Convergence of the Restricted Additive Schwarz method with impedance boundary conditions for the Helmholtz equation

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Joint work with

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and Martin Gander (Geneva) and David Lafontaine (Bath)

Frequency domain wave problems - Strathclyde - 24 June 2022

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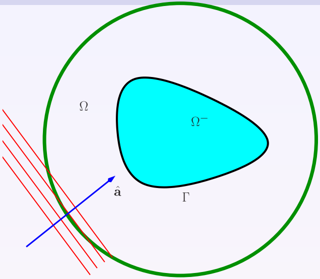
Outline of talk

- Helmholtz equation and discretization
- Overlapping DD for solving discrete system: ‘ORAS’
- It’s cousin SORAS and some theory - inspiring but limited
- A new result on convergence of ORAS
- Bootstrapped from theory of a related ‘Parallel Schwarz method’ method at the PDE (non-discrete) level

Helmholtz sound-soft scattering problem



Model problem:

$$\Delta u + k^2 u = f \text{ in } \Omega$$

$$u = 0 \text{ on } \Gamma$$

Impedance B.C. $\quad \partial_\nu u - \mathrm{i}ku = g, \quad$ on far field boundary

Most of the theory:

$$\Omega^- = \emptyset \quad \longrightarrow \quad \text{Interior impedance problem}$$

Finite Element Method

Variational formulation $u \in H^1(\Omega)$

$$\underbrace{\int_\Omega (\nabla u \cdot \nabla \overline{v} - k^2 u \overline{v}) - \mathrm{i}k \int_{\partial\Omega} u\overline{v}}_{a(u,v)} = \int_\Omega f\overline{v} + \int_{\partial\Omega} g\overline{v}, \quad v \in H^1(\Omega)$$

Finite element discretization (degree p, meshsize h)

$$\mathbf{A}\mathbf{u} := (\mathbf{S} - k^2\mathbf{M} - \mathrm{i}k\mathbf{N})\mathbf{u} = \mathbf{f}$$

A non-Hermitian, indefinite

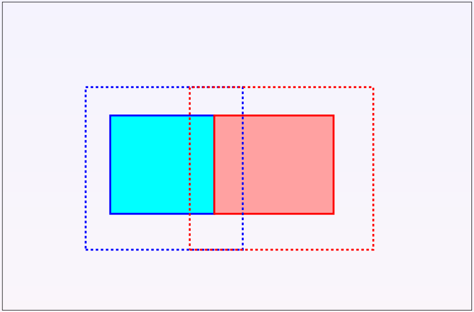
For existence/bounded error as $k \to \infty$: $h \sim k^{-1-1/2p}$ Du & Wu, 2015

To accurately compute 100 waves in Ω using linear elements:

$$\#\text{DoF} \quad \sim 10^6 \quad \text{in 2D,} \quad \sim 10^9 \quad \text{in 3D}$$

Domain decomposition - one level basics

overlapping subdomains Ω_ℓ



Partition of unity: $\{\chi_\ell\}$.

$$\mathrm{supp}(\chi_\ell) \subset \Omega_\ell \quad \text{and} \quad \sum_\ell \chi_\ell = 1.$$

'Local' impedance matrices $\mathbf{A}_\ell$, discretization of

$$a_\ell(u,v) = \int_{\Omega_\ell} (\nabla u . \nabla \overline{v} - k^2 u \overline{v}) - \mathrm{i} k \int_{\partial \Omega_\ell} u \overline{v}$$

Restricted Additive Schwarz (RAS) methods

	On nodal vectors
Restriction (chopping) Extension (by zero)	$\mathbf{R}_\ell$ $\mathbf{R}_\ell^\top$
Weighting by POU	$\mathbf{D}_\ell = \mathrm{diag}(\chi_\ell)$

$\mathbf{B}^{-1} := \sum_\ell \mathbf{R}_\ell^\top \mathbf{D}_\ell \mathbf{A}_\ell^{-1} \mathbf{R}_\ell$ 'Optimised' RAS = 'ORAS'

Using Impedance BC

$\mathbf{B}^{-1} := \sum_\ell \mathbf{R}_\ell^\top \mathbf{D}_\ell \mathbf{A}_\ell^{-1} \mathbf{D}_\ell \mathbf{R}_\ell$ 'Symmetric ORAS' = 'SORAS'

Talk by M. Bonazzoli

One approach to the theory (indefinite operator)

- Introduce absorption $k^2 \rightsquigarrow k^2 + \mathrm{i}\varepsilon, \quad \varepsilon > 0$

  a_ε is now coercive!

- $\mathbf{A} \rightsquigarrow \mathbf{A}_\varepsilon, \qquad \mathbf{B}^{-1} \rightsquigarrow \mathbf{B}_\varepsilon^{-1}$

- Analyse: $\mathbf{B}_\varepsilon^{-1}$ as a preconditioner for $\mathbf{A}_\varepsilon$ use coercivity!

  and hence : $\mathbf{B}_\varepsilon^{-1}$ as a preconditioner for $\mathbf{A}$ (?)

Theory with ε [Gong, IGG, Spence - IMAJNA 2021]

Assumptions:

- $kh \to 0$, as $k \to \infty$
- allows general geometries and variable A and n

Theorem: for the SORAS preconditioner,

$$\|\mathbf{B}_\varepsilon^{-1}\mathbf{A}_\varepsilon\| \lesssim (1 + C(p)kh),$$

$$\mathrm{dist}\left(0, \mathbf{FoV}(\mathbf{B}_\varepsilon^{-1}\mathbf{A}_\varepsilon)\right) \gtrsim \left(1 - (1 + C(p)kh)\frac{k}{\varepsilon}\right),$$

- $C(p)hk \to 0$ for fixed $p \implies$ no p dependence
- Need key parameter $\frac{k}{\varepsilon}$ small enough to get to $\mathbf{A}$

GMRES with SORAS: $\varepsilon = 0$, overlap $\delta \sim H = k^{-0.3}$

$\Omega = (0,1)^2$, rectangular subdomains

$k \backslash p$	1 (6)	2 (6)	3 (6)	4 (6)
40	13 (6)	14 (7)	13 (7)	13 (6)
80	12 (5)	13 (6)	12 (6)	12 (5)
120	13 (7)	14 (8)	14 (8)	13 (7)

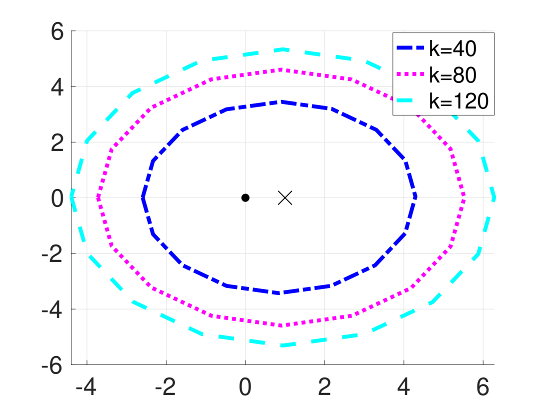
GMRES with SORAS (ORAS) $\varepsilon = 0,\ \delta = k^{-0.3}$

$\Omega = (0,1)^2$, rectangular subdomains

$k \backslash p$	1	2	3	4
40	13 (6)	14 (7)	13 (7)	13 (6)
80	12 (5)	13 (6)	12 (6)	12 (5)
120	13 (7)	14 (8)	14 (8)	13 (7)

Can our FoV estimates be improved?

Boundary of FoV of $\mathbf{B}^{-1}\mathbf{A}$, for ORAS with $\varepsilon = 0$:

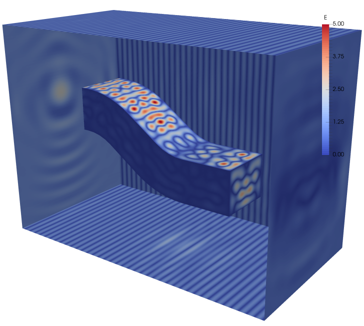


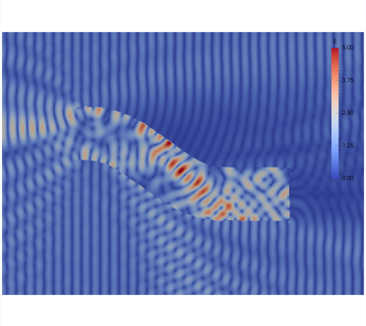
It appears not!

But ORAS is a very useful algorithm.....

3D Maxwell ("cobra cavity" at 10 GHz):

[Bonazzoli, Dolean, IGG, Spence, Tournier, Math Comp 2019]





Nédélec elements, degree 2: $\sim 107,000,000$ DOFs

DD method: ORAS, applied hierarchically:

Coarse grid: 3.3 pts/wavelength (inner GMRES, $\varepsilon_{\mathrm{prec}} = k$)

cores	outer GMRES iterations	Total time
1536	31	515.8
3072	32	285.0

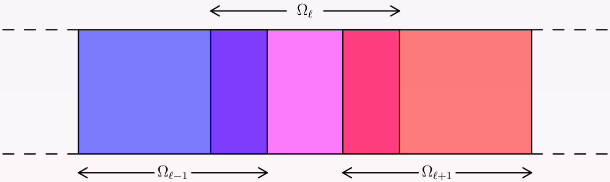
There should be a theory for ORAS

$$\mathbf{u}^n = \mathbf{u}^{n-1} + \mathbf{B}^{-1}(\mathbf{f} - \mathbf{A}\mathbf{u}^{n-1}) \iff \mathbf{e}^n = (\mathbf{I} - \mathbf{B}^{-1}\mathbf{A})^n \mathbf{e}^0$$ error equation

Discussion with Martin Gander: Look at power contractivity

If $\|(\mathbf{I} - \mathbf{B}^{-1}\mathbf{A})^n\| < 1$ then convergence in $\mathcal{O}(n)$ iterations for stationary iteration and GMRES

Expt:
N subdomains



Average # GMRES iterations

$k\backslash N$	4	8	16
20	6.0	12.3	25.1
40	5.6	10.2	17.0
80	4.4	8.0	15.9

Convergence analysis strategy

- Analyse a parallel Schwarz method at PDE level and show it is a power contraction [Gong, Gander, IGG, Lafontaine, Spence, 2021]

- show ORAS (as an iterative method) 'converges' to the parallel Schwarz method as mesh diameter $h \to 0$ [Gong, IGG, Spence, Math Comp 2022]

Parallel Schwarz method - N subdomains Ω_ℓ

- given u^{n-1} on Ω, solve for local components u_ℓ^n on each Ω_ℓ s.t.

$$-(\Delta + k^2)u_\ell^n = f \quad \text{in } \Omega_\ell,$$

$$(\partial_\nu - \mathrm{i}k)\, u_\ell^n = (\partial_\nu - \mathrm{i}k)\, u^{n-1} \quad \text{on } \partial\Omega_\ell \backslash \partial\Omega \quad \text{exchange of data}$$

$$(\partial_\nu - \mathrm{i}k)\, u_\ell^n = g \quad \text{on } \partial\Omega_\ell \cap \partial\Omega,$$

- new global iterate

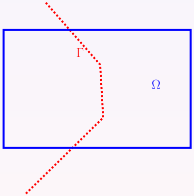
$$u^n = \sum_\ell \chi_\ell u_\ell^n.$$

Well-posedness of the parallel Schwarz method

For general Lipschitz domains, and χ_ℓ smooth enough,

Function space setting:

$$\mathrm{U}(\Omega) := \left\{ u \in H^1(\Omega) \mid \Delta u \in L^2(\Omega), \partial_\nu u \in L^2(\partial\Omega) \right\}$$



Lemma If $u \in U(\Omega)$ then $(\partial_\nu - \mathrm{i}k)u \in L^2(\Gamma)$

Theorem

If $u^{n-1} \in \mathrm{U}(\Omega)$, then $u^n \in \mathrm{U}(\Omega)$.

Error propagation operator $\mathcal{T}$

Error vector: $\boldsymbol{e}^n = (e_1^n, e_2^n, \cdots e_N^n)^T$, where $e_\ell^n := u|_{\Omega_\ell} - u_\ell^n$

$$\boldsymbol{e}^n = \mathcal{T}\boldsymbol{e}^{n-1}$$

where

$$\begin{aligned}(\Delta + k^2)(\mathcal{T}_{\ell,j} e_j) &= 0 \quad \text{in } \Omega_\ell,\\ (\partial_{\nu_\ell} - \mathrm{i}k)(\mathcal{T}_{\ell,j} e_j) &= (\partial_{\nu_\ell} - \mathrm{i}k)(\chi_j e_j), \quad \text{on } \partial\Omega_\ell \backslash \partial\Omega\\ (\partial_\nu - \mathrm{i}k)(\mathcal{T}_{\ell,j} e_j) &= 0, \quad \text{on } \partial\Omega_\ell \cap \partial\Omega\end{aligned}$$

Function space for errors

$$U_0(\Omega_\ell) := \{v_\ell \in U(\Omega_\ell) : (\Delta + k^2)v_\ell = 0\} \qquad \text{Helmholtz harmonic}$$

$$\mathbb{U}_0 := \prod_\ell U_0(\Omega_\ell) \qquad \text{tensor product}$$

$$\|\boldsymbol{v}\|^2_{\mathbb{U}_0} := \sum_\ell \int_{\partial\Omega_\ell} \left|(\partial_{\nu_\ell} - \mathrm{i}k)v_\ell\right|^2 ds \qquad \text{Boundary impedance norm}$$

Després, 1997

Structure of $\boldsymbol{\mathcal{T}}$

$\boldsymbol{\mathcal{T}}$ is sparse and related to the connectivity of the DD

For example in the 'strip domain' case:

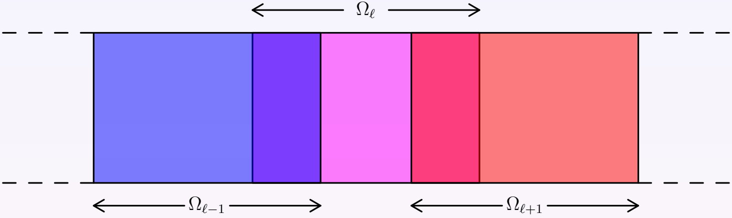


Fig 1. strip-type DD

$$\boldsymbol{\mathcal{T}} = \begin{pmatrix} 0 & \mathcal{T}_{1,2} & & & \\ \mathcal{T}_{2,1} & 0 & \mathcal{T}_{2,3} & & \\ & \mathcal{T}_{3,2} & 0 & \mathcal{T}_{3,4} & \\ & & \ddots & \ddots & \ddots \\ & & \mathcal{T}_{N-1,N-2} & 0 & \mathcal{T}_{N-1,N} \\ & & & \mathcal{T}_{N,N-1} & 0 \end{pmatrix}$$

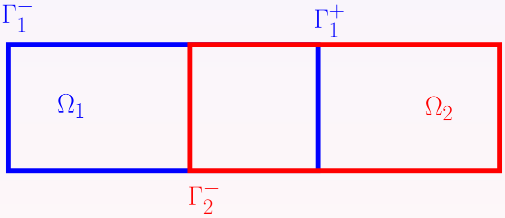
Impedance-to-impedance maps - strip domain

Consider $\mathcal{T}_{2,1} v_1$, with $v_1 \in U_0(\Omega_1)$.

Case : v_1 has impedance data given on Γ_1^-

Then $\mathrm{imp}_{\Gamma_2^-}(\mathcal{T}_{2,1} v_1) = \mathcal{I}_{\Gamma_1^- \to \Gamma_2^-} \mathrm{imp}_{\Gamma_1^+}(v_1)$

Left-to-left impedance map $\mathcal{I}_{\Gamma_1^- \to \Gamma_2^-}$



Impedance-to-impedance maps

Consider $\mathcal{T}_{2,1} v_1$, with $v_1 \in U_0(\Omega_1)$.

Case : v_1 has impedance data given on Γ_1^+

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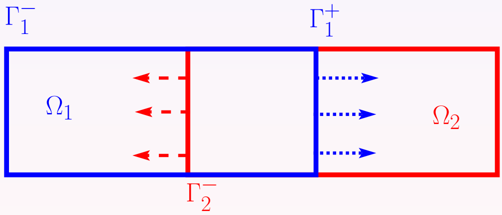
Right-to-Left impedance map $\mathcal{I}_{\Gamma_1^+ \to \Gamma_2^-}$



Impedance-to-impedance maps

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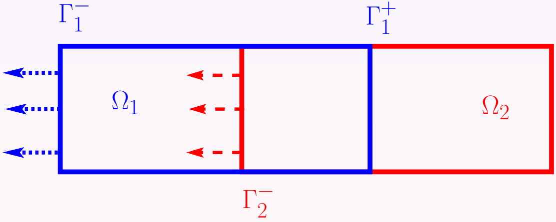

Power contractivity of $\mathcal{T}$

Gong, Gander, IGG, Lafontaine, Spence, 2021

Key parameters:

$$\rho := \max\{\|\mathcal{I}_{R\to L}\|, \|\mathcal{I}_{L\to R}\|\}, \quad \gamma := \max\{\|\mathcal{I}_{R\to R}\|, \|\mathcal{I}_{L\to L}\|\}$$

Computable by solving local EVPs

in 1-D, $\rho = 0$, $\gamma = 1$,
in general: $\gamma \leq \sqrt{1+\rho^2}$

Theorem (Power contractivity when ρ small)
For strip with N subdomains

$$\|\mathcal{T}^N\|_{\mathbb{U}_0} \leq 4\gamma^{N-1}(N-1)\rho + \mathcal{O}(\rho^2)$$

$(*) = N-1$ 'one-switch' products, e.g. $\mathcal{I}_{L\to L}\mathcal{I}_{L\to L}\cdots\mathcal{I}_{L\to L}\mathcal{I}_{L\to R}$

Dependence on N can be removed by estimating the iterated products

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Power contractivity of $\mathcal{T}$

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Computable by solving local EVPs

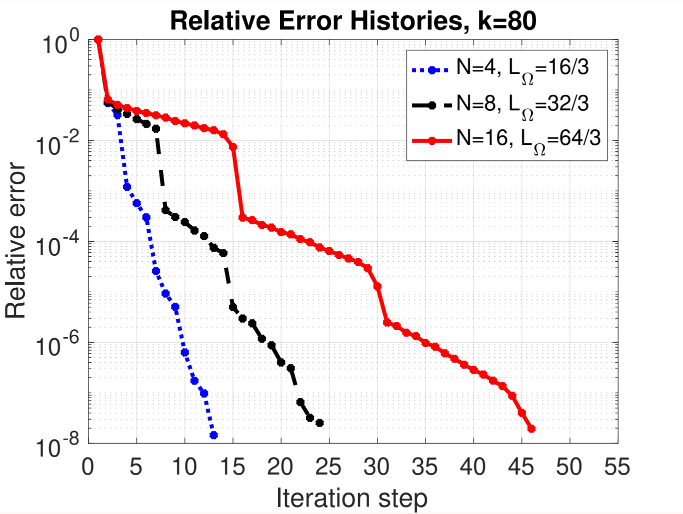
in 1-D, $\quad \rho = 0, \quad \gamma = 1,$
in general: $\quad \gamma \leq \sqrt{1+\rho^2}$

Theorem (Power contractivity)
For strip with N subdomains

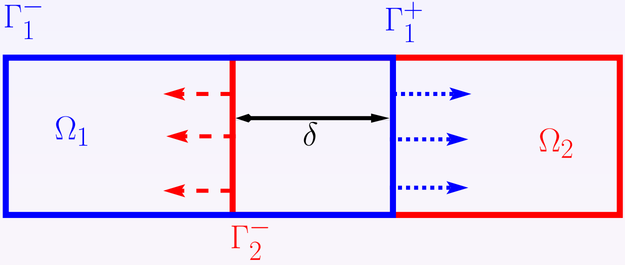
$$\|\mathcal{T}^{sN}\|_{\mathbb{U}_0} \;\leq\; C(N,\gamma)\rho^s \;+\; \mathcal{O}(\rho^{s+1}),$$

$(*) = N-1$ 'one-switch' products, e.g. $\mathcal{I}_{L\to L}\mathcal{I}_{L\to L}\cdots\mathcal{I}_{L\to L}\mathcal{I}_{L\to R}$

Convergence history



Benefit of overlap



Theorem [Lafontaine and Spence 2021] There exists a constant C independent of k such that

$$\|\mathcal{I}_{R\to L}\|, \|\mathcal{I}_{L\to R}\| \leq C\delta^{-2}, \quad \text{for all} \quad k \text{ sufficiently large}$$

(But assumes perfect ABC not impedance on outer boundary.)

$\mathcal{I}$ is computable

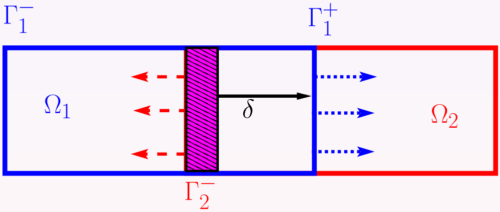
- $\mathcal{I} \mapsto \mathcal{I}^h$ via a variational formulation
- The norm of $\mathcal{I}^h$ is computable (eigenvalue problem)

- like a condition number

Theorem [Gong, IGG & Spence, 2021]

$$\|\mathcal{I} - \mathcal{I}^h\|_{L^2} \to 0, \quad \text{as} \quad h \to 0.$$

Proof uses interior regularity for u.



ρ_h is small

Computation of $\rho_h \approx \rho, \quad p = 2, \; h \sim k^{-5/4}$

	$k \backslash \delta$	$1/3$	$2/3$	$4/3$	$8/3$	$16/3$
	20	0.190	0.0997	0.0382	0.0175	0.00909
ρ_h	40	0.234	0.116	0.0434	0.0205	0.00884
	80	0.284	0.148	0.0557	0.0231	0.0115

Parallel Schwarz $\Longrightarrow$ ORAS [Gong, IGG & Spence, 2021]

	Parallel Schwarz	ORAS
error equation	$\mathbf{e}^n = \mathcal{T}\mathbf{e}^{n-1}$	$\mathbf{e}_h^n = \mathcal{T}_h \mathbf{e}_h^{n-1}$
function space on Ω_ℓ	'Helmholtz harmonic' with L^2 impedance data $\mathbb{U}_0$	Discrete 'Helmholtz harmonic' $\mathbb{V}_0$
impedance map (e.g.)	$\mathrm{imp}_{\Gamma_2^-}(\mathcal{T}_{2,1}v_1) = \mathcal{I}_{\Gamma_1^+ \to \Gamma_2^-}\mathrm{imp}_{\Gamma_1^+}(v_1)$	$\mathrm{imp}_{h,\Gamma_2^-}(\mathcal{T}_{h,2,1}v_{h,1}) = \mathcal{I}^h_{\Gamma_1^+ \to \Gamma_2^-}\mathrm{imp}_{h,\Gamma_1^+}(v_{h,1})$

Main Result

Theorem

for all n, $\|(\mathcal{T}_h)^n\|_{\mathbb{V}_0} \to \|(\mathcal{T})^n\|_{\mathbb{U}_0}$, as $h \to 0$.

Corollary: For h sufficiently small...

ORAS has the same power contractivity property as the Schwarz method
ORAS convergence is independent of h and p

ORAS preconditioned GMRES - independence of h and p

$\Omega = (0,1)^2$, square subdomains, diameter $H \sim k^{-0.4}$

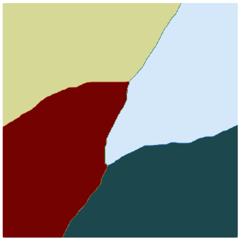
Iteration counts :

$k \backslash h$	$\frac{1}{k}$	$\frac{1}{2k}$	$\frac{1}{4k}$	$\frac{1}{8k}$
40	13	13	13	13
80	17	17	16	15
120	19	19	18	17
160	22	22	21	19

$k \backslash p$	1	2	3	4
40	14	13	13	12
80	17	16	15	14
120	20	18	17	16
160	22	21	19	17

ORAS preconditioned GMRES – general DD

$\Omega = (0,1)^2$, $p = 2$, $h \sim k^{-5/4}$, mesh partitioning via METIS



(a) 4 subdomains

(b) 16 subdomains

(c) 64 subdomains

$k \backslash N$	4	16	64
40	7	17	39
80	7	17	37
120	6	16	33
160	6	15	33

Summary

- Both the parallel Schwarz and ORAS are analysed as fixed point operators in Helmholtz harmonic spaces.
- For strip domains the parallel Schwarz method is power contractive at the PDE level (under conditions).
- For h small enough, ORAS has the same power contraction property.
- ORAS converges independently of h and polynomial degree p.
- Dependence on number of subdomains N is similar to the Laplace case