Gauge Fields in Arithmetic, Topology & Physics

ICMS, Edinburgh

Banach Algebraic Geometry

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Articles, Preprints conversations with Kobi Kremnizer
Jack Kelly
and others

Goals

1) Find common language for different flavors of (derived) geometry
2) Find applications to arithmetic and the other topics of this workshop!!

Higher Category Viewpoint on Geometry

Toën and Vezossi, Lurie.

$\boxed{\text{Raksit merged these to some extent}}$

$\mathcal{C}$ = stable presentable symmetric monoidal ∞-category, $\otimes$

$(\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0})$ a compatible t-structure: $\mathcal{C}_{\leq 0}$ closed under filtered colimits

the unit object of $\mathcal{C}$ lies in $\mathcal{C}_{\geq 0}$

$\mathcal{C}_{\geq 0} \otimes_{\mathcal{C}} \mathcal{C}_{\geq 0} \subseteq \mathcal{C}_{\geq 0}$

$\mathcal{C}^{0} \subseteq \mathcal{C}^{\heartsuit}$ a full symmetric monoidal subcategory closed under finite coproducts in $\mathcal{C}$ whose objects form compact projective generators. Some technical conditions

There are actually two notions of algebra objects relative to $\mathcal{C}$!

$$\mathrm{CAlg}_{\mathcal{C}} \quad \text{and} \quad \mathrm{DAlg}_{\mathcal{C}}$$

However they agree when $\mathcal{C}$ lives over a field of characteristic zero.

Let $\mathrm{Aff}_{\mathcal{C}}$ be the opposite category of one of the above.

What kind of topoligies can we define on $\mathrm{Aff}_{\mathcal{C}}$?

$\boxed{A \in CAlg_{\mathbb{C}} \text{ or } DAlg_{\mathbb{C}}}$

1) Zariski Topology

$f_1, \ldots, f_n \in A$ generate the unit ideal

Covers $A \to A_{f_1} \times \cdots \times A_{f_n}$

$A \to B$ is flat/faithfully flat if it is flat/faithfully flat on π_0 and

$$\pi_i(B) \simeq \pi_0(B) \otimes_{\pi_0(A)} \pi_i(A)$$

2) faithfully flat topology

$$A \longrightarrow B$$

is an epimorphism if $B \otimes_A B \simeq B$

3) Monomorphism topology

$$A \to A_1 \times \cdots \times A_n = B$$

where $A \to A_i$ is an epimorphism

and $M \sim 0 \iff M \otimes_A B \sim 0$

faithful

<u>Def</u> (Bhatt)

$A \to B$ is weakly étale if it is flat and

$$\begin{array}{c} B \otimes_A B \\ \downarrow \\ B \end{array}$$

is flat

4) Weakly étale topology

Covers by finite weakly étale jointly faithful

$$
\begin{tikzcd}
 & \text{Zar Covers} \arrow[r, phantom, "\subseteq"] \arrow[d, phantom, "\cap"] & \text{Mono Covers} \arrow[d, phantom, "\cap"] \\
\text{Faithfully Flat Covers} \arrow[r, phantom, "\supseteq"] \arrow[dr, phantom, "\cup"] & \text{flat étale covers} \arrow[r, phantom, "\subseteq"] \arrow[d, phantom, "\cap"] & \text{étale covers} \arrow[r, phantom, "\subseteq"] \arrow[dl, phantom, "\cup"] & \text{T.B.A.} \\
 & \text{W. étale covers} & &
\end{tikzcd}
$$

Examples

1) $\mathcal{C} = \mathrm{Mod}_{\mathbb{Z}}^{\otimes}$ (e.g. as complexes)

$\mathcal{C} = \mathrm{Mod}_R^{\otimes}$ for a usual ring R

As algebraic examples

$\longrightarrow$ Derived Alg. Geometry

In terms of analytic and smooth geometry we will be dealing with larger rings and so we want to consider them as rings with either

- extra algebraic structure

$\approx$

- extra analytic e.g. Fréchet or Dual Fréchet structure

We need to do this for formulae like $\mathcal{O}(X \times Y) = \mathcal{O}(X) \hat{\otimes} \mathcal{O}(Y)$

Fix a Banach ring R

e.g. $\mathbb{Z}, \mathbb{Z}_p, \mathbb{R}, \mathbb{Q}_p, k\langle \frac{x}{r} \rangle$

for k non-archimedean

$\mathbb{C}((t))$ etc.

1) Consider the monoidal category

$\mathrm{Ban}_R^{\hat{\otimes}_R}$ of complete normed R-modules and bounded R-linear maps. It is not closed under $\hat{\otimes}$ infinite limits and colimits which arrise in functional analysis.

$\operatorname{cok}(V \xrightarrow{f} W) = W/\overline{f(V)}$

2) Adjoin formal filtered colimits so objects are written

$V = \text{"colim"}_{i \in I} V_i$, $W = \text{"colim"}_{j \in J} W_j$

Categorical operations extend

$$V \hat{\otimes}_R W = \underset{(i,j)\in I\times J}{\text{"colim"}} V_i \otimes_R W_j$$

we never use uncompleted tensor products in the analytic context

$$\underline{\mathrm{Hom}}_R(V,W) = \lim_{i\in I} \underset{j\in J}{\text{"colim"}} \underline{\mathrm{Hom}}_R(V_i, W_j)$$

closed symmetric monoidal structure $\mathrm{Hom}_R(V,W)$ defined similarly

<u>Definition</u>:

$V \xrightarrow{f} W$ is

<u>strict</u> if

$$\mathrm{cok}(\ker f \to V) \simeq \ker(W \to \mathrm{cok} f)$$

A <u>projective object</u> in ${}^{pe}\underline{\mathrm{Ind}}(\mathrm{Ban}_R)$ is an object such that for any strict epimorphism

$$V \xrightarrow{f} W$$

we get a surjective map

$$\mathrm{Hom}(P,V) \to \mathrm{Hom}(P,W).$$

A <u>flat object</u> $F \in \underline{\mathrm{Ind}}(\mathrm{Ban}_R)$ is an object such that for any strict mono $V \to W$, $F \hat{\otimes}_R V \to F \hat{\otimes}_R W$ is a strict mono

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<u>Definition</u> (for Jack's talk)

$$\mathrm{CBorn}_R \subseteq \underline{\mathrm{Ind}}(\mathrm{Ban}_R)$$

is the full subcategory of objects $\text{"colim"}_{i \in I} V_i$ where the transition maps are injective.

when $R = \mathbb{C}$ or $\mathbb{Z}_p$ this is equivalent to to the category of vectorspaces w/ bornology (collection of subsets thought of as bounded satisfying some axioms.

$\mathcal{C}^0$ = category of nice projectives in $\mathrm{Ind}(\mathrm{Ban}_R)$

e.g. direct sums of projectives in Ban_R

$\mathcal{C}_{\geq 0}$ = the sifted cocompletion

$\mathcal{C} = \mathrm{Stab}(\mathcal{C}_{\geq 0})$ (modeled by simplicial objects in $\mathrm{Ind}(\mathrm{Ban}_R)$)

Building Blocks

$R\left\{\frac{x}{r}\right\}$

as formal power series $\sum_{i=0}^{\infty} a_i x^i$ with condition

$\sum_{i=0}^{\infty} |a_i| r^i < \infty$

The monoidal str will now be written $\hat{\otimes}_R^{\mathbb{L}}$

Which topology is appropriate?

1. The open sets in the Zariski topology are too big
2. Many topologies in algebraic geometry have covers of the form $\{X_a \xrightarrow{\phi_a} X\}$ where each ϕ_a is flat.

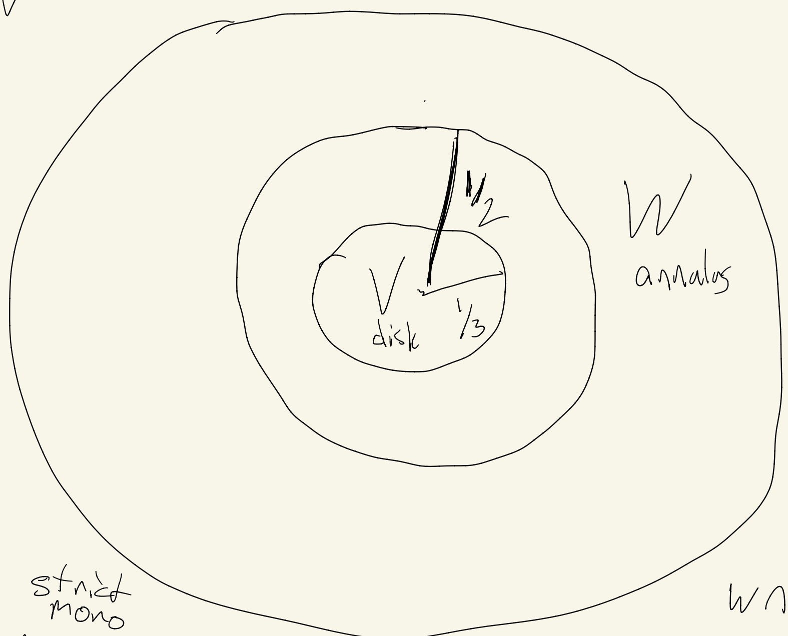
These are interesting here but as the following example shows the X_a are also too big for analytic purposes using $\hat{\mathcal{O}}_R$

Let $A = \mathbb{Q}_p\langle x\rangle$

$$A_W = A\left\langle \frac{z}{2}\right\rangle \Big/ (xz-1)$$

$$A_V = \mathbb{Q}_p\langle 3x\rangle$$

A_V is not flat over A



$W \cap V = \emptyset$

$A_V \hat{\otimes}^{\mathbb{L}}_A A_W = \{0\}$

strict mono

$$0 \to A \to A_W \to A_W/A \to 0$$

$$(A_W/A)\, \hat{\otimes}^{\mathbb{L}}_A A_V \simeq A_V[1]$$

cohomological grading

3 So we consider the monomorphism topology

$$\{A \to B_\alpha\}_{\text{finite}}$$

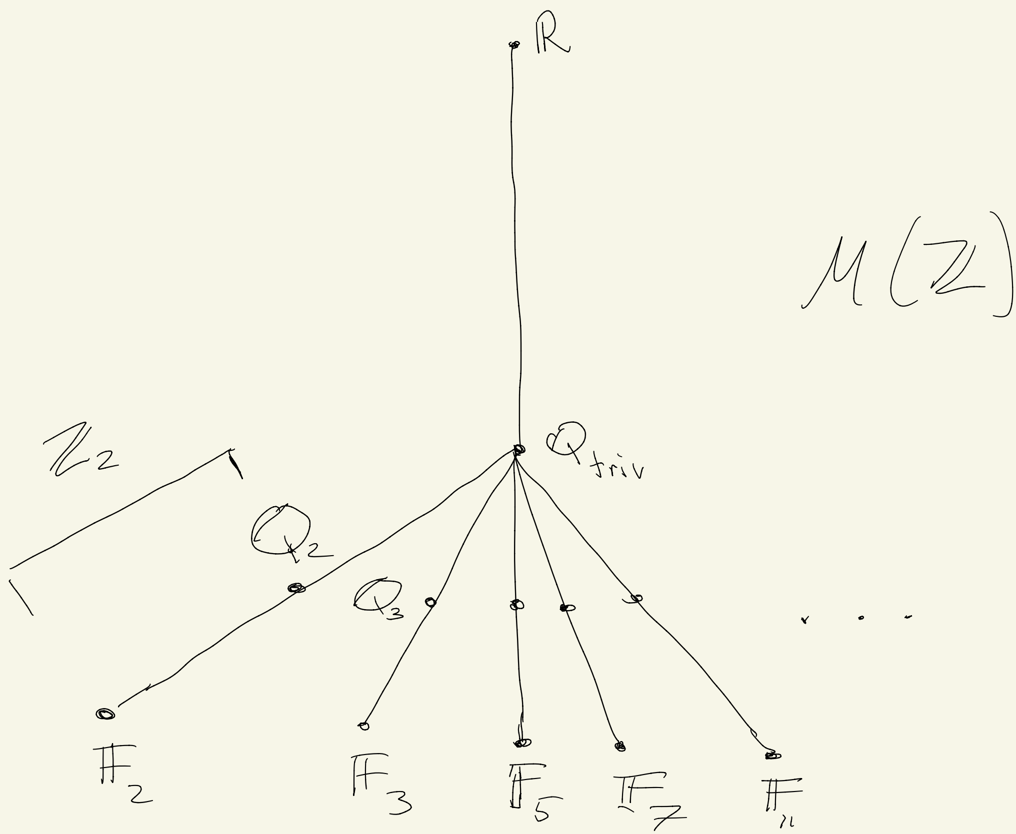
as a conservative cover where $B_\alpha \hat{\otimes}^{\mathbb{L}}_A B_\alpha \cong B_\alpha$

Descent Results

Recover Rigid Geometry on Affinoids (2013) and other types of analytic geometry (dagger, Stein etc)

This topology also makes sense and is interesting in algebraic geometry.
(w/usual tensor)

My hope is to analyize/ maybe enlarge the Berkovich Spectrum of $\mathbb{Z}$.



- Find interesting covers by homotopy monos $\mathbb{Z} \to R$

$$\boxed{R \hat{\otimes}^{\mathbb{L}}_{\mathbb{Z}} R \cong R}$$

• Idea that if $S \subseteq \mathcal{M}(\mathbb{Z})$

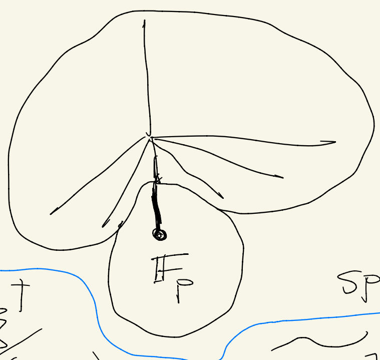
$R\underline{\mathrm{Hom}}_{\mathbb{Z}}(\mathcal{O}(S), \mathbb{Z}[1])$ of a certain form

$$= \underline{\mathrm{Ext}}^1_{\mathbb{Z}}(\mathcal{O}(S), \mathbb{Z})$$

$$\cong \mathcal{O}(\mathcal{M}(\mathbb{Z}) \setminus S) \Big/ \mathbb{Z}$$

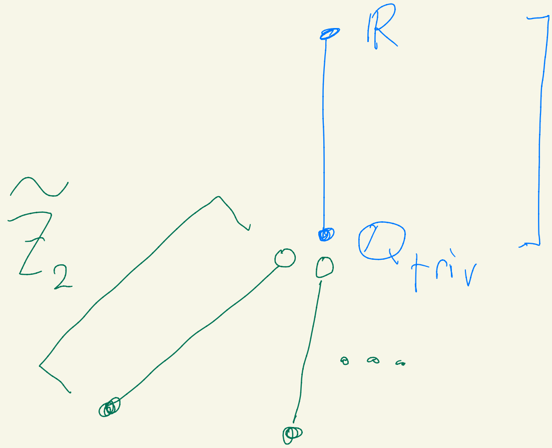
<u>Example of such a cover:</u>

$$\widetilde{\mathbb{Z}[\tfrac{1}{p}]} \longleftarrow \mathbb{Z} \longrightarrow \mathbb{Z}_p^\dagger$$



$\mathbb{Z}_p^\dagger := \mathbb{Z}\{px\}^\dagger \big/ (x-p)$

$\widetilde{\mathbb{Z}[\frac{1}{p}]} := \left(\lim\limits_{r<p} \mathbb{Z}\{\frac{y}{r}\}\right) \Big/ (py-1)$



completion of $\mathbb{Q}$ wrt to $\max\{|\,|_\infty, |\,|_0\}$

$$\tilde{\mathbb{Z}}_p := \left(\lim_{r<1} \mathbb{Z}\left\{\frac{x}{r}\right\}\right) \Big/ (x-p)$$

$$\mathbb{R}^\dagger := \text{"colim"}_{n\in\{1,2,3,\dots\}} \mathbb{Z}\{y\}^\dagger \big/ (ny-1)$$

$$\mathbb{Z} \longrightarrow \mathbb{R}^\dagger \times \prod_p \tilde{\mathbb{Z}}_p$$

like integral adeles