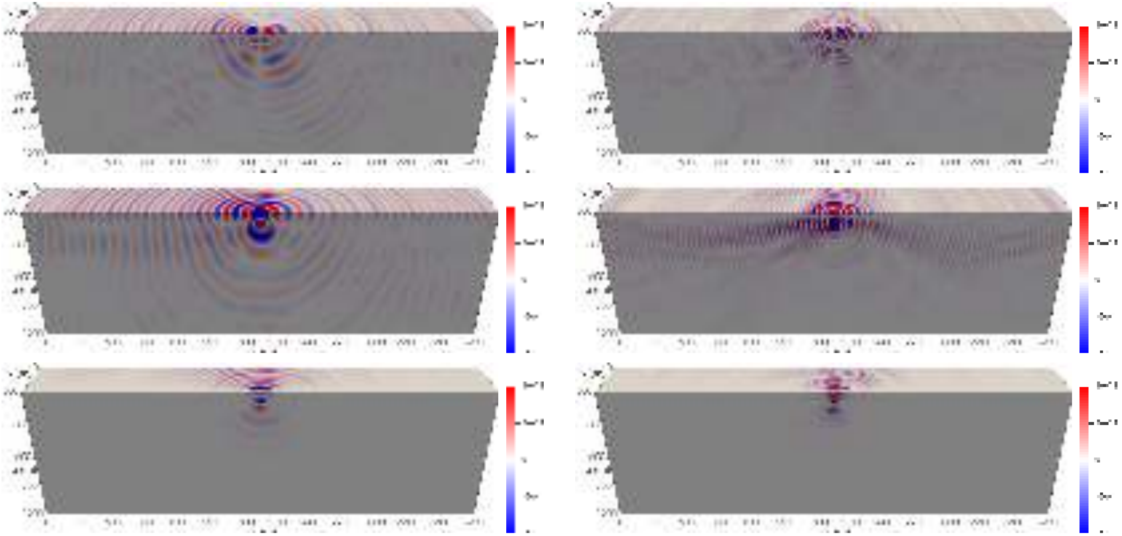


⇒ **(Super) linear scaling**

3D-Elastic Marmousi model (2): CMCG solutions (10Hz and 20Hz)

10Hz

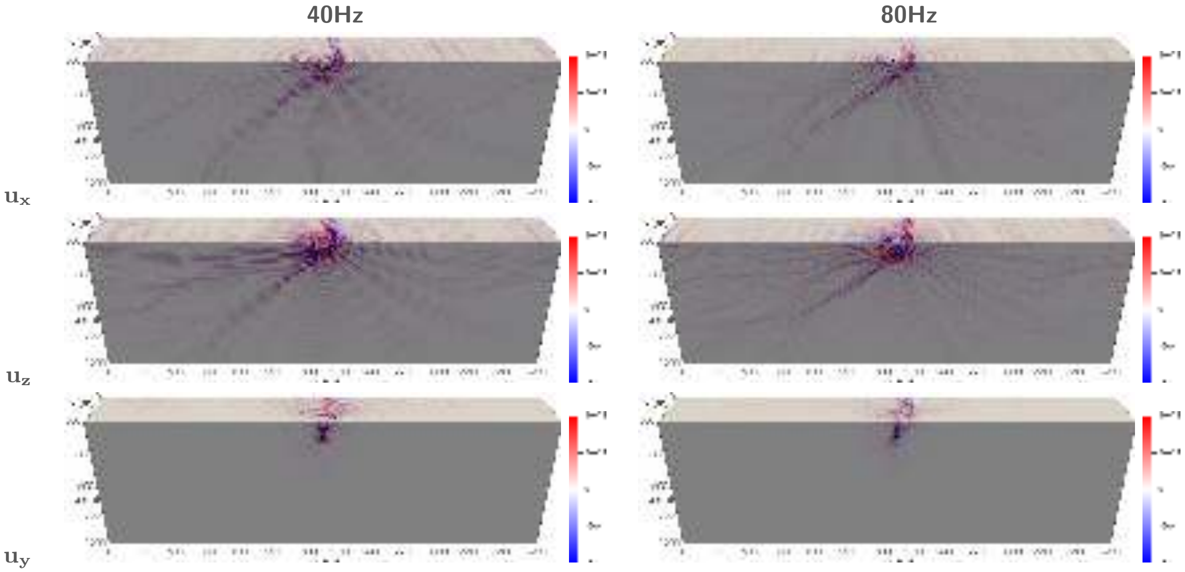
20Hz



$\mathbf{u_x}$

$\mathbf{u_z}$

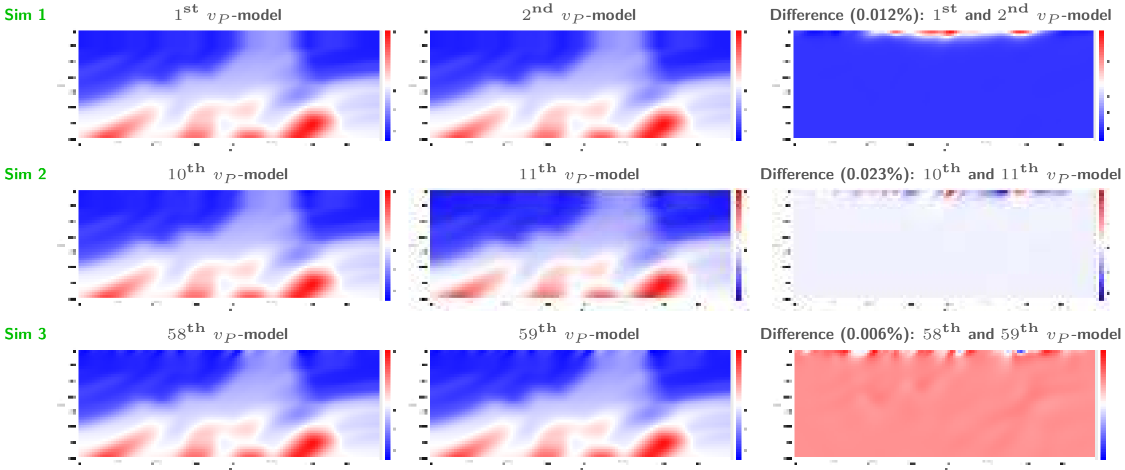
$\mathbf{u_y}$


40Hz
80Hz
u_x
u_z
u_y

Outline

As shown in the previous experiment, the convergence of the CMCG method is accelerated by the good initial estimate. In practice, e.g., FWI, the wavefields from two similar models are quite close.

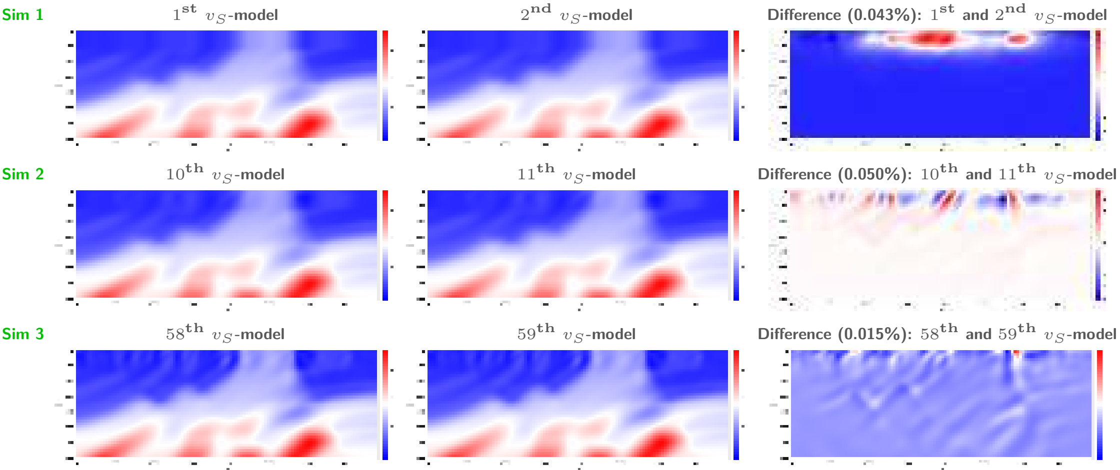
Question: Suppose we know the solution from a similar model, e.g., the wavefields from the previous FWI iteration. Can we benefit from it?



Marmousi models from FWI

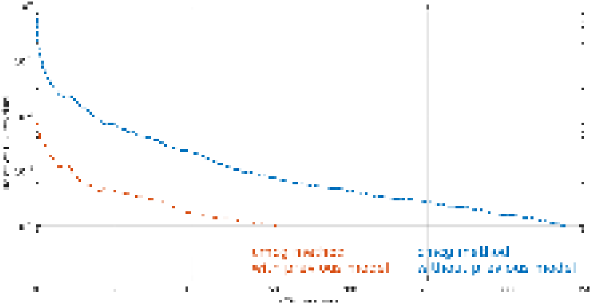
As shown in the previous experiment, the convergence of the CMCG method is accelerated by the good initial estimate. In practice, e.g., FWI, the wavefields from two similar models are quite close.

Question: Suppose we know the solution from a similar model, e.g., the wavefields from the previous FWI iteration. Can we benefit from it?

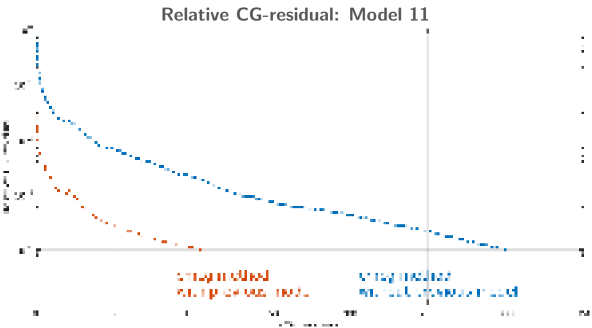


Marmousi models from FWI(2)

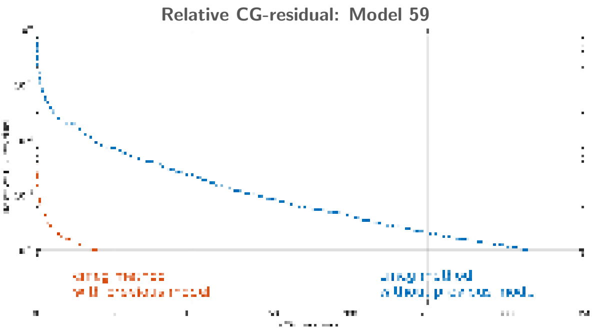
Relative CG-residual: Model 2



⇒ The CMCG method is significantly accelerated by a good initial estimate (reduction: 55%).



⇒ The CMCG method is significantly accelerated by a good initial estimate (reduction: 65%).



⇒ The CMCG method is significantly accelerated by a good initial estimate (reduction: 87%).

CMCG with multiple-frequencies

Elastic Model:

$$-\omega_j^2 \rho \mathbf{u}_j - \nabla \cdot \sigma(\mathbf{u}_j) \;=\; \mathbf{f}_j, \qquad j = 1, \ldots, N_\omega$$

where $\omega_1 < \omega_2 < \ldots < \omega_{N_\omega}$ and $\frac{\omega_j}{\omega_1} \in \mathbb{N} \;\forall j$

Linear system:

$$\mathbf{L}_j \mathbf{u}_j \;=\; \mathbf{f}_j, \qquad j = 1, \ldots, N_\omega \quad \Rightarrow \quad N_\omega \text{ linear systems}$$

CM approach: Let $\mathbf{U}$ be the solution of the time-dependent elastic wave equation with

$$\mathbf{F}(x,t) = \sum_{j=1}^{N_\omega} \mathrm{Re}\{\mathbf{f}_j(x) \exp(-i\omega_j t)\}.$$

Then a (single) CM yields in one shot the superposition

$$\mathbf{U}(x,t) = \sum_{j=1}^{N_\omega} \mathrm{Re}\{\mathbf{u}_j(x) \exp(-i\omega_j t)\}.$$

Elastic Model:

$$-\omega_j^2 \rho \mathbf{u}_j - \nabla \cdot \sigma(\mathbf{u}_j) \;=\; \mathbf{f}_j, \qquad j = 1, \ldots, N_\omega$$

where $\omega_1 < \omega_2 < \ldots < \omega_{N_\omega}$ and $\frac{\omega_j}{\omega_1} \in \mathbb{N} \;\forall j$

CM approach: A single CM yields the solution

$$\mathbf{U}(x,t) = \sum_{j=1}^{N_\omega} \mathrm{Re}\{\mathbf{u}_j(x)\exp(-i\omega_j t)\}.$$

Filtering procedure:

$$\mathbf{u}_j(x) = \frac{2}{T}\int_0^T \mathbf{U}(x,t)\exp(i\omega_j t)\; dt, \quad j = 1, \ldots, N_\omega \quad \Rightarrow \quad \mathbf{u}_1, \ldots, \mathbf{u}_{N_\omega}.$$

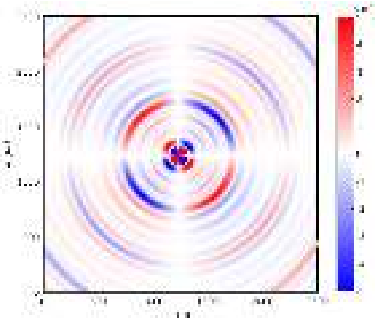
$\Rightarrow$ CM yields in one shot N_ω solutions

Elastic model ($\mathcal{P}^5$-SEM + RK4) :

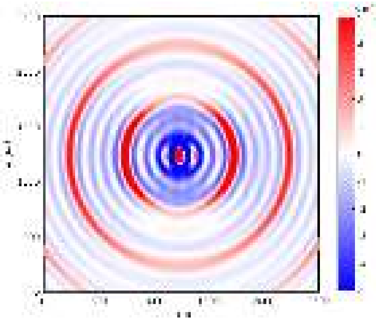
$$-\omega_j^2 \rho(x)\mathbf{u}_j(x) - \nabla \cdot \sigma(\mathbf{u}_j) \quad = \quad \mathbf{f}_j(x) \quad \text{in} \quad \Omega = (0, 2500m)^3$$

- First-order absorbing boundary conditions with sponge layers (length=4)
- Frequencies **5Hz**, **10Hz**, **15Hz** and **20Hz** $\Rightarrow \omega_j = 10\pi j,\ j = 1, \ldots, N_\omega = 4$
- $v_P = 5000$m/s, $v_S = 2500$m/s
- $\mathbf{f}_j = \mathbf{f}$ single (Dirac) point source

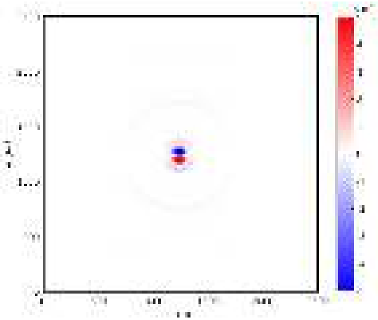
The CM, combined with the initial run-up phase (**25**), yields in one-shot the superposition $\mathbf{u} = (u_x, u_z, u_y)$ (**4** CG-iterations):



u_x



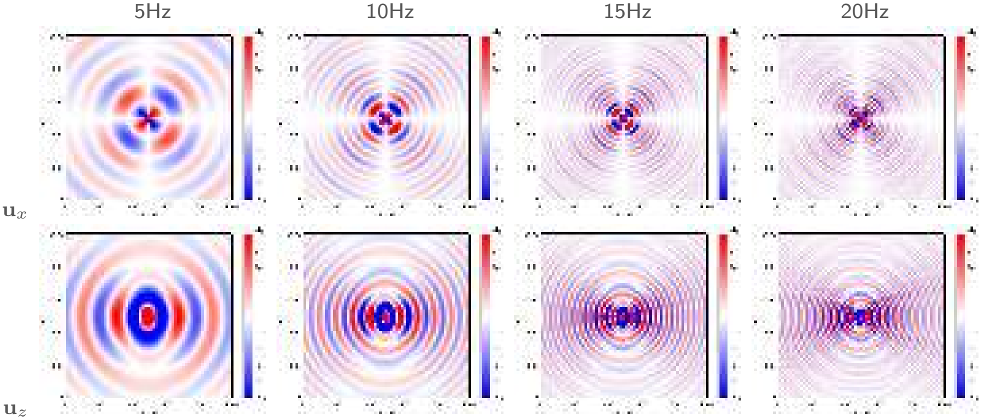
u_z



u_y

CMCG with Multiple-frequencies (5)

The filtering procedure yields



Rel. L^2-error (u_x, u_z, u_y)				
cmcg (superposition)	$2.16\%, 1.93\%, 2.16\%$	$1.55\%, 1.48\%, 1.55\%$	$1.54\%, 1.51\%, 1.54\%$	$4.02\%, 8.99\%, 4.02\%$
cmcg (single `freq`)	$2.09\%, 1.89\%, 2.09\%$	$1.55\%, 1.50\%, 1.55\%$	$1.52\%, 1.50\%, 1.52\%$	$3.61\%, 7.28\%, 3.61\%$

Outline

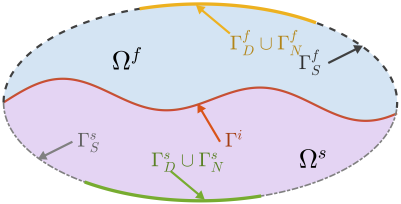
Outline

Acoustic wave equation (fluid)

$$
\begin{aligned}
-\frac{\omega^2}{\kappa(x)} P^f(x) - \nabla \cdot \left(\frac{1}{\rho_f(x)} \nabla P^f(x) \right) &= F(x) \quad \text{in } \Omega^f \\
c(x) \nabla P^f(x) \cdot \mathbf{n} - i\omega P^f(x) &= G_S(x) \quad \text{on } \Gamma_S^f \\
P^f(x) &= G_D(x) \quad \text{on } \Gamma_D^f \\
\nabla P^f(x) \cdot \mathbf{n} &= G_N(x) \quad \text{on } \Gamma_N^f
\end{aligned}
$$

Elastic wave equation (solid)

$$
\begin{aligned}
-\omega^2 \rho_s(x) \mathbf{U}^s(x) - \nabla \cdot \sigma(x, \mathbf{U}^s) &= \mathbf{F}(x) \quad \text{in } \Omega^s \\
-i\omega \rho_s(x) \mathcal{B} \mathbf{U}^s(x) + \sigma(x, \mathbf{U}^s) \mathbf{n} &= \mathbf{G}_S(x) \quad \text{on } \Gamma_S^s \\
\mathbf{U}^s(x) &= \mathbf{G}_D(x) \quad \text{on } \Gamma_D^s \\
\sigma(x, \mathbf{U}^s) \mathbf{n} &= \mathbf{G}_N(x) \quad \text{on } \Gamma_N^s
\end{aligned}
$$



Interface

$$
\begin{aligned}
\omega^2 \rho_f(x) \mathbf{U}^s(x) \cdot \mathbf{n} &= \frac{\partial P^f(x)}{\partial \mathbf{n}}, \quad x \in \Gamma^i \\
\sigma(x, \mathbf{U}^s) \mathbf{n} &= -P^f(x) \mathbf{n}, \quad x \in \Gamma^i
\end{aligned}
$$

Figure 2: Computational domain consisting of solid Ω^s and fluid Ω^f region with an interaction Γ^i between both regions, artificial boundaries Γ_S^f and Γ_S^s as well as some physical boundaries Γ_D^f, Γ_N^f and Γ_D^s, Γ_N^s.

S. Mönkölä, *"Time-harmonic solution for acousto-elastic interaction with controllability and spectral elements"*, J. Comput. Appl. Math., 2010.

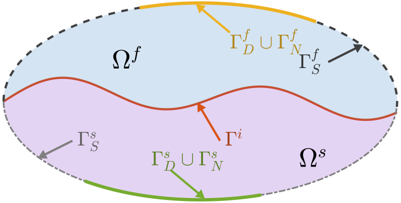
J. Cao, R. Brossier, and L. Métivier, *"3D acoustic-(visco)elastic coupled formulation and its spectral-element implementation on a Cartesian-based hexahedral mesh"*, SEG Abstract 2020.

Acoustic wave equation (fluid)

$$
\begin{aligned}
\frac{1}{\kappa(x)}\ddot{p}^f(x,t) - \nabla\cdot\left(\frac{1}{\rho_f(x)}\nabla p^f(x,t)\right) &= \mathrm{Re}\left\{F(x)\overline{\phi}(t)\right\} && \text{in } \Omega^f \\
c(x)\nabla p^f(x,t)\cdot\mathbf{n} + \dot{p}^f(x,t) &= \mathrm{Re}\left\{G_S(x)\overline{\phi}(t)\right\} && \text{on } \Gamma_S^f \\
p^f(x,t) &= \mathrm{Re}\left\{G_D(x)\overline{\phi}(t)\right\} && \text{on } \Gamma_D^f \\
\nabla p^f(x,t)\cdot\mathbf{n} &= \mathrm{Re}\left\{G_N(x)\overline{\phi}(t)\right\} && \text{on } \Gamma_N^f
\end{aligned}
$$

Elastic wave equation (solid)

$$
\begin{aligned}
\rho_s(x)\ddot{\mathbf{u}}^s(x,t) - \nabla\cdot\sigma(x,\mathbf{u}^s(\cdot,t)) &= \mathrm{Re}\left\{\mathbf{F}(x)\overline{\phi}(t)\right\} && \text{in } \Omega^s \\
\rho_s(x)\mathcal{B}\dot{\mathbf{u}}^s(x,t) + \sigma(x,\mathbf{u}^s(\cdot,t))\mathbf{n} &= \mathrm{Re}\left\{\mathbf{G}_S(x)\overline{\phi}(t)\right\} && \text{on } \Gamma_S^s \\
\mathbf{u}^s(x,t) &= \mathrm{Re}\left\{\mathbf{G}_D(x)\overline{\phi}(t)\right\} && \text{on } \Gamma_D^s \\
\sigma(x,\mathbf{u}^s(\cdot,t))\mathbf{n} &= \mathrm{Re}\left\{\mathbf{G}_N(x)\overline{\phi}(t)\right\} && \text{on } \Gamma_N^s
\end{aligned}
$$



Interface

$$
\begin{aligned}
\rho_f(x)\ddot{\mathbf{u}}^s(x,t)\cdot\mathbf{n} &= -\frac{\partial p^f(x,t)}{\partial \mathbf{n}}, && x\in\Gamma^i \\
\sigma(x,\mathbf{u}^s(\cdot,t))\mathbf{n} &= -p^f(x,t)\mathbf{n}, && x\in\Gamma^i
\end{aligned}
$$

Figure 2: Computational domain consisting of solid Ω^s and fluid Ω^f region with an interaction Γ^i between both regions, artificial boundaries Γ_S^f and Γ_S^s as well as some physical boundaries Γ_D^f, Γ_N^f and Γ_D^s, Γ_N^s.

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J. Cao, R. Brossier, and L. Métivier, *"3D acoustic-(visco)elastic coupled formulation and its spectral-element implementation on a Cartesian-based hexahedral mesh"*, SEG Abstract 2020.

Theorem

Let $(P^f, \mathbf{U}^s)$ be the unique solution to frequency-domain problem and $(p^f, \mathbf{u}^s)$ be any T-time periodic solution to time-domain problem. Then

$$P^f(x) = 2 \fint_0^T p^f(x,t)\exp(i\omega t)\,dt, \qquad \mathbf{U}^s(x) = 2 \fint_0^T \mathbf{u}^s(x,t)\exp(i\omega t)\,dt,$$

when $|\Gamma_S^s|, |\Gamma_S^f| > 0$.

$$\min_{(p_0^f, \dot{p}_0^f, \mathbf{u}_0^s, \dot{\mathbf{u}}_0^s)} \mathcal{J}(p_0^f, \dot{p}_0^f, \mathbf{u}_0^s, \dot{\mathbf{u}}_0^s)$$

The energy cost functional $\mathcal{J}$,

$$\mathcal{J}(p_0^f, \dot{p}_0^f, \mathbf{u}_0^s, \dot{\mathbf{u}}_0^s) = \frac{\omega^{-2}}{2}|p^f(T) - p_0^f|^2_{\rho_f^{-1}} + \frac{\omega^{-2}}{2}\|\dot{p}^f(T) - \dot{p}_0^f\|^2_{\kappa^{-1}} + \frac{1}{2}|\mathbf{u}^s(T) - \mathbf{u}_0^s|^2_{\sigma} + \frac{1}{2}\|\dot{\mathbf{u}}^s(T) - \dot{\mathbf{u}}_0^s\|^2_{\rho_s},$$

where the state variables $p^f, \mathbf{u}^s$ solve the coupled wave equations with the initial values (control variables) $p_0^f, \dot{p}_0^f, \mathbf{u}_0^s, \dot{\mathbf{u}}_0^s$.

$$\begin{aligned} |u|^2_{\rho^{-1}} &= (u,u)_{\rho^{-1}}, & \|v\|^2_{\kappa^{-1}} &= (v,v)_{\kappa^{-1}}, & u \in H^1(\Omega^f),\ v \in L^2(\Omega^f), \\ |\mathbf{u}|^2_{\sigma} &= (\mathbf{u},\mathbf{u})_{\sigma}, & \|\mathbf{v}\|^2_{\rho_s} &= (\mathbf{v},\mathbf{v})_{\rho_s}, & \mathbf{u} \in (H^1(\Omega^s))^d,\ \mathbf{v} \in (L^2(\Omega^s))^d \end{aligned}$$

$$\begin{aligned} (u_1,u_2)_{\rho^{-1}} &= \int_\Omega \frac{1}{\rho_f(x)} \nabla u_1(x) \cdot \nabla u_2(x)\, dx, & (v_1,v_2)_{\kappa^{-1}} &= \int_\Omega \frac{1}{\kappa(x)} v_1(x) v_2(x)\, dx, & u_1,u_2 \in H^1(\Omega^f), v_1,v_2 \in L^2(\Omega^f), \\ (\mathbf{u}_1,\mathbf{u}_2)_{\sigma} &= \int_\Omega \sigma(\mathbf{u}_1) : \varepsilon(\mathbf{u}_2)\, dx,, & (\mathbf{v}_1,\mathbf{v}_2)_{\rho_s} &= \int_\Omega \rho_s(x) \mathbf{v}_1(x) \cdot \mathbf{v}_2(x)\, dx, & \mathbf{u}_1,\mathbf{u}_2 \in (H^1(\Omega^s))^d, \mathbf{v}_1,\mathbf{v}_2 \in (L^2(\Omega^s))^d. \end{aligned}$$

Outline

The second-order stress σ and strain ε tensors:

$$\sigma(x,t) = \mathcal{M}(x,t) *_t \varepsilon(x,t) + \mathcal{T}(x,t)$$

where the tensor $\mathcal{T}$ is the possible stress failure and the tensor $\mathcal{M}$ the relaxation rate.

L standard-linear-solid systems (SLS)

$$\Rightarrow \sigma(x,\mathbf{u}(\cdot,t)) = \sigma^0(x,\mathbf{u}(\cdot,t)) - \sum_{\nu=1}^{L} \sigma^R(x,\boldsymbol{\psi}_\nu(\cdot,t))$$

with

$$\sigma^R(x,\boldsymbol{\phi}_\nu) = \mathbf{C}^R(x) : \boldsymbol{\phi}_\nu(x), \quad \sigma^0(x,\mathbf{v}) = \mathbf{C}(x) : \varepsilon^0(\mathbf{v}), \quad \varepsilon^0(\mathbf{v}) = \frac{1}{2}\left(\nabla\mathbf{v} + \nabla\mathbf{v}^\top\right),$$

where $\mathbf{C}$ is the fourth-order elastic (or unrelaxed) stiffness tensor, $\mathbf{C}^R$ the fourth-order relaxed stiffness tensor, and ψ_ν memory variables, $\nu = 1,\ldots,L$.

P. Trinh, R. Brossier, L. Métivier, L. Tavard, and J. Virieux *“Efficient time-domain 3D elastic and viscoelastic full-waveform inversion using a spectral-element method on flexible Cartesian-based mesh”*, Geophysics, 2019.

Frequency-domain elastic wave equation

$$
\begin{aligned}
-\omega^2\rho(x)\mathbf{U}(x) - \nabla\cdot\sigma^0(x,\mathbf{U}) + \nabla\cdot\sum_{\nu=1}^{L}\sigma^R(x,\boldsymbol{\Psi}_\nu) &= \mathbf{F}(x) \quad \text{in } \Omega\\
\mathbf{U}(x) &= \mathbf{G}_D(x) \text{ on } \Gamma_D\\
\sigma^0(x,\mathbf{U})\mathbf{n} - \sum_{\nu=1}^{L}\sigma^R(x,\boldsymbol{\Psi}_\nu)\mathbf{n} &= \mathbf{G}_N(x) \text{ on } \Gamma_N\\
-i\omega\rho\mathcal{B}\mathbf{U}(x) + \sigma^0(x,\mathbf{U})\mathbf{n} - \sum_{\nu=1}^{L}\sigma^R(x,\boldsymbol{\Psi}_\nu)\mathbf{n} &= \mathbf{G}_S(x) \text{ on } \Gamma_S
\end{aligned}
$$

and

$$
(-i\omega+\omega_\nu)\boldsymbol{\Psi}_\nu(x) = \mathbf{Y}_\nu : \varepsilon(x,\mathbf{U}) \text{ in } \Omega
$$

Time-domain elastic wave equation

$$
\begin{aligned}
\rho(x)\ddot{\mathbf{u}}(x,t) - \nabla\cdot\sigma^0(x,\mathbf{u}(\cdot,t)) + \nabla\cdot\sum_{\nu=1}^{L}\sigma^R(x,\boldsymbol{\psi}_\nu(\cdot,t)) &= \mathbf{f}(x,t) \quad \text{in } \Omega\\
\mathbf{u}(x,t) &= \mathbf{g}_D(x,t) \text{ on } \Gamma_D\\
\sigma^0(x,\mathbf{u}(\cdot,t))\mathbf{n} - \sum_{\nu=1}^{L}\sigma^R(x,\boldsymbol{\psi}_\nu(\cdot,t))\mathbf{n} &= \mathbf{g}_N(x,t) \text{ on } \Gamma_N\\
\rho\mathcal{B}\dot{\mathbf{u}}(x,t) + \sigma^0(x,\mathbf{u}(\cdot,t))\mathbf{n} - \sum_{\nu=1}^{L}\sigma^R(x,\boldsymbol{\psi}_\nu(\cdot,t))\mathbf{n} &= \mathbf{g}_S(x,t) \text{ on } \Gamma_S\\
\mathbf{u}(x,0) &= \mathbf{u}_0(x) \quad \text{in } \Omega\\
\dot{\mathbf{u}}(x,0) &= \dot{\mathbf{u}}_0(x) \quad \text{in } \Omega
\end{aligned}
$$

and

$$
\begin{aligned}
\dot{\boldsymbol{\psi}}_\nu(x,t) + \omega_\nu\boldsymbol{\psi}_\nu(x,t) &= \mathbf{Y} : \varepsilon(x,\mathbf{u}(\cdot,t)) \text{ in } \Omega\\
\boldsymbol{\psi}_\nu(x,0) &= \boldsymbol{\psi}_{\nu,0}(x) \quad \text{in } \Omega
\end{aligned}
$$

Theorem

Let $(\mathbf{u}, \psi_\nu)$, $\nu = 1, \ldots, L$, be any solution to time-domain problems and satisfying the periodic conditions. Then there are $\gamma_\ell^{\mathbf{u}}, \boldsymbol{\eta}^{\mathbf{u}} \in (H^1(\Omega))^d$ and $\gamma_{\nu,\ell}^{\psi} \in (L^2(\Omega))^d$, such that

$$\begin{aligned} \mathbf{u}(x,t) &= \operatorname{Re}\left\{\mathbf{U}(x)\overline{\phi}(t)\right\} + \sum_{\ell\neq 1} \gamma_\ell^{\mathbf{u}}(x)\overline{\phi}_\ell(t) + \boldsymbol{\eta}^{\mathbf{u}}(x)\frac{t}{T}, \\ \psi_\nu(x,t) &= \operatorname{Re}\left\{\boldsymbol{\Psi}_\nu(x)\overline{\phi}(t)\right\} + \sum_{\ell\neq 1} \gamma_{\nu,\ell}^{\psi}(x)\overline{\phi}_\ell(t), \end{aligned}$$

with $\phi_\ell(t) = \exp(i\omega\ell t)$ and

$$\boldsymbol{\eta}^{\mathbf{u}}(x) = \mathbf{u}(x,T) - \mathbf{u}_0(x).$$

Moreover, $\gamma_\ell^{\mathbf{u}}, \gamma_{\nu,\ell}^{\psi}$, $\nu = 1, \ldots, L$, solves frequency-domain visco-elastic wave equation with $\mathbf{F} = \mathbf{G}_D = \mathbf{G}_N = \mathbf{G}_S = \mathbf{0}$.

If $\mathcal{H}^{d-1}(\Gamma_S \cup \Gamma_D) > 0$ then $\boldsymbol{\eta}^{\mathbf{u}} = \mathbf{0}$.

$$\min_{\mathbf{u}_0,\dot{\mathbf{u}}_0,\boldsymbol{\psi}_{\nu,0}} \mathcal{J}(\mathbf{u}_0,\dot{\mathbf{u}}_0,\boldsymbol{\psi}_{\nu,0})$$

The energy cost functional $\mathcal{J}$,

$$\mathcal{J}(\mathbf{u}_0,\dot{\mathbf{u}}_0,\boldsymbol{\psi}_{\nu,0}) = \frac{1}{2}|\mathbf{u}(T)-\mathbf{u}_0|^2_{\mathbf{C}} + \frac{1}{2}\|\dot{\mathbf{u}}(T)-\dot{\mathbf{u}}_0\|^2_{\rho} + \frac{1}{2}\sum_{\nu=1}^{L}\|\boldsymbol{\psi}_{\nu}(T)-\boldsymbol{\psi}_{\nu,0}\|^2_{\mathbf{C}^R},$$

where the state variables $(\mathbf{u},\boldsymbol{\psi}_\nu)$ solve the wave equation with the initial values (control variables) $\mathbf{u}_0,\dot{\mathbf{u}}_0,\boldsymbol{\psi}_{\nu,0},\ \nu=1,\ldots,L$.

$$|\mathbf{V}|_{\mathbf{C}} = \sqrt{(\mathbf{V},\mathbf{V})_{\mathbf{C}}}, \quad \|\mathbf{W}\|_{\rho} = \sqrt{(\mathbf{W},\mathbf{W})_{\rho}}, \quad \|\boldsymbol{\Phi}\|_{\mathbf{C}^R} = \sqrt{(\boldsymbol{\Phi},\boldsymbol{\Phi})_{\mathbf{C}^R}}$$

and

$$(\mathbf{V}_1,\mathbf{V}_2)_{\mathbf{C}} = \int_{\Omega}\mathbf{C}(x):\varepsilon(x,\mathbf{V}_1):\varepsilon(x,\mathbf{V}_2)\,dx, \qquad (\mathbf{W}_1,\mathbf{W}_2)_{\rho} = \int_{\Omega}\rho(x)\mathbf{W}_1(x)\cdot\mathbf{W}_2(x)\,dx$$

$$(\boldsymbol{\Phi}_1,\boldsymbol{\Phi}_2)_{\mathbf{C}^R} = \int_{\Omega}\mathbf{C}^R(x):\boldsymbol{\Phi}_1(x):\boldsymbol{\Phi}_2(x)\,dx$$

Summary

- The CM offers an alternative for the solution of the frequency-domain wave equations and requires only solutions of time-dependent wave equations $\Rightarrow$ easy to parallelize.
- Explicit time integration methods has less memory requirement.
- Achieved high parallel performance and (super-)linear speed up.
- Extend the original formulation to more general boundary conditions / PML.
- Combined CM with explicit local time-stepping (LTS) methods to overcome CFL restriction due to local refinement.
- Combined CM with filtering procedures to solve multiple frequency-domain problems with different frequencies in one shot.
- For FWI, the solution of the previous iteration provides a good initial estimate for the CMCG method, which significantly speeds up convergence.

Future works:

- CM for acoustic-elastic coupled wave equation
- CM for viscoelastic wave equation
- CMCG method applied to the frequency-domain FWI

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