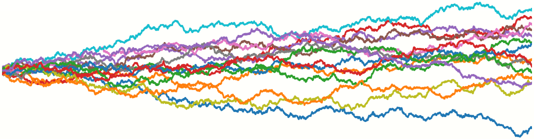
The rough journey of Brownian motion: from pollen particles, to Avogadro number, to stock markets

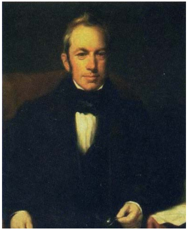


Ilya Chevyrev

The University of Edinburgh

21 June 2022

Harmonic Analysis, Stochastics and PDEs
ICMS, Edinburgh



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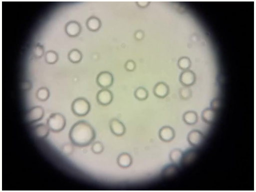

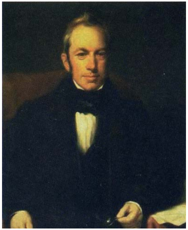

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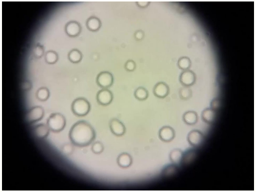

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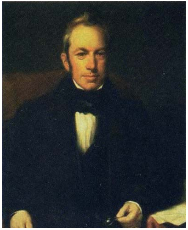

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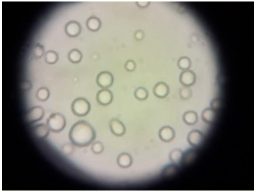

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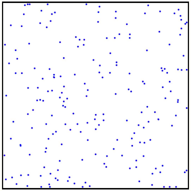
Even inside **quartz** with trapped water!

Molecular motion

James Maxwell (1859), **Ludwig Boltzmann** (1871): kinetic theory.



Motivation: explain the ideal gas law $PV \propto T$.

Important link to Brownian motion...

Molecular motion

- **Amedeo Avogadro** (1811): "number of integral molecules in any gases is always the same for equal volumes"

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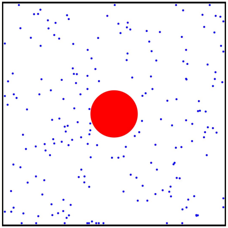
- Many were sceptical of kinetic theory: seen only as **explanatory**.

Brownian motion explained

Albert Einstein (1905),
Marian Smoluchowski (1906):
bombardment by tiny particles causes observable movement.



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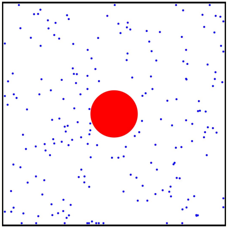

Idea not new, but Einstein and Smoluchowski made it **quantitative**.

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Key insight: mean square displacement is proportional to time.

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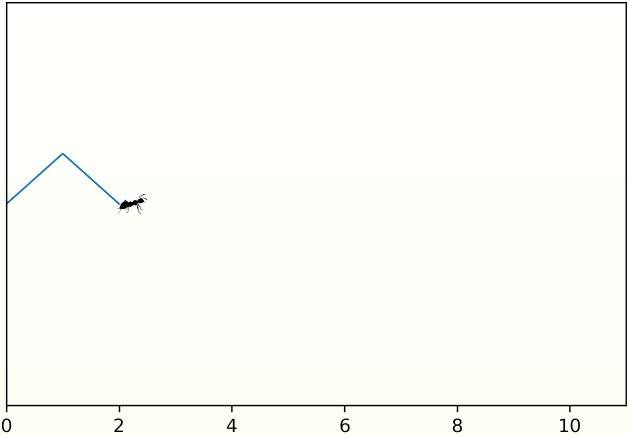

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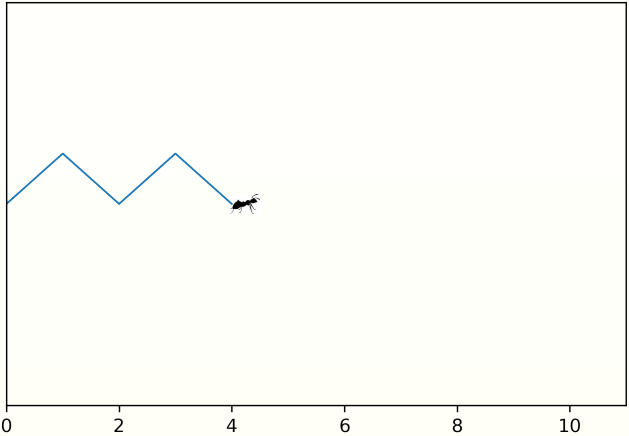

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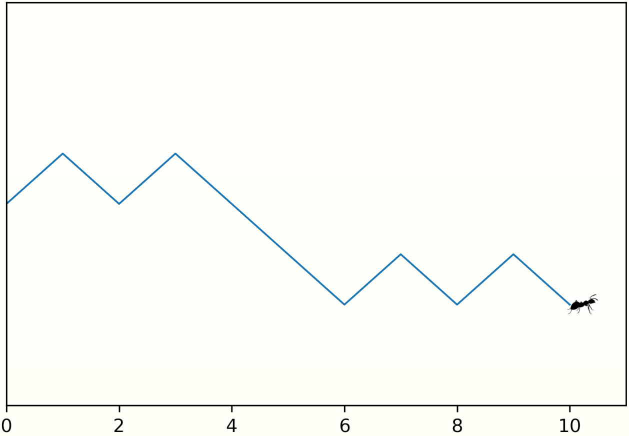

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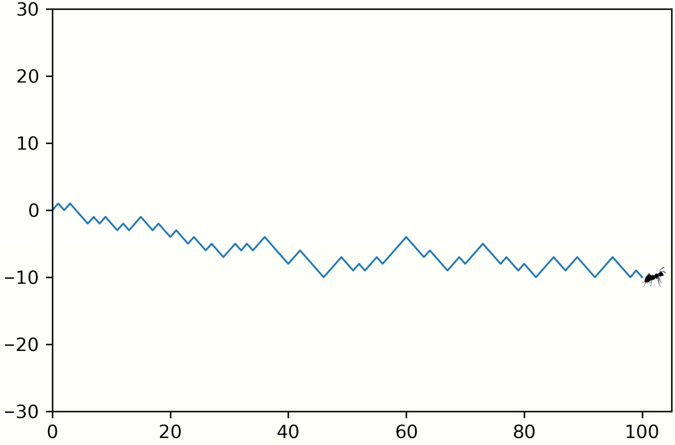
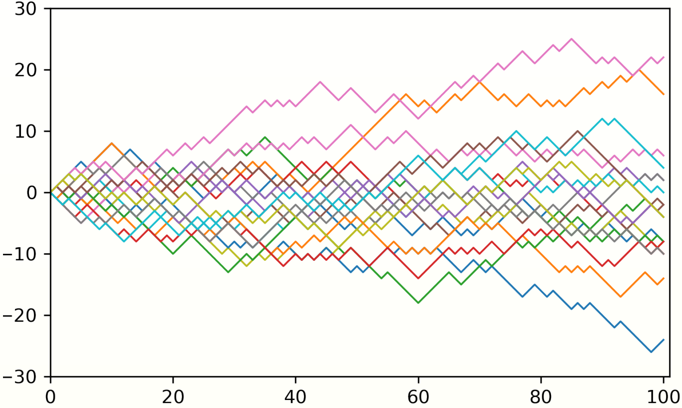


But on average, **10 steps** sideways.

20 ants moving together:



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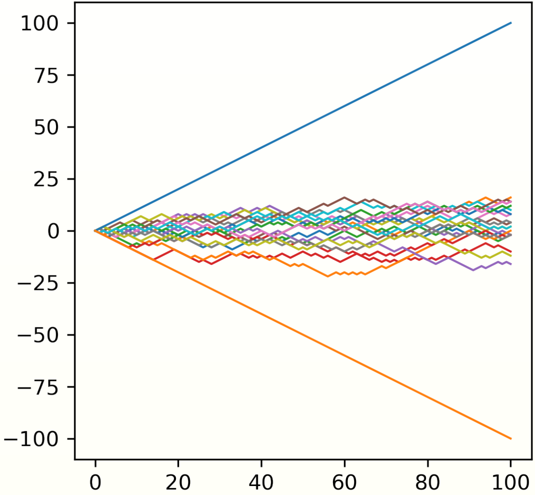
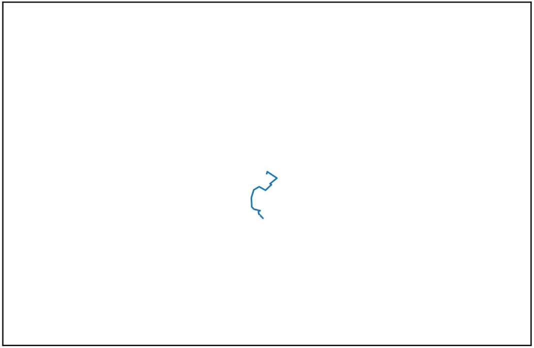
Much slower than directed movement.



Random walk in 2D



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- The average of $\frac{1}{2}mv^2$ is T/N – thermal equilibrium.
- Using symmetry and m small, deduce at time t:

$$\text{average of } x^2 = 4t\frac{T}{rN} \; .$$

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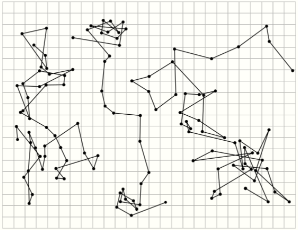
Jean Perrin (1909):
measured Avogadro's number as 6.4×10^{23}
(true value is 6.02×10^{23})
$\rightsquigarrow$ Nobel Prize 1926.



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Brownian motion in gas is **deterministic**.

Nonetheless we gain a lot by treating it as **random**.

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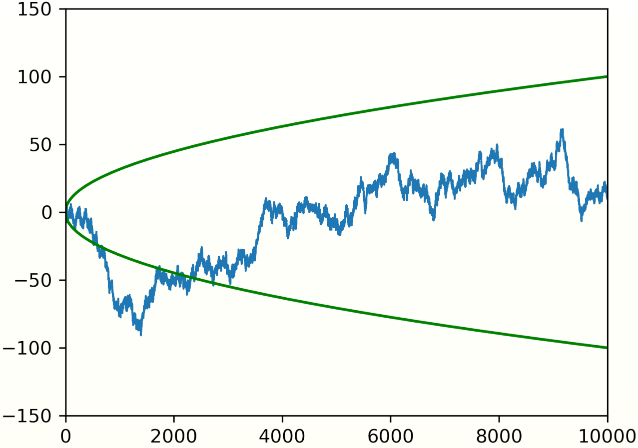
Norbert Wiener (1928): mathematical construction. Then Paul Lévy, Kiyosi Itô, ...

Brownian motion is **rough** and **fractal**.



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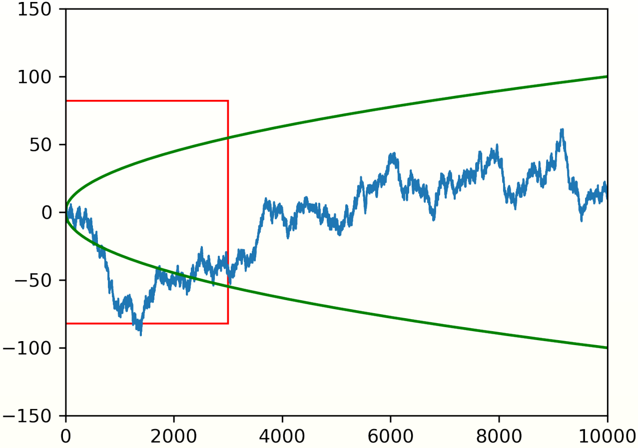

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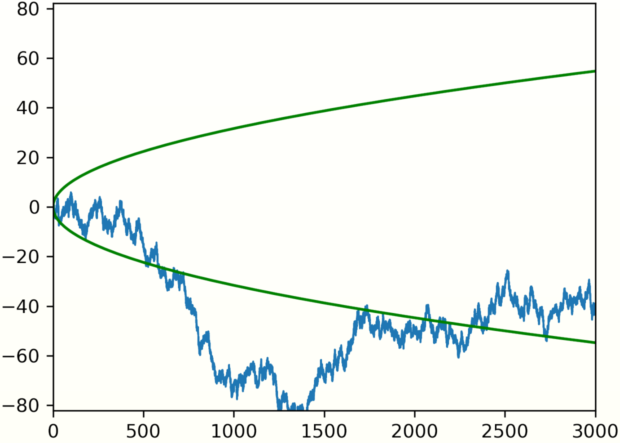

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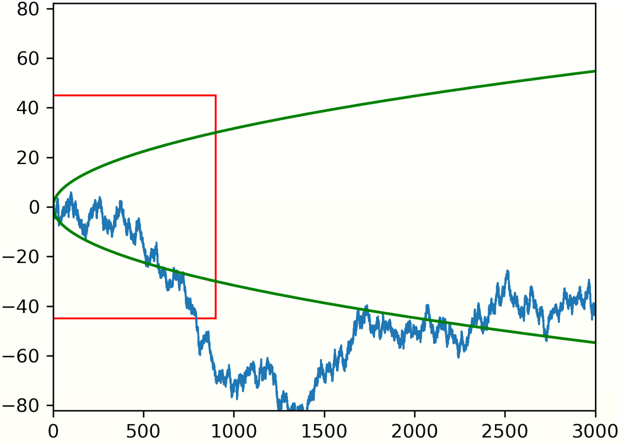

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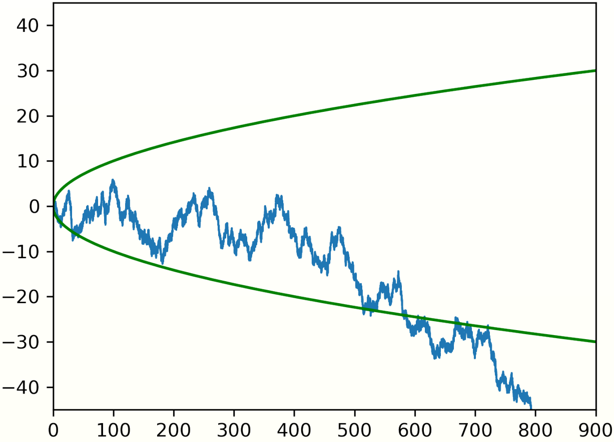

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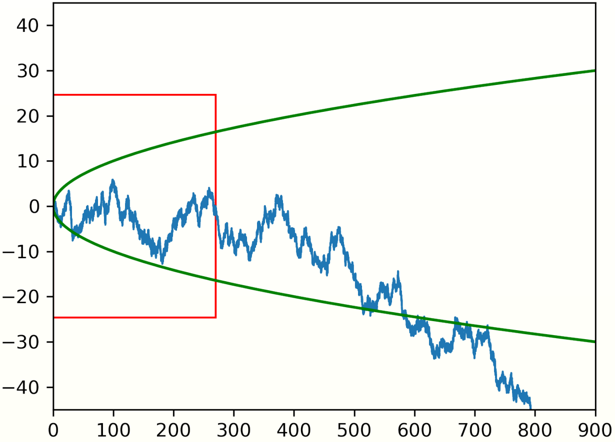

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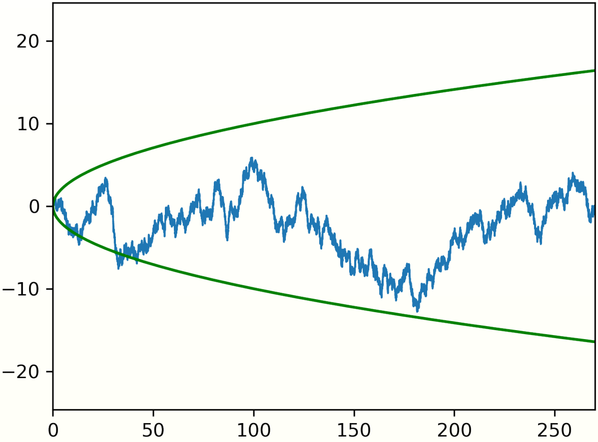

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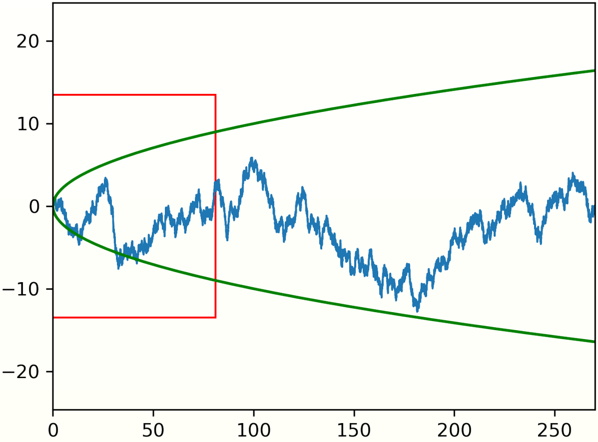

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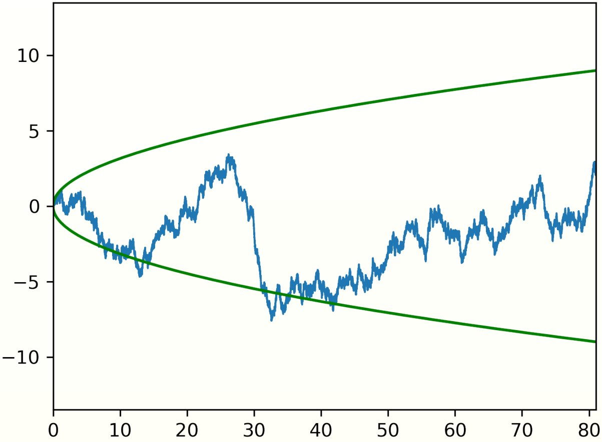

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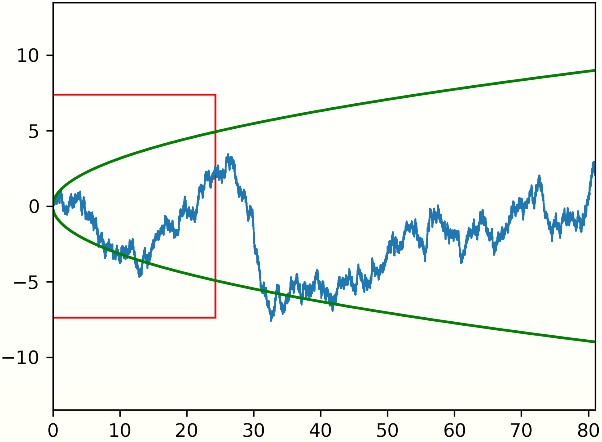

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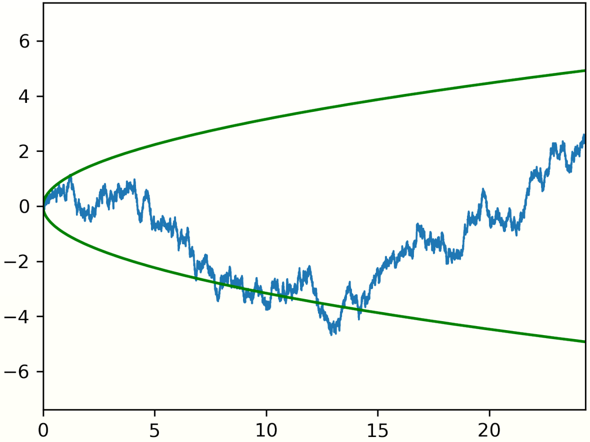

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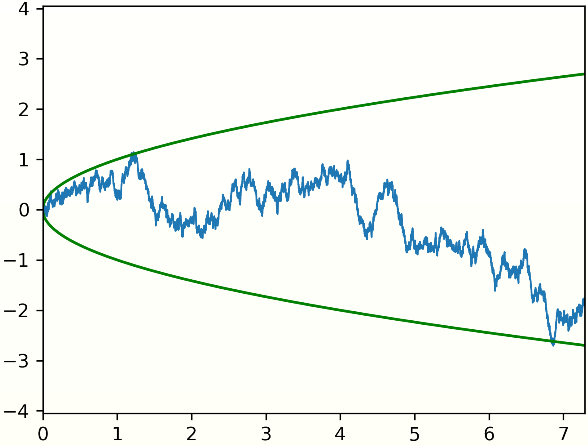
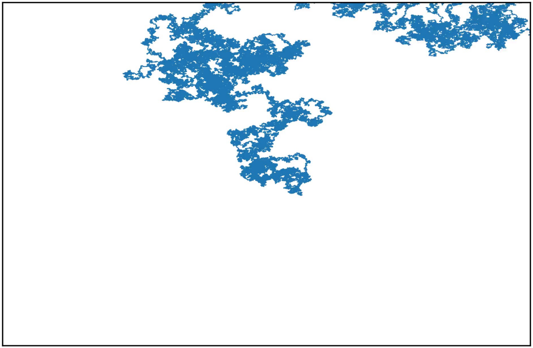


Mathematics of Brownian motion

Take 2D Brownian motion and zoom in. It stays a Brownian motion!



Stock prices

Brownian motion models randomness in **finance**.

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- (Revival of Bachelier model in 2020 with negative oil prices!)

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- Change in price proportional to current price:

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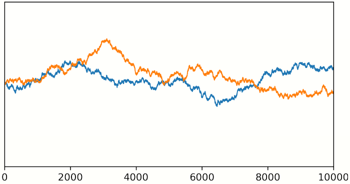
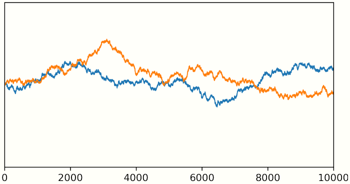
- Brownian motion does **not** respect rules of calculus.

Multiple sources of randomness

Order of events and microscopic differences are **crucial**.





Multiple sources of randomness

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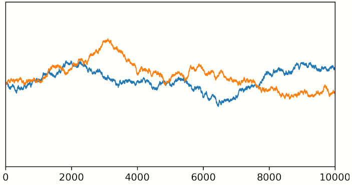

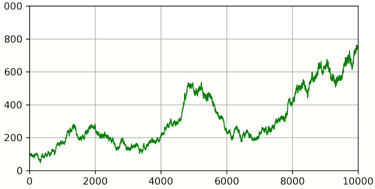

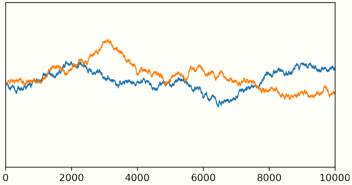

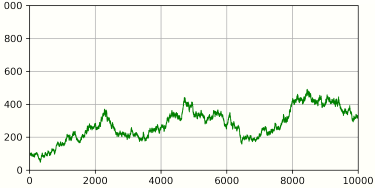

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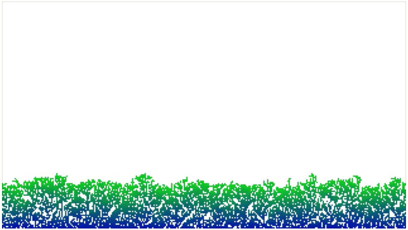

Credit: Nils Berglund

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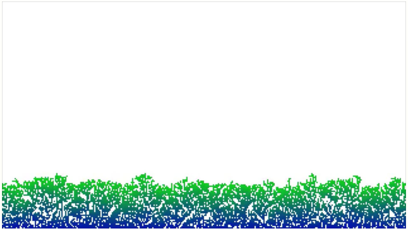

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- Connections between **Harmonic Analysis, Stochastics, and PDEs** are being explored.

Thank you for listening!

Happy Summer Solstice