Sobolev estimates for degenerate Kolmogorov equations

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Harmonic Analysis, Stochastics and PDEs in Honour of the 80th Birthday of Nicolai Krylov, ICMS

Abstract

I will present some recent results about degenerate (ultra-parabolic) Kolmogorov equations (also known as linear kinetic Fokker–Planck equations) with rough coefficients. Such equations appear often in the kinetic theory.

We consider equations in both divergence and non-divergence form and our proof does not rely on any kernel estimates, in spirit of Krylov's work in 2005 about parabolic equations with VMO_x coefficients.

This is based on joint work with Timur Yastrzhembskiy (Brown University).

Kolmogorov equations

Consider linear Kolmogorov (kinetic Fokker-Planck) equations in nondivergence form

$$u_t + v \cdot D_x u - a^{ij} D_{v_i v_j} u + b^i D_{v_i} u + \lambda u = f$$

and in divergence form

$$u_t + v \cdot D_x u - D_{v_i}(a^{ij} D_{v_j} u + \tilde{b}^i u) + b^i D_{v_i} u + \lambda u = f_0 + \operatorname{div}_v f,$$

where $(t, x, v) \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d$ and the leading coefficients $a^{ij} = a^{ij}(t, x, v)$ satisfy the ellipticity condition $\delta|\xi|^2 \le a^{ij}\xi_i\xi_j$, $|a^{ij}| \le \delta^{-1}$

Second-order parabolic, nondegenerate in v, but degenerate in x.
Also known as "ultra-parabolic" (or "hypo-elliptic") equations.
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Motivations

A fundamental collision plasma model is given the Landau equation:

$$F_t + v \cdot \nabla_x F = \mathcal{Q}[F, F],$$

where F is the density of the distribution function, v is the velocity, the Landau (Fokker-Planck) collision operator $\mathcal{Q}$ (Landau, 1936) is given by

$$\mathcal{Q}[F_1, F_2](v) := \nabla_v \cdot \int_{\mathbb{R}^3} \Phi(v - v') \left[F_1(v')\nabla_v F_2(v) - F_2(v)\nabla_v F_1(v')\right] dv',$$

and the Landau (Fokker-Planck) collision kernel Φ is defined as

$$\Phi(v) := (I_3 - |v|^{-2} v \otimes v) \cdot |v|^{\gamma+2}, \quad \gamma \in [-3, 1].$$

(Space homogeneous) stationary solution (Maxwellian): $\mu(v) = \pi^{-3/2} e^{-|v|^2}$. We are interested in the stability of such solution. Denote $F = \mu + \mu^{1/2} f$.

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In the Coulomb case (i.e., $\gamma = -3$), after linearization (or applying the Picard iteration), we get the Kolmogorov equation in divergence form

$$\partial_t f_{n+1} + v \cdot \nabla_x f_{n+1} - \nabla_v \cdot (a_n \nabla_v f) + b_n \cdot \nabla_v f_{n+1} + c_n f_{n+1} = h_n$$

where $a_n = \Phi * (\mu + \mu^{1/2} f)$ is nondegenerate whenever v is any bounded open set:

$$(|v| + 1)^{-3} I_3 \leq [a^{ij}] \leq (|v| + 1)^{-1} I_3.$$

If f_n is sufficiently smooth, we get a nondivergence form equation.
Kolmogorov equations also arise in

- Diffusion processes: Langevin equation.
- Mathematical Finance.

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Scalings and translations

The equation satisfies the following scaling and translation properties

$$(t, x, v) \to (\lambda^2 t, \lambda^3 x, \lambda v),$$

$$(t, x, v) \to (t - t_0, x - x_0 - (t - t_0)v_0, v - v_0).$$

In the matrix form: denoting

$$A = \begin{bmatrix} 0 \cdot I_d & I_d \\ 0 \cdot I_d & 0 \cdot I_d \end{bmatrix},$$

then,

$$(x - x_0 - (t - t_0)v_0, v - v_0) = (x, v) - e^{(t-t_0)A}(x_0, v_0).$$

Formally, $\partial_t \sim D_v^2$ and $D_x \sim D_v^3$.

A quasi-metric

If we define

$$\rho(z, z_0) := \max\left\{|t - t_0|^{1/2}, |x - x_0 - (t - t_0)v_0|^{1/3}, |v - v_0|\right\},$$

then ρ satisfies

$$\rho(z, z_0) \le 2\rho(z_0, z), \quad \rho(z, z_0) \le 2(\rho(z, z_1) + \rho(z_1, z_0)).$$

Therefore, $\tilde{\rho}(z, z_0) := \rho(z, z_0) + \rho(z_0, z)$ is a quasi-metric.

With an induced doubling measure, we get a space of homogeneous type associated with the operator.

Previous results in the literature

- Lanconelli–Polidoro (1993): constant coefficients, a Harnack type inequality.
- Bramanti–Cerutti–Manfredini (1996): an interior S_p estimate for nondivergence form equations with VMO (vanishing mean oscillation) coefficients. Here

$$\|u\|_{S_p} = \|u\|_{L_p} + \|D_v u\|_{L_p} + \|D_v^2 u\|_{L_p} + \|u_t + vD_x u\|_{L_p}.$$

  See also Polidoro–Ragusa (1998) for an extension of the estimate in Morrey spaces.
- Manfredini–Polidoro (1998): an interior estimate of $\|D_v u\|_{L_p}$ for divergence form equations with VMO coefficients.

Previous results in the literature

- Bramanti–Cupini–Lanconelli–Priola (2010): constant coefficients, a global S_p estimate in $\{t \in [-1, 1]\}$ for solutions with compact support.
- Bramanti–Cupini–Lanconelli–Priola (2013): uniformly continuous coefficients, a global S_p estimate in $\{t \in [-T, T]\}$ for solutions with compact support for a small T.
- Bouchut (2002): A result in spirit of the averaging lemma, i.e., if $u_t + v \cdot D_x u = g$, then

$$\text{regularity in } v \Rightarrow \text{ regularity in } x.$$

$$\text{In particular, } D_v^2 u, g \in L_p \Rightarrow (-\Delta_x)^{1/3} u \in L_p.$$

Previous results in the literature

- Golse–Imbert–Mouhot–Vasseur (2019): a Harnack type inequality for divergence form equations with measurable coefficients by using De Giorgi's method.
- Z.-Q. Chen and X. Zhang (2019): Nondivergence form equations with coefficients $a^{ij} = a^{ij}(t)$. Global a priori estimate in $\mathbb{R}^{1+2d}$ in the form

$$\|D_v^2 u\|_{L_p} + \|(-\Delta_x)^{1/3} u\|_{L_p} \le N\|f\|_{L_p}$$

  by using the fundamental solution.
- Niebel–Zacher (2020): a priori estimates and unique solvability in $L_p((0,T), L_q(\mathbb{R}^{2d}))$ with the power weight in time $t^\mu, \mu \in (-1,p)$, assuming that the functions a^{ij} are uniformly continuous

Our goals

- We are interested in developing an argument without using any kernel estimates, which works for both divergence form and nondivergence form equations.
- Global a priori estimates and also the unique solvability of equations (including the Cauchy problem).
- Coefficients are measurable in t and VMO in (x, v). This is crucial in our applications to the boundary value problem of the Landau equation because after taking certain extension the coefficients are no longer uniformly continuous in (t, x, v).
- Weighted mixed-norm Lebesgue spaces.

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- Weighted mixed-norm Lebesgue spaces.

For simplicity, we assume $b = \tilde{b} = 0$. Denote

$$Lu = a^{ij} D_{v_i v_j} u - v \cdot D_x u,$$

where a^{ij} satisfy

$$\delta|\xi|^2 \le a^{ij}\xi_i\xi_j, \quad |a^{ij}| \le \delta^{-1}.$$

We impose a condition on a^{ij}, which is weaker than the uniform continuity condition (and the full VMO condition) used by Bramanti et. al., Polidoro–Ragusa.

The VMO$_{x,v}$ condition

There exists $R_0 > 0$ such that for any $z_0 \in \mathbb{R}^{1+2d}$ and $r \in (0, R_0]$, $\text{osc}_{x,v}(a, Q_r(z_0)) \le \gamma_0$, where $\gamma_0 \in (0,1)$ is a small constant,

$$\text{osc}_{x,v}(a, Q_r(z_0)) = \fint_{t_0-r^2}^{t_0} \fint_{D_r(t,z_0)\times D_r(t,z_0)} |a(t,x,v) - a(t,x',v')|\, dxdvdx'dv'dt,$$

$$Q_r(z_0) = \{(t,x,v) : -r^2 < t - t_0 < 0, |x - x_0 - (t-t_0)v_0| < r^3, |v - v_0| < r\}.$$

$$D_r(t,z_0) = \{(x,v) : |x - x_0 - (t-t_0)v_0| < r^3, |v - v_0| < r\} \quad \text{(time slice)}.$$

- Even if r is small, $Q_r(z_0)$ can be very long in the x-direction.
- If $a^{ij} = a^{ij}(t)$, then $\text{osc}_{x,v}(a, Q_r(z_0)) \equiv 0$.
- If a^{ij} are uniformly continuous in (x,v), then they satisfy the VMO$_{x,v}$ condition.

Main result for nondivergence form equations

Theorem (D-Yastrzhembskiy (2021))

Let $p, r_1, \dots, r_d, q > 1$ *and* $T \in (-\infty, \infty]$.
Then $\exists \gamma_0 \in (0, 1)$ *sufficiently small such that the following hold.*
(i) $\exists \lambda_0 \geq 0$ *such that* $\forall \lambda > \lambda_0$ *and* $u \in S_{p,r_1,\dots,r_d,q}((-\infty, T) \times \mathbb{R}^{2d}, w)$,

$$\lambda \|u\| + \sqrt{\lambda}\|D_v u\| + \|D_v^2 u\| + \|(-\Delta_x)^{1/3} u\| + \|(-\Delta_x)^{1/6} D_v u\| \leq N\|f\|,$$

where $\|\cdot\| = \|\cdot\|_{L_{p,r_1,\dots,r_d,q}((-\infty,T)\times\mathbb{R}^{2d},w)}$ *and* $w = w_0(t)\prod_{i=1}^d w_i(v_i)$ *is certain Muckenhoupt weight.*
Moreover, $\forall f \in L_{p,r_1,\dots,r_d,q}((-\infty, T) \times \mathbb{R}^{2d}, w)$,
the equation has a unique solution $u \in S_{p,r_1,\dots,r_d,q}((-\infty, T) \times \mathbb{R}^{2d}, w)$.

Theorem (cont'ed)

(ii) If $T \in (0,\infty)$ and $f \in L_{p,r_1,\dots,r_d,q}((0,T)\times\mathbb{R}^{2d},w)$, the Cauchy problem

$$u_t - Lu = f \quad \text{in } (0,T)\times\mathbb{R}^{2d}$$

with $u(0,\cdot,\cdot)=0$ has a unique solution $u \in S_{p,r_1,\dots,r_d,q}((0,T)\times\mathbb{R}^{2d},w)$.
(iii) The assertions (i) and (ii) hold with $S_{p,r_1,\dots,r_d,q}((-\infty,T)\times\mathbb{R}^{2d},w)$ replaced with $S_{p;r_1,\dots,r_d}(\mathbb{R}^{1+2d}_T,|x|^\alpha\prod_{i=1}^d w_i(v_i))$, where $\alpha \in (-1,p-1)$.

Muckenhoupt weights

For any $p \in (1, \infty)$, recall that $A_p = A_p(\mathbb{R}^d)$ is the collection of all nonnegative functions w on $\mathbb{R}^d$ such that

$$[w]_{A_p} := \sup_{x_0 \in \mathbb{R}^d, r>0} \left(\fint_{B_r(x_0)} w(x) \right) \left(\fint_{B_r(x_0)} w^{-\frac{1}{p-1}}(x) \right)^{p-1} < \infty.$$

- By Hölder's inequality,

$$A_p \subset A_q, \quad 1 \le [w]_{A_q} \le [w]_{A_p}, \quad 1 < p < q < \infty.$$

- The maximal operator is bounded in $L_{p,w}$ for any $w \in A_p$.

By $L_{p,r_1,\ldots,r_d,q}(G, w)$, we denote the space of all Lebesgue measurable functions on $\mathbb{R}^{1+2d}$ such that

$$||f||_{L_{p,r_1,\ldots,r_d,q}(G,w)}$$
$$= \Big| \int_{\mathbb{R}} \Big| \ldots \Big| \int_{\mathbb{R}} \Big| \int_{\mathbb{R}^d} |f|^p(z) 1_G(z)\, dx \Big|^{\frac{r_1}{p}} w_1(v_1) dv_1 \Big|^{\frac{r_2}{r_1}} \ldots w_d(v_d) dv_d \Big|^{\frac{q}{r_d}} w_0(t) dt \Big|^{\frac{1}{q}}.$$

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Remarks

- We obtained a similar result for divergence form equations. In that case, we can estimate

$$\lambda\|u\| + \sqrt{\lambda}\|D_v u\| + \lambda^{1/2}\|(-\Delta_x)^{1/6}u\|$$

- We used these results to prove the existence and uniqueness of weak and strong solutions to the Vlasov-Poisson-Landau equations near Maxwellian with the specular boundary condition.

The mean oscillation method

Krylov (2005): $W^{1,2}_p(\mathbb{R}^d)$ solvability of both divergence and nondivergence form parabolic equations with a_{ij} in VMO_x.

Idea of the proof:
establish pointwise estimate of sharp functions of Du (or D^2u), and use the Fefferman–Stein theorem on sharp functions and the Hardy–Littlewood maximal function theorem: i.e.,

$$\|\mathcal{M}f\|_{L_p} \le N\|f\|_{L_p}, \quad \|f\|_{L_p} \le N\|f^{\#}\|_{L_p}.$$

In some sense, the proof is based on an interpolation between $C^{2,\alpha}$-estimate and W^2_2-estimate (for equations with $a_{ij} = a_{ij}(t)$), as well as a perturbation argument.

The proof

- We first consider the case when $a^{ij} = a^{ij}(t)$.

When $p = 2$, the global estimate follows from the Fourier transform.

Starting point: local L_2 estimate of $D_v u$ (Cacciopoli type inequality)

Let $0 < r \le R$ and $u \in S_{2,\mathrm{loc}}(\mathbb{R}^{1+2d})$. Let $\psi \in C_0^\infty(B_1)$ and $\eta(t, v) \in C_0^\infty((-1, 0] \times B_1)$. Denote

$$\varphi_1(x) = \psi(x/R^3), \quad \varphi_2(t, v) = \eta(t/r^2, v/r), \quad \varphi(t, x, v) = \varphi_1\varphi_2.$$

Then

$$\int_{Q_{r,R}} |D_v u|^2 \varphi^2 \, dz \le N \int_{Q_{r,R}} (r^2 |\partial_t u - Lu|^2 + r^{-2} u^2) \, dz,$$

$$\int_{\mathbb{R}^{1+2d}} |D_v u|^2 \varphi_2^2 \, dz \le N \int_{(-r^2,0)\times\mathbb{R}^d\times B_r} (r^2 |\partial_t u - Lu|^2 + r^{-2} u^2) \, dz.$$

where $\quad Q_{r,R}(z_0) = \{z : -r^2 < t - t_0 < 0, |v - v_0| < r, |x - x_0 - (t - t_0)v_0|^{1/3} < R\}.$

The proof (cont'ed)

- For higher derivative estimate ($D_v^2 u$, etc.), a major difficulty is that if we differentiate $u_t - Lu = 0$ in v, then

$$(D_v u)_t - L(D_v u) = -D_x u.$$

Thus, we first need to estimate $D_x u$. Recall that by scaling $D_x u \sim D_v^3 u$.

Local estimate of $D_x u$ (key lemma)

Let $u \in S_{2,\mathrm{loc}}(\mathbb{R}^{1+2d})$ be a function such that $\partial_t u - Lu = 0$ in Q_1. Then for any numbers $0 < r < R \leq 1$,

$$\|D_x u\|_{L_2(Q_r)} \leq N(d, \delta, r, R)\|u\|_{L_2(Q_R)}.$$

Idea: apply the Caccioppoli inequality and the global L_2 estimate to $\tilde{u} := (-\Delta_x)^{1/3}(u\varphi)$, which is a nonlocal quantity.

The proof (cont'ed)

- From the estimate of $D_x u$ and the Caccioppoli inequality, we can further bound
$$\|D_v^2 u\|_{L_2(Q_r)} \le N\|u\|_{L_2(Q_R)}.$$
- Differentiating the eq. in x gives $\partial_t D_x u - L D_x u = 0$, so we can estimate $D_{vx} u$, $D_v^2 D_x u$, and $D_x^2 u$.
  Differentiating the eq. of $D_v u$ in v again, we bound $D_v^3 u$, $D_v^3 D_x u$...
- From the eq. $u_t = Lu$,
$$\|D_t^i D_x^j D_v^k u\|_{L_2(Q_r)} \le N\|u\|_{L_2(Q_R)}, \quad i = 0, 1, \quad j, k \ge 0$$
  By the Sobolev embedding,
$$\|D_t^i D_x^j D_v^k u\|_{L_\infty(Q_r)} \le N\|u\|_{L_2(Q_R)}, \quad i = 0, 1, \quad j, k \ge 0$$

The proof (cont'ed)

- Next, we replace the RHS with the norms of $D_v^2 u$ and $(-\Delta_x)^{1/3}u$. Note that $u_1(z) := u(z) - (u)_{Q_r} - v^i(D_{v^i}u)_{Q_r}$ satisfies the same equation. Then,

$$\sup_{Q_{1/2}} |D_x^j D_v^{k+2} u| + \sup_{Q_{1/2}} |\partial_t D_x^j D_v^{k+2} u| \le N \|u_1\|_{L_2(Q_r)}.$$

We then apply the Poincaré inequality to bound the RHS by

$$N(\|D_v^2 u\|_{L_2(Q_r)} + \|D_x u\|_{L_2(Q_r)}).$$

The proof (cont'ed)

To further bound the second term above, we use

A nonlocal inequality

Let $r \in (0,1)$ and $u \in S_{2,\text{loc}}(\mathbb{R}^{1+2d})$. Assume that $\partial_t u - Lu = 0$ in $(-1,0) \times \mathbb{R}^d \times B_1$. Then,

$$\|D_x u\|_{L_2(Q_r)} \le N \sum_{k=0}^{\infty} 2^{-k} (|(-\Delta_x)^{1/3} u|^2)^{1/2}_{Q_{1,2^k}}.$$

Proof. Apply the L_2 estimate to $(-\Delta_x)^{1/3} u$.

Consequently, we get

$$[D_v^2 u]_{C^\alpha(Q_{1/2})} \le N(\|D_v^2 u\|_{L_2(Q_1)} + \sum_{k=0}^{\infty} 2^{-k} (|(-\Delta_x)^{1/3} u|^2)^{1/2}_{Q_{1,2^k}}).$$

The proof (cont'ed)

For solutions to nonhomogeneous equations, we prove

Let $R \geq 1$ and $f \in L_2(\mathbb{R}^{1+2d})$ vanish outside $(-1,0) \times \mathbb{R}^d \times B_1$. Let $u \in S_2((-1,0) \times \mathbb{R}^d \times \mathbb{R}^d)$ be the unique solution to

$$\partial_t u - Lu = f, \quad u(-1,\cdot) = 0.$$

Then

$$\||u| + |D_v u| + |D_v^2 u|\|_{L_2((-1,0)\times B_{R^3}\times B_R)} \leq N \sum_{k=0}^{\infty} 2^{-k(k-1)/2} R^{-k} \|f\|_{L_2(Q_{1,2^{k+1}R})},$$

$$(|(-\Delta_x)^{1/3} u|^2)^{1/2}_{Q_{1,R}} \leq N \sum_{k=0}^{\infty} 2^{-2k} (f^2)^{1/2}_{Q_{1,2^k R}}.$$

The proof (cont'ed)

Combining these estimates, we get the following mean oscillation estimate: Let $r > 0$, $\nu \geq 2$, $z_0 \in \mathbb{R}^{1+2d}$, and $u \in S_2(\mathbb{R}^{1+2d})$. Then,

$$\fint_{Q_r(z_0)} \left| D_v^2 u - (D_v^2 u)_{Q_r(z_0)} \right|^2 dz \leq N\nu^{-2}(|D_v^2 u|^2)_{Q_{\nu r}(z_0)}$$

$$+ N\nu^{-2}\left(\sum_{k=0}^{\infty} 2^{-2k} (|(-\Delta_x)^{1/3} u|^2)^{1/2}_{Q_{\nu r, 2^k \nu r}(z_0)} \right)^2 + N\nu^{1+2d}\left(\sum_{k=0}^{\infty} 2^{-k} (f^2)^{1/2}_{Q_{\nu r, 2^k \nu r}(z_0)} \right)^2,$$

$$\fint_{Q_r(z_0)} \left| (-\Delta_x)^{1/3} u - ((-\Delta_x)^{1/3} u)_{Q_r(z_0)} \right|^2 dz$$

$$\leq N\nu^{-2}(|(-\Delta_x)^{1/3} u|^2)_{Q_{\nu r}(z_0)} + N\nu^{2+4d}\left(\sum_{k=0}^{\infty} 2^{-k} (f^2)^{1/2}_{Q_{\nu r, 2^k \nu r}(z_0)} \right)^2.$$

The proof (cont'ed)

We conclude the proof by using the weighted mixed-norm Hardy-Littlewood maximal function theorem and the Fefferman-Stein sharp function theorem (cf. D-D. Kim (2018)).

For $c > 0$, we construct a family of maximal and sharp functions.

$$\mathbb{M}_c f(z_0) = \sup_{r>0: z_0 \in Q_{r,rc}(z_1)} \fint_{Q_{r,rc}(z_1)} |f(z)|\, dz,$$

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Let $c \geq 1$, $p, q, r_1, \ldots, r_d > 1$ be numbers, $w = w_0(t)w_1(v)$ be certain Muckenhoupt weight, and $f \in L_{p,r_1,\ldots,r_d,q}(\mathbb{R}^{1+2d}, w)$. Then,

$$\|\mathbb{M}_c f\|_{L_{p,r_1,\ldots,r_d,q}(\mathbb{R}^{1+2d},w)} \leq N \|f\|_{L_{p,r_1,\ldots,r_d,q}(\mathbb{R}^{1+2d},w)}$$

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The proof (cont'ed)

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Applications: equations with boundary conditions

We consider the initial-boundary value problem of two types of equations.

- Linear Kinetic Fokker-Planck equations:

$$\partial_t f + v \cdot \nabla_x f - \partial_{v_i}(a^{ij}(t,x,v)\partial_{v_j} f) + b(t,x,v) \cdot \nabla_v f = g \text{ in } (0,T) \times \Omega \times \mathbb{R}^3$$

- Linear Landau equation with Coulomb interaction:

$$\partial_t f + v \cdot \nabla_x f = \nabla_v \cdot (\sigma_G \nabla_v f) + a_g \cdot \nabla_v f + \overline{K}_g f \text{ in } (0,T) \times \Omega \times \mathbb{R}^3,$$

where

$$\sigma_G = \Phi * (\mu + \mu^{1/2} g), \quad a_g^i = -\Phi^{ij} * (v_j \mu^{1/2} g + \mu^{1/2} \partial_{v_j} g).$$

The second equation is obtained by the Picard iteration to the (nonlinear) Landau equation, with $g = f_n$ and $f = f_{n+1}$.

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The boundary condition

We impose the **specular reflection** boundary condition

$$f(t,x,v) = f(t,x,R_x v), \quad (x,v) \in \gamma_-.$$

Here $R_x v = v - 2(n_x \cdot v)n_x$ is the specular reflected velocity, and n_x is the outward unit normal vector at $x \in \partial\Omega$, and

$$\gamma_\pm = \{(x,v) : x \in \partial\Omega, \pm n_x \cdot v > 0\}$$

is the outgoing/incoming set.
Generally, solutions are not regular near the grazing set

$$\gamma_0 = \{(x,v) : x \in \partial\Omega, n_x \cdot v = 0\}$$

.

Existence of finite energy weak solutions

In a recent joint work with Timur Yastrzhembskiy and Yan Guo, we first obtained the existence of weak solutions with finite energy.

Idea: Finite difference approximations by using an idea to Krylov (2011) to approximate the second-order derivatives by pure second-order directional finite-difference quotients. Then apply an existence theory for Vlasov equations.

Difficulty: The equation does not satisfies the energy identity because we only know that $f_t + v \cdot D_x f \in \mathbb{H}_2^{-1}$, so the uniqueness is an issue.

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Existence of finite energy strong solutions and the uniqueness

We assume that $g, D_v g \in L_\infty((0,T); C^{\alpha/3,\alpha}_{x,v})$ and g satisfies the specular boundary condition.

Theorem (D., Guo, Yastrzhembskiy)

Suppose that Ω is a bounded C^3 domain, $\|g\|_{L_\infty} \le \varepsilon$, and the initial data is sufficiently small (in certain function spaces). Then the linear Landau equation has a unique finite energy strong solution for any $T > 0$, and its norms are bounded by its initial data. Moreover, any two finite energy weak solutions must coincide.

Some remarks

- The key step of the proof is to use a boundary extension argument. Thus it is important to check that the coefficients are still regular (with respect to (x, v)) after the extension (but not in (t, x, v)).
- After the extension, some drift coefficients are discontinuous, so we need to apply the Sobolev theory.
- The above results are crucial in the proof of the global existence and stability of solutions to the (nonlinear) Landau equation near the Maxwellian.

Further questions

- Harnack (Krylov–Safonov) estimate for nondivergence form equations with measurable coefficients.
- Boundary estimate with prescribed incoming boundary condition.
- Replace $a^{ij}D_{v_iv_j}$ with a fractional (nonlocal) operator. Applications to the Bolzmann equations.

Reference

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Happy Birthday, Nick!!!

Hang Zhou, 6/5/2005