Paracontrolled calculus and regularity structures

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Two approaches to singular PDEs

Singular SPDEs contain ill-posed multiplications, e.g., generalized KPZ equation

$$\partial_t h = \partial_x^2 h + \underbrace{f(h)}_{\frac{1}{2}-}\underbrace{(\partial_x h)}_{-\frac{1}{2}-}{}^2 + \underbrace{g(h)}_{\frac{1}{2}-}\underbrace{\xi}_{-\frac{3}{2}-}$$

Multiplication $C^\alpha \times C^\beta \to C^{\alpha\wedge\beta}$ is well-posed iff $\alpha + \beta > 0$.

$\to$ We need **renormalizations**.

Two approaches

- Regularity structure (Hairer '14)
  $\to$ "Black box" (Bruned-Hairer-Zamotti '19, Chandra-Hairer '16, & Bruned-Chandra-Chevyrev-Hairer '21)
- Paracontrolled calculus (Gubinelli-Imkeller-Perkowski '15)
  $\to$ High order PC (Bailleul-Bernicot '19)

The two approaches are different but believed to be equivalent.

Goal of our work

Regularity structure

- Based on (local) Euclid structures.
- Highly algebraic.

Paracontrolled calculus

- Based on Fourier analysis.
- Familiar PDE tools.

Both are extensions of the **rough path theory** for SDEs

$$dX = F(X)dB.$$

- RS provides a pointwise description

$$X_t - X_s = F(X_s)(B_t - B_s) + O(|t-s|^{1-}).$$

- PC provides a spectral description

$$X = F(X) \varolessthan B + (C^{1-}) \qquad \varolessthan\text{: Bony's paraproduct}$$

Goal: pointwise ⇔ spectral

Main result (rough)

Rough path theory	RS		PC
Rough path	Model	$\Leftrightarrow$	Pararemainders
Controlled path	Modelled distribution	$\Leftrightarrow$	Paracontrolled distribution
Stochastic integral	Chandra-Hairer	future work	No systematic theories

Theorem (Bailleul-H '21)

- *(JMSJ '21a) RS $\Rightarrow$ PC in general settings.*
- *(JEP '21b) PC $\Rightarrow$ RS under additional (but harmless) assumptions satisfied by* **another** *basis of Bruned-Hairer-Zambotti's algebra.*

- Gubinelli-Imkeller-Perkowski '15 & Martin-Perkowski '20: Fourier representation of the reconstruction operator.
- Tapia-Zambotti '20: a geometry of the space of branched rough paths.

Rough story of RS

Regularity structure

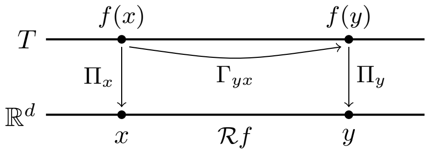
- $T = \langle \mathcal{B} \rangle$ **model space** $\cdots$ $\mathcal{B}$ is the set of "trees".
- $G \subset \mathrm{Aut}(T)$ $\cdots$ **Structure group**.

Model

- $\Pi_x : T \to \mathcal{S}'(\mathbb{R}^d)$ $\cdots$ Recentering operators.
- $\Gamma_{yx} : T \to T$ $\cdots$ Compatibility conditions.

Modelled distribution & Reconstruction

- $f : \mathbb{R}^d \to T$ $\cdots$ Local Taylor expansions.
- $\mathcal{R}f \in \mathcal{S}'(\mathbb{R}^d)$ $\cdots$ Original distribution.



More specific situation

We consider more specific situations containing Bruned-Hairer-Zamotti ('19).

$$(T, G) \leftrightarrow (T, T^+).$$

Assumption

- *$T^+ = \bigoplus_{\alpha \in A^+} T^+_\alpha$ is a graded Hopf algebra, where A^+ is a countable and locally finite set of nonnegative numbers containing 0.*
- *$T = \bigoplus_{\alpha \in A} T_\alpha$ is graded by a countable, locally finite, and bounded below set $A \in \mathbb{R}$. A right comodule structure $\Delta : T \to T \otimes T^+$ is given and compatible with the grading.*
- *Each $T^{(+)}_\alpha$ is finite dimensional.*

G is the group of all algebra maps from T^+ to $\mathbb{R}$. Then $g \in G$ acts on $\tau \in T$ by

$$g \cdot \tau = (\mathrm{id} \otimes g)\Delta\tau.$$

Specific situation $(\Pi, \Gamma) \leftrightarrow (\mathbf{\Pi}, g)$.

Definition (Model)

- $g \in C(\mathbb{R}^d; G)$ *such that*

$$g_{yx}(\tau) := (g_y \otimes g_x^{-1})\Delta^+\tau = O(|y-x|^\alpha), \quad \tau \in T_\alpha^+.$$

- $\mathbf{\Pi} \in \mathcal{L}(T; \mathcal{S}'(\mathbb{R}^d))$ *such that*

$$\Pi_x\tau(y) := (\mathbf{\Pi} \otimes g_x^{-1})\Delta\tau(y) = O(|y-x|^\beta), \quad \tau \in T_\beta$$

(in a distributional sense).

Definition (Modelled distribution)

For $\gamma \in \mathbb{R}$, the space $\mathcal{D}^\gamma(g)$ consists of all functions $f : \mathbb{R}^d \to T$ such that

$$\big(f(y) - g_{yx} \cdot f(x)\big)_{T_\alpha} = O(|y-x|^{\gamma-\alpha}), \quad \alpha < \gamma.$$

If $\gamma > 0$, then $\exists_1 \mathcal{R}f \in \mathcal{S}'(\mathbb{R}^d)$ such that $\mathcal{R}f = \Pi_x f(x) + O(|\cdot - x|^\gamma)$. (Hairer '14, Caravenna-Zambotti '20)

RS $\Rightarrow$ PC

$(\mathbf{\Pi}, g)$, $\mathcal{D}^\gamma(g)$ (pointwise) $\Longrightarrow$ PC counterparts (spectral)

$\nwarrow \quad \nearrow$

(T, T^+)

Notations

- Each $T_\alpha^{(+)}$ is finite dimensional.
- Fix a basis $\mathcal{B}_\alpha^{(+)}$ of $T_\alpha^{(+)}$, and set $\mathcal{B}^{(+)} = \bigcup_{\alpha \in A^{(+)}} \mathcal{B}_\alpha^{(+)}$.
- $|\tau| := \alpha \overset{\text{def}}{\Leftrightarrow} \tau \in \mathcal{B}_\alpha^{(+)}$.
- For any $\tau, \sigma \in \mathcal{B}^{(+)}$, we define the element $\tau/\sigma \in T^+$ by

$$\Delta^{(+)}\tau = \sum_{\sigma \in \mathcal{B}^{(+)}} \sigma \otimes (\tau/\sigma).$$

e.g. in Bruned-Hairer-Zambotti's (BHZ) algebra,

- $\mathcal{B}^{(+)}$ consists of rooted decorated trees.
- σ is a subtree of τ and τ/σ is a quotient graph.

We actually consider the Hölder space with polynomial weights, but we omit the details here.

Theorem (Bailleul-H '21a)

1. *For any model* $(\mathbf{\Pi}, g)$ *for* (T, T^+) *and for any* $\tau \in \mathcal{B}^+$*, one has*

$$g(\tau) = \sum_{\sigma \in \mathcal{B}^+,\, |\sigma| < |\tau|} g(\tau/\sigma) \otimes [\sigma] + [\tau],$$

with $[\tau] \in C^{|\tau|}$*. One has a similar expansion for* $\mathbf{\Pi}$*.*

2. *Let* $\gamma \in \mathbb{R}$*. For any modelled distribution* $f = \sum_{\tau \in \mathcal{B},\, |\tau| < \gamma} f_\tau \tau \in \mathcal{D}^\gamma(g)$*, one has*

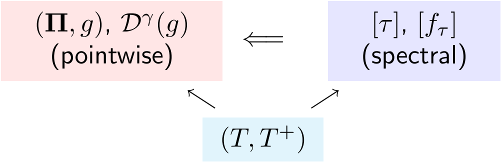
$$f_\tau = \sum_{\tau \in \mathcal{B},\, |\tau| < |\sigma| < \gamma} f_\sigma \otimes [\sigma/\tau] + [f_\tau],$$

with $[f_\tau] \in C^{\gamma - |\tau|}$*. One has a similar expansion for* $\mathcal{R} f$*.*

These formulas give an algebraic meaning to the paracontrolled systems (Gubinelli-Imkeller-Perkowski '15 and Bailleul-Bernicot '19).

RS $\Leftarrow$ PC

Next problem



"RS $\Rightarrow$ PC" holds for general pairs (T, T^+), but we need stronger assumption to state "PC $\Rightarrow$ RS".

Assumption (rough)

1. *$\mathcal{B}^+$ is freely generated by a finite set $\mathcal{G}_\circ^+ \subset \mathcal{B}^+$, polynomials $\{X^k\}_{k\in\mathbb{N}^d}$, and derivatives.*
2. *$\mathcal{B}$ is freely generated by a finite set $\mathcal{B}_\bullet \subset \mathcal{B}$ and polynomials $\{X^k\}_{k\in\mathbb{N}^d}$.*
3. *(g, Π) canonically applies to X^k.*

e.g. in BHZ algebra,

- $\mathcal{B}_\bullet$: all strongly conforming trees with $\mathfrak{n} = 0$ at roots.
- $\mathcal{G}_\circ^+$: all "planted" trees with $\mathfrak{n} = 0$ at roots and $\mathfrak{e} = 0$ at the edges leaving from roots.

Under the above assumptions,

Theorem (Bailleul-H '21b)

Subfamilies $\{[\tau] \in C^{|\tau|}\,;\, \tau \in \mathcal{G}_\circ^+\}$ *and* $\{[\sigma] \in C^{|\sigma|}\,;\, \sigma \in \mathcal{B}_\bullet,\ |\sigma| < 0\}$ *uniquely determine the original model* $(\mathbf{\Pi}, g)$*. This inverse map is continuous, hence the space of all models is homeomorphic to*

$$\prod_{\tau \in \mathcal{G}_\circ^+} C^{|\tau|} \times \prod_{\sigma \in \mathcal{B}_\bullet,\, |\sigma|<0} C^{|\sigma|}$$

To recover modelled distributions, we assume

Assumption

For any $\tau \in \mathcal{B}_\bullet$, *its coproduct* $\Delta\tau$ *does not have terms of the form* $\sigma \otimes X^k$ *with* $k \neq 0$.

then we have

Theorem (Bailleul-H '21b)

A subfamily $\{[f_\sigma]\,;\,\sigma \in \mathcal{B}_\bullet,\ |\sigma| < \gamma\}$ *uniquely determines the original modelled distribution* $f \in \mathcal{D}^\gamma(g)$. *This inverse map is continuous, hence*

$$\mathcal{D}^\gamma(g) \simeq \prod_{\tau\in\mathcal{B}_\bullet,\,|\tau|<\gamma} C^{\gamma-|\tau|}.$$

The canonical basis of BHZ algebra does not satisfy this assumption, but

Proposition (Bailleul-H '21b)

There is another basis of BHZ algebra which satisfies the above assumption.

Future works

- Renormalizations in the space $\prod_\tau C^{|\tau|}$?
  - Bailleul-Bruned '21: parametrization of the BPHZ model by $[\tau]$'s.
  - Fourier approach to Chandra-Hairer '16?.
- **First motivation**: Can we complete anything within PC (only Fourier picture)?
- In Riemannian manifolds? (cf. Dahlqvist-Diehl-Driver '19, Bailleul-Bernicot '19)
- If T_α are infinite dimensional? (cf. Gerencsér-Hairer '19, Chandra-Chevyrev-Hairer-Shen '22)