Recent Advances in Robust Algebraic DD

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ICMS@Strathclyde: Solvers for frequency-domain wave problems and applications
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Outline

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- **Black box:** prefer non-intrusive solvers. Only provide A and b
- **Easy setup:** Few knowledge on linear solvers. Not worry how to set it up perfectly. As minimal parameters as possible. (Type: Hermitian, saddle-point, etc; Accuracy; Max iter)

Ingredients: Overlapping Subdomain

The sparsity graph of A has n nodes.
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$$\sum_{i=1}^{N} R_i^T D_i R_i = I_n$$

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$$D_1 = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 0 \end{pmatrix} \qquad D_2 = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 0 \end{pmatrix}$$

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One-Level Schwarz

Four stages:

1. Restrict
2. Solve locally
3. Augment
4. Update

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It	42	53	66	74	84	97

Table: 2D Poisson on 300×300 mesh. Metis partitioning.

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Need a second level (coarse space correction) to maintain robustness

$$M_2^{-1} = R_0^H A_{00}^{-1} R_0 + \sum_{i=1}^{N} R_i^T A_{ii}^{-1} R_i.$$

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Local HPSD Splitting [H.A, L. Grigori '19]

B HPD

$$B = \begin{pmatrix} B_{11} & B_{12} & \\ B_{21} & B_{22} & B_{23} \\ & B_{32} & B_{33} \end{pmatrix}.$$

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- $\alpha \neq 1$ very expensive (bad)

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Local HPSD Splitting $\iff$ Robust (w.r.t N) Two-Level DD

Spectral Coarse Space

$D_i A_{ii} D_i z = \lambda \widetilde{A}_{ii} z$, $Z_i =$ basis for $span\{z, \lambda > \tau\}$

$$R_0^H = [R_1^T D_1 Z_1, \ldots, R_N^T D_N Z_N], A_{00} = R_0 A R_0^H$$

$$M_2^{-1} = R_0^H A_{00}^{-1} R_0 + \sum_{i=1}^{N} R_i^T A_{ii}^{-1} R_i$$

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Sparse SPD Matrix [H.A. P. Jolivet '21]

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Diagonally Dominant HPD [H.A., P. Jolivet, T. Rees '22]

B HPD diagonally dominant $b_{ii} \geq \sum_{j \neq i} |b_{ij}|$.

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$$B = \underbrace{\begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} - \mathrm{diag}(s) \end{pmatrix}}_{\text{Diag Dom HPSD}} + \underbrace{\begin{pmatrix} \mathrm{diag}(s) & B_{23} \\ B_{32} & B_{33} \end{pmatrix}}_{\text{Diag Dom HPSD}}$$

PDE-CO

Solve

$$\min_{y} \|y - \hat{y}\|_{\Omega_1}^2 + \beta \|u\|_{\Omega_2}^2 \quad \text{subject to } \mathcal{L}y = u \in \Omega$$

The resulting matrix

$$\begin{pmatrix} M & & K^* \\ & \beta R & L^* \\ K & L & \end{pmatrix}$$

Mass lumping yields an equivalent diagonal matrix W to the (1:2,1:2)-block. $\widetilde{S} = J^* J$, where $J^* = [KL] W^{-1/2}$.

Poisson PDE-CO

IFISS: Grid $2^8 \times 2^8$, $\beta = 0.01$, Q_2-FE, matrix length $\approx 200K$.

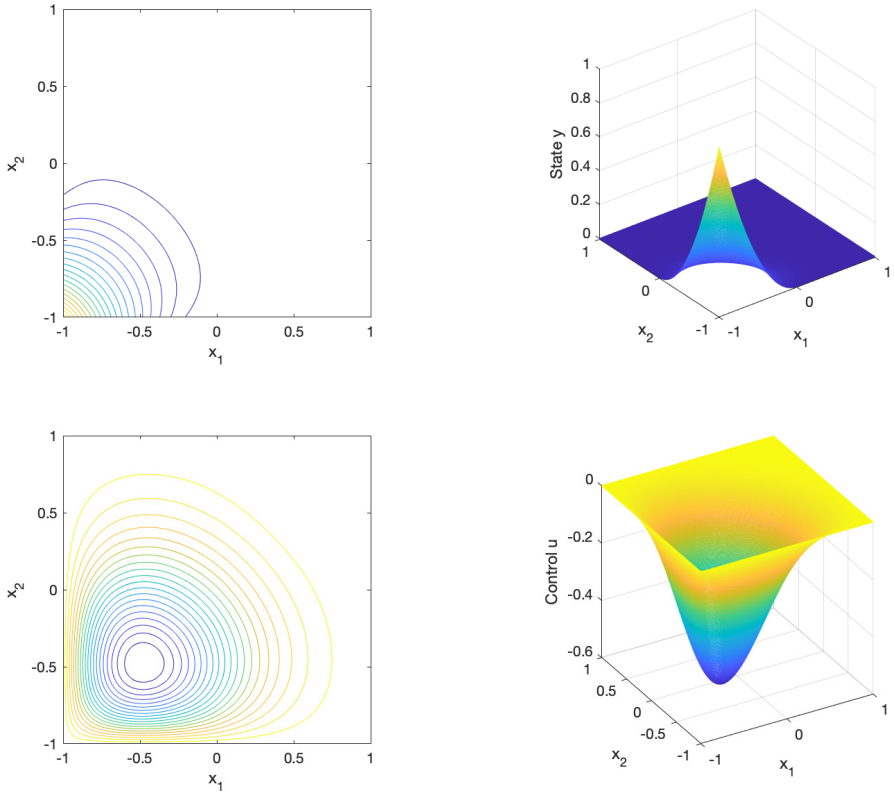

Figure: Plots

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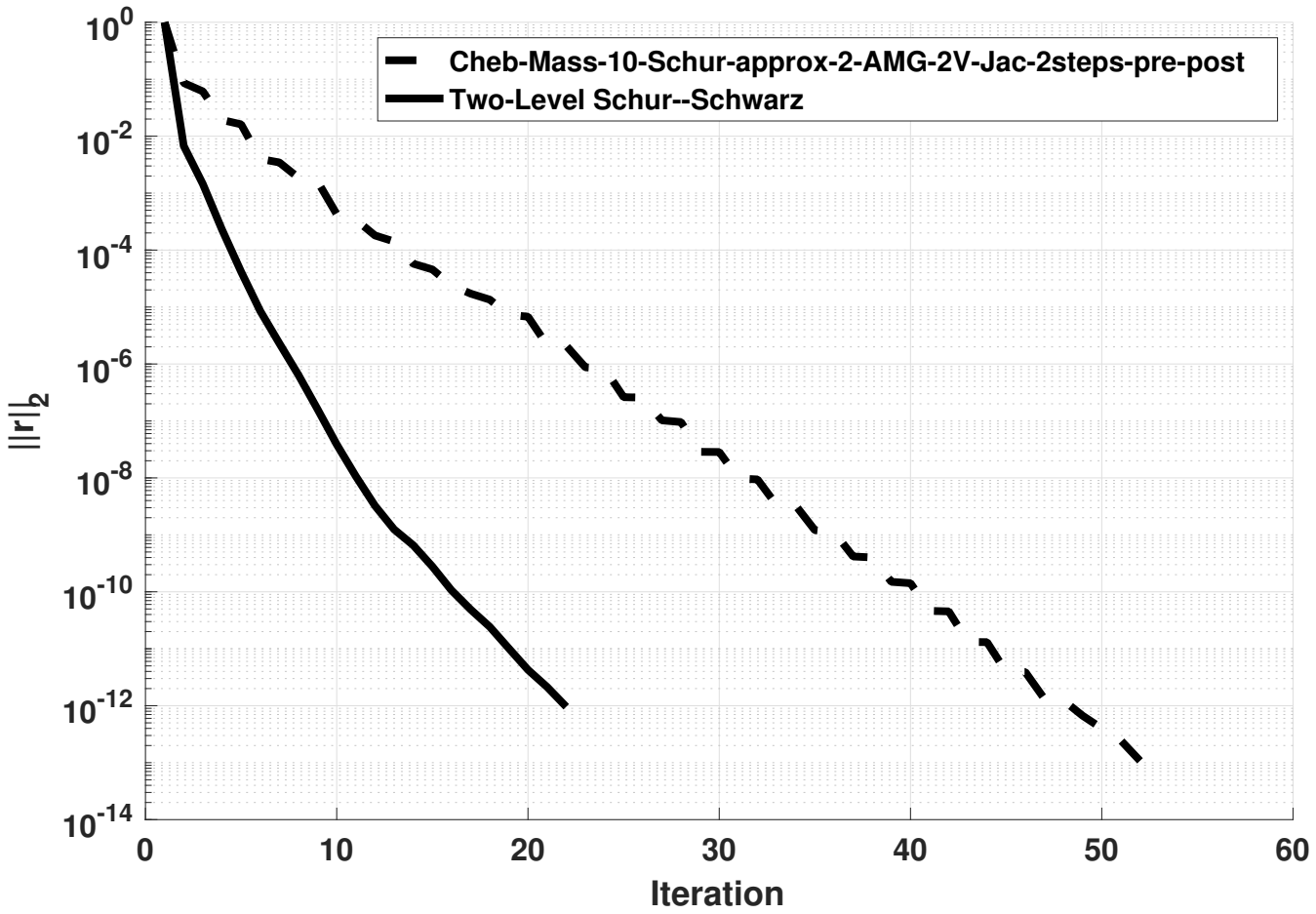

Figure: Residual history

Helmholtz PDE-CO: I

Test case inspired by [Kouri et al. SISC 21']
Grid 80×80, $\beta = 0.01$, P_1-FE, matrix length $\approx 80K$.

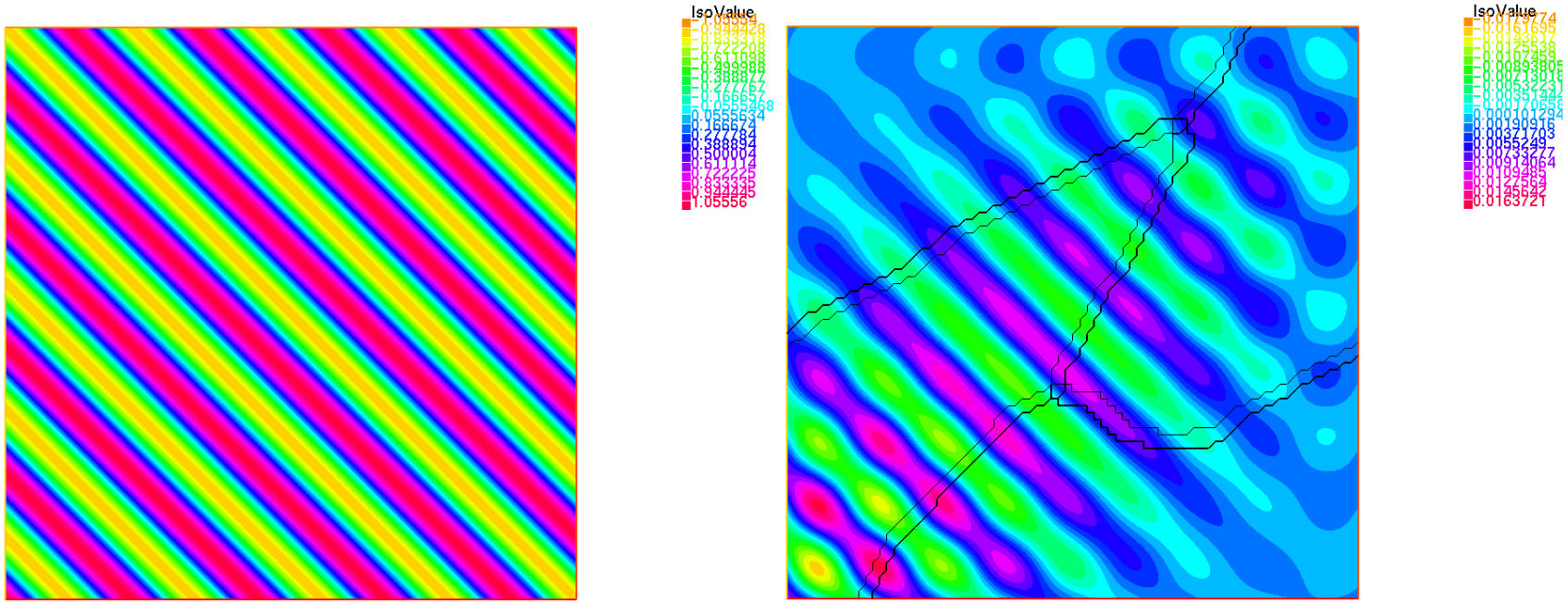

Figure: State (real part): Desired (left), solution (right)

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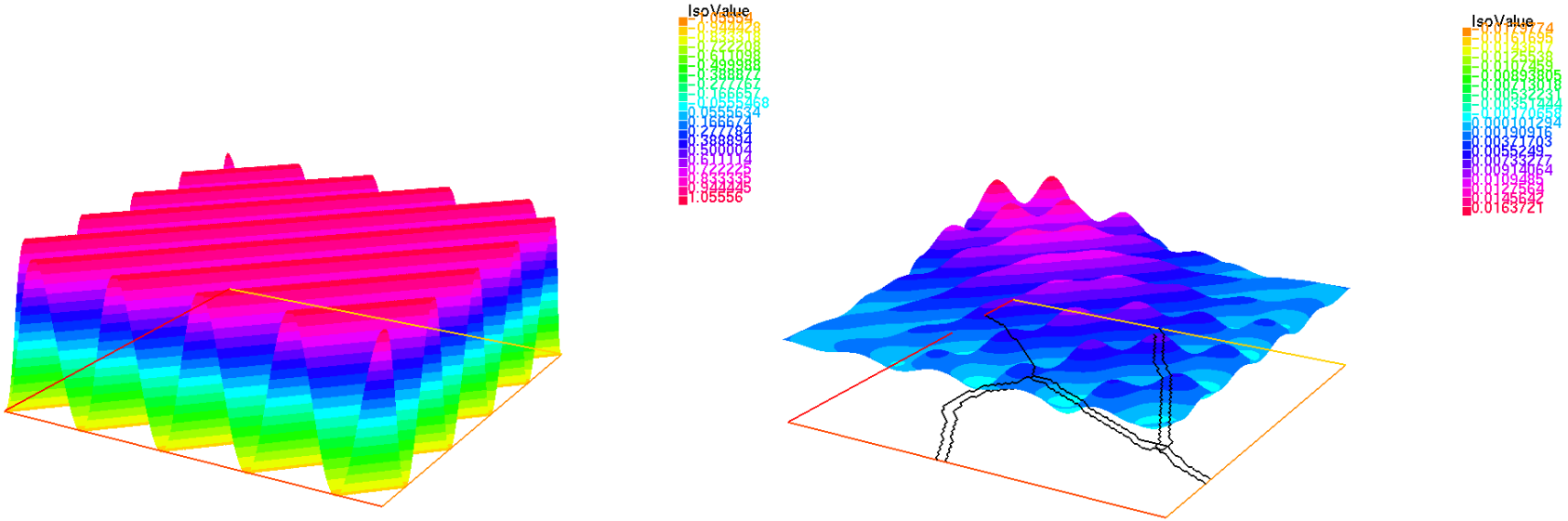

Figure: 3D view of the state (real part)

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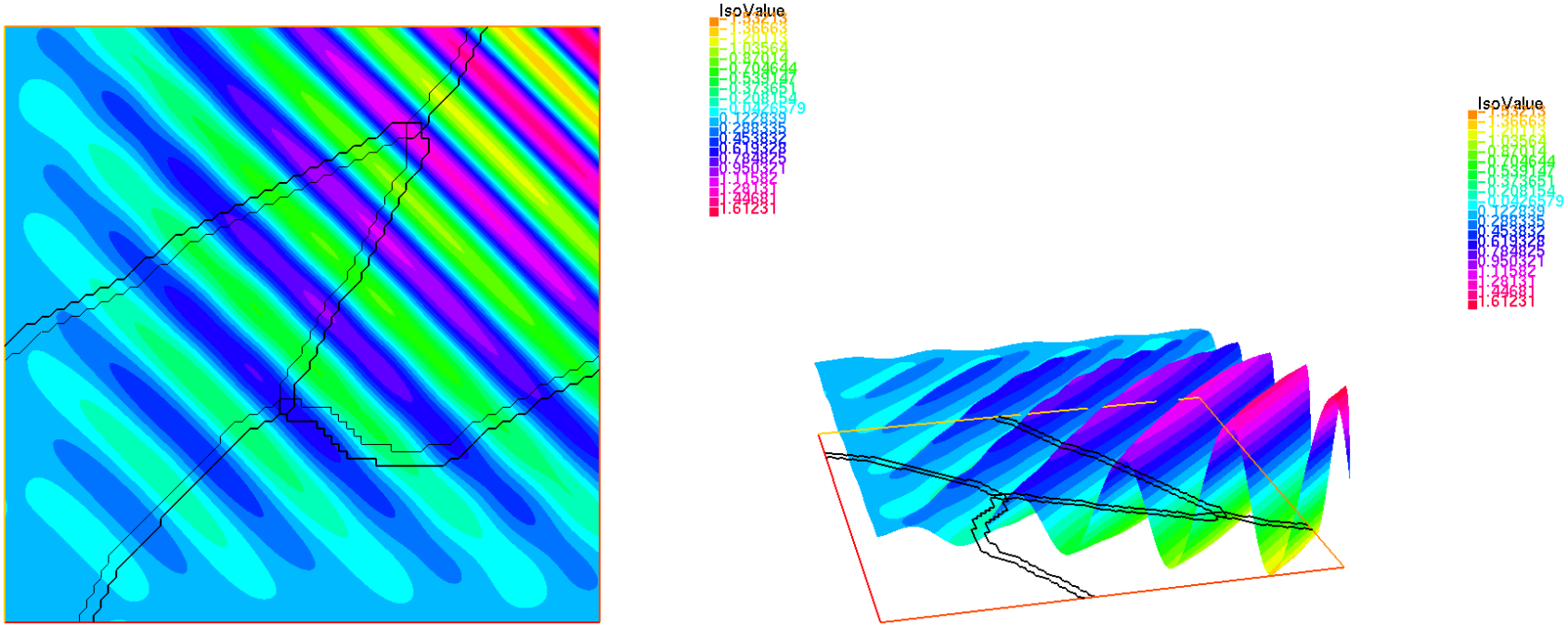

Figure: Control (real part)

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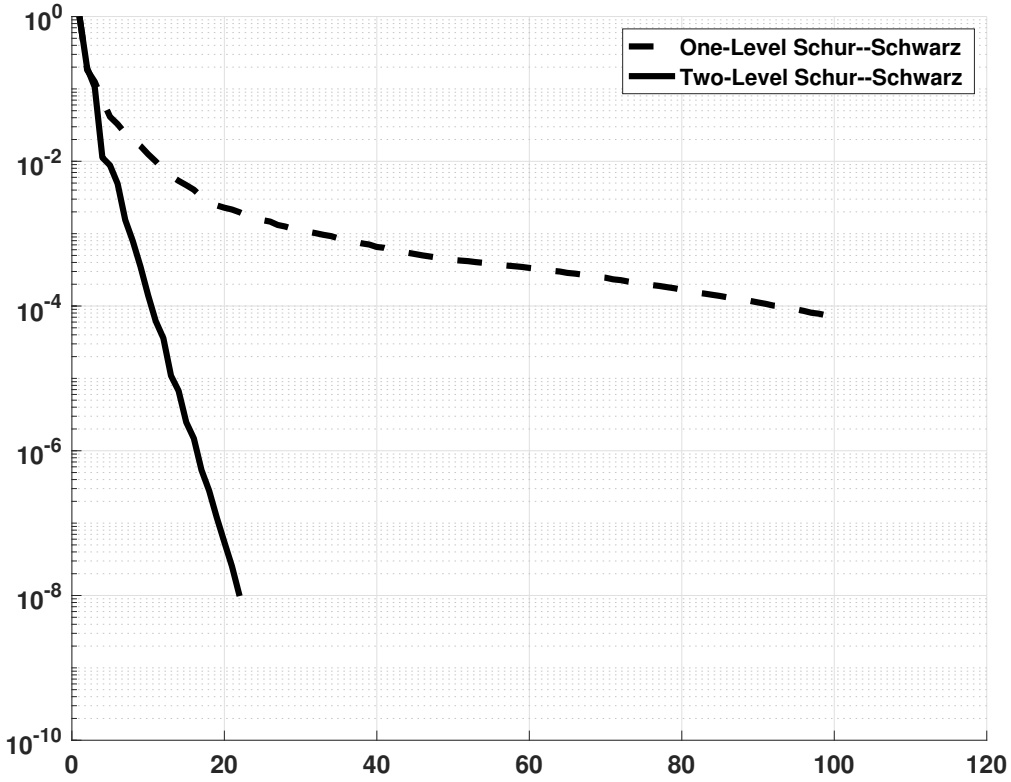

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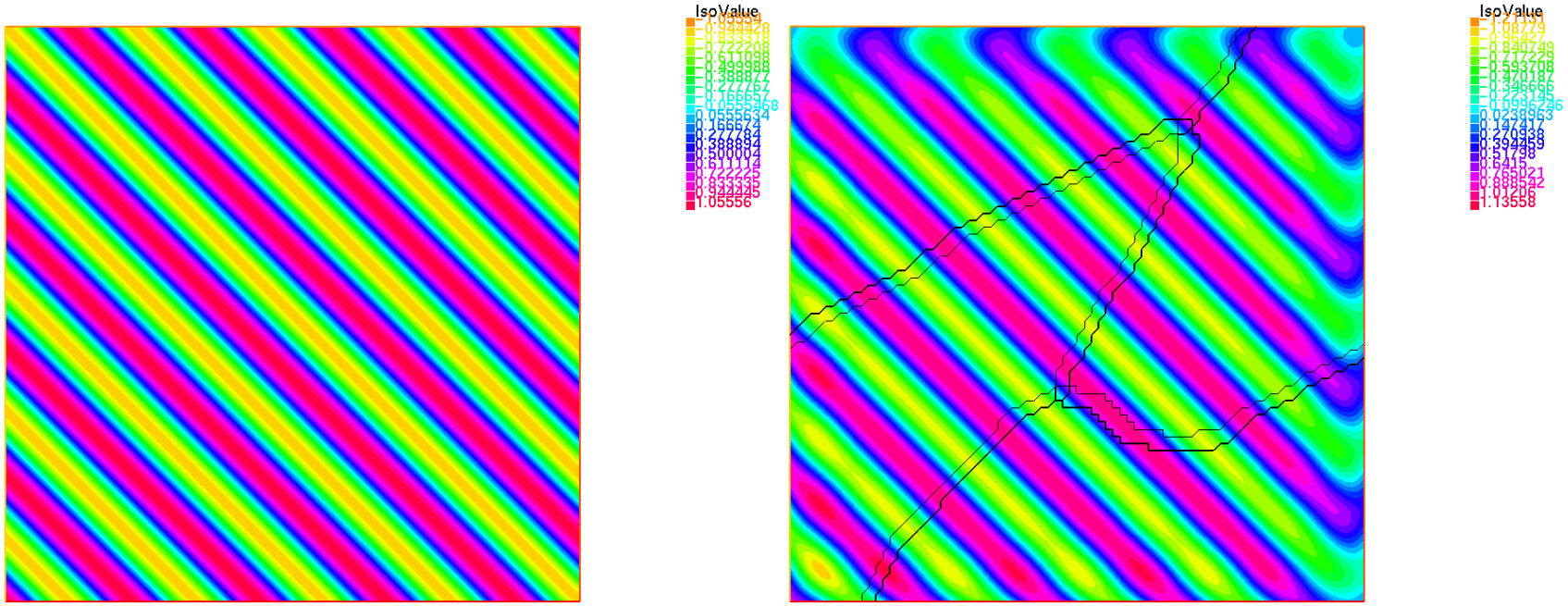

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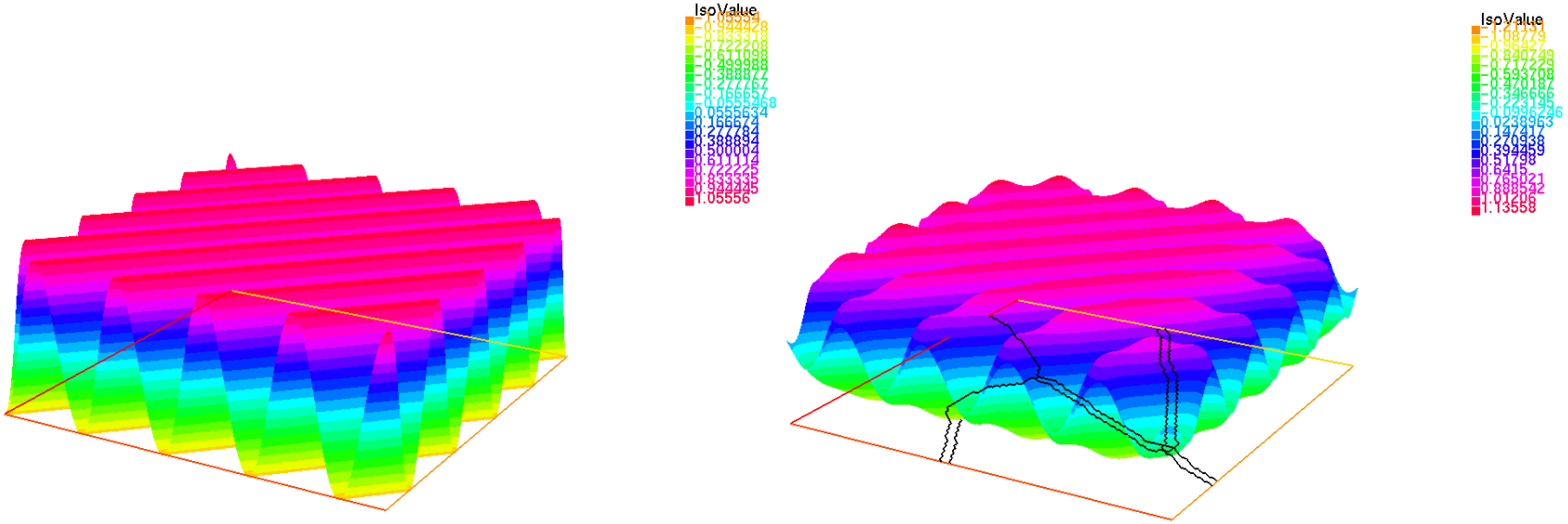

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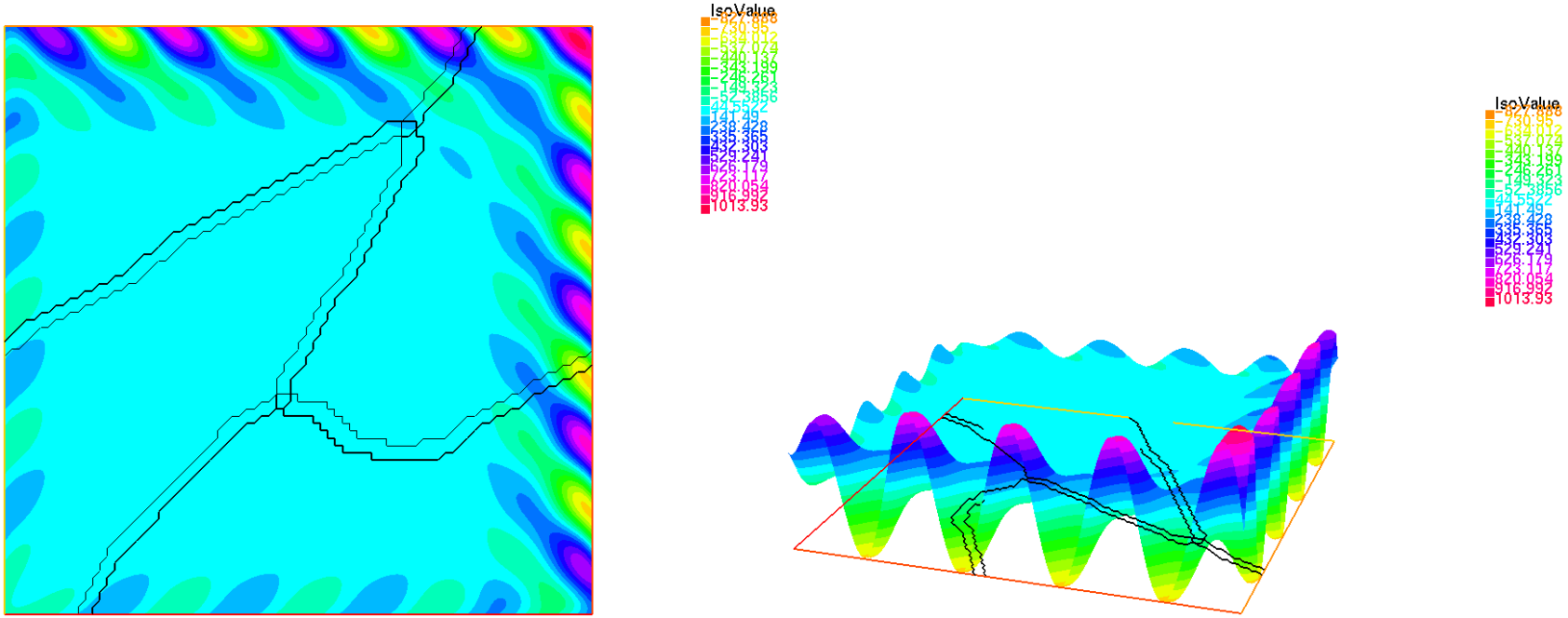

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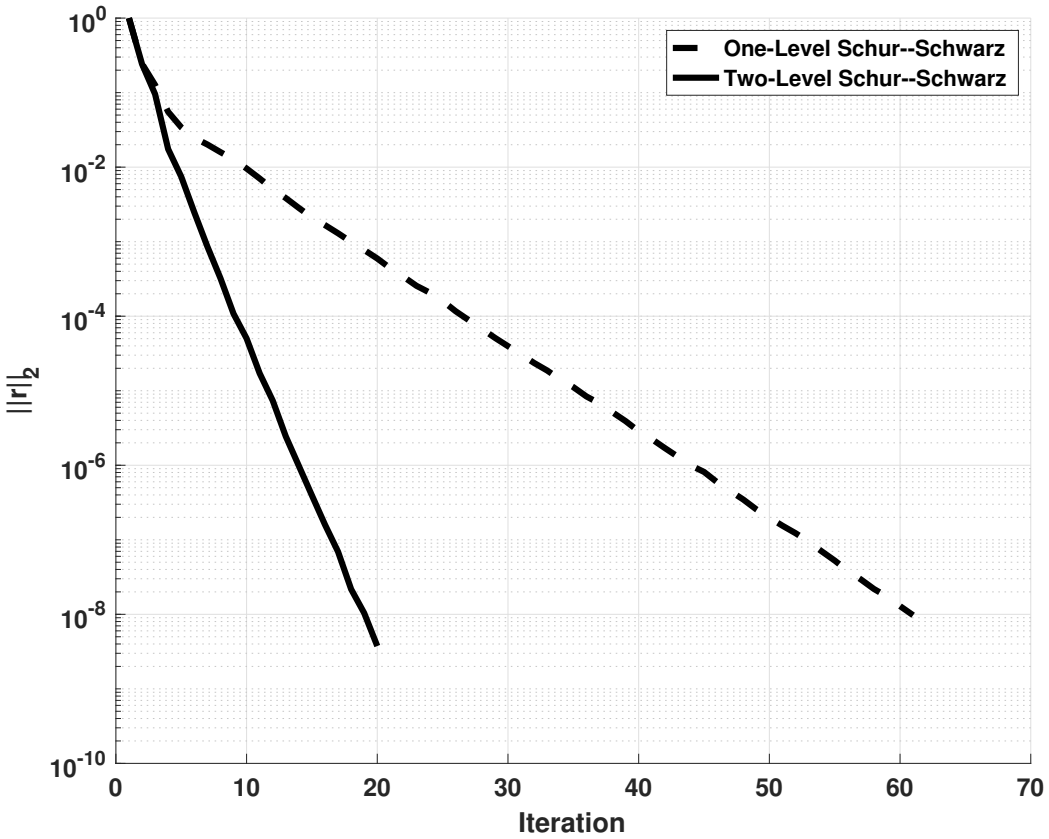

Figure: Residual history

Helmholtz PDE-CO: II

Test case inspired by Kouri et al. 21'
Grid 160×160, $\beta = 10^{-5}$, P_1-FE, matrix length $\approx 50K$.

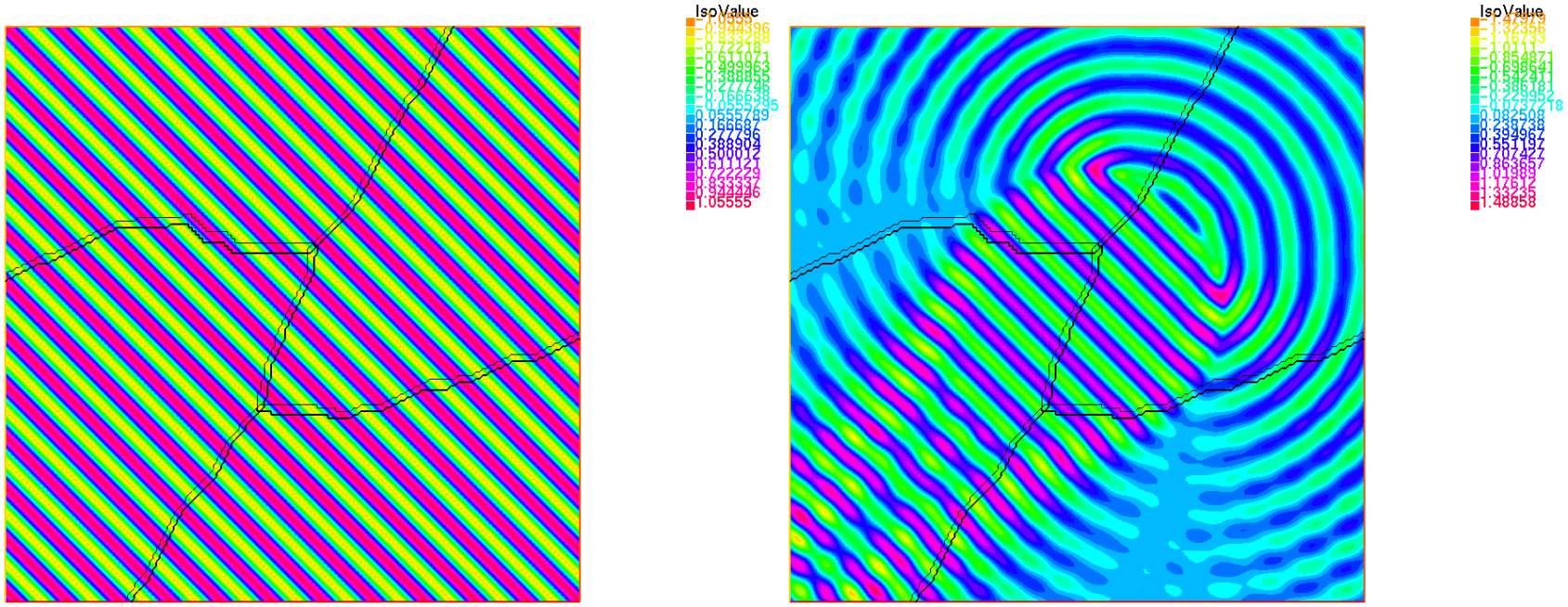

Figure: State (real part): Desired (left), solution (right)

Helmholtz PDE-CO: II

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Grid 160×160, $\beta = 10^{-5}$, P_1-FE, matrix length $\approx 50K$.

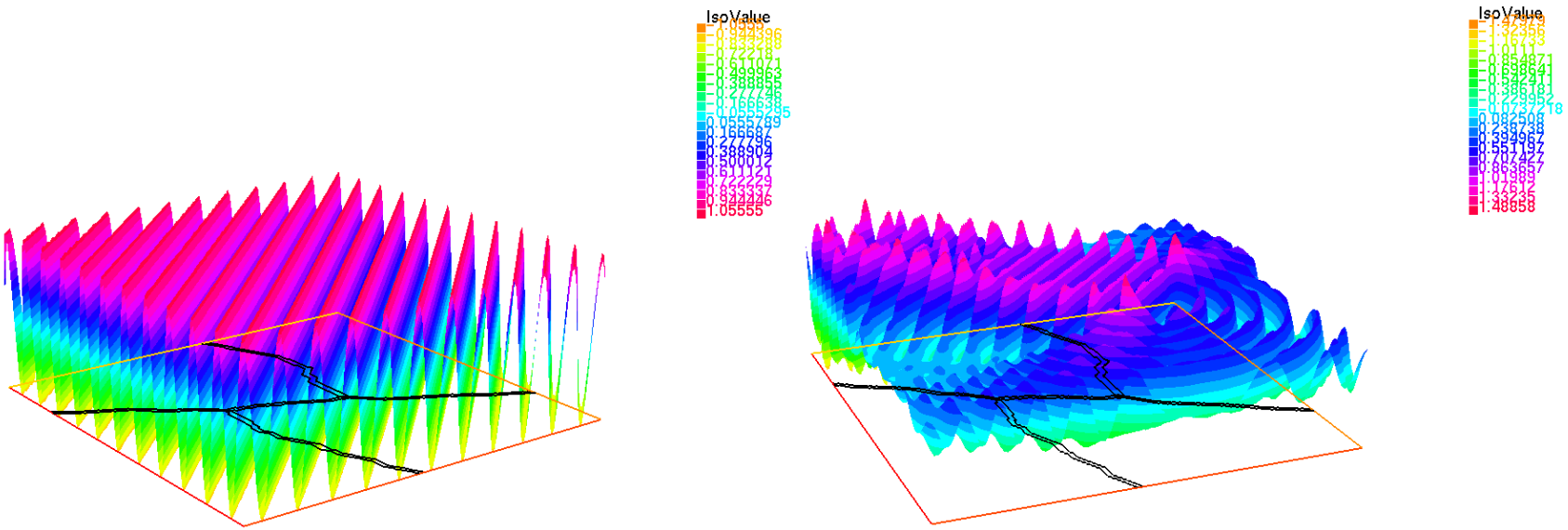

Figure: 3D view of the state (real part)

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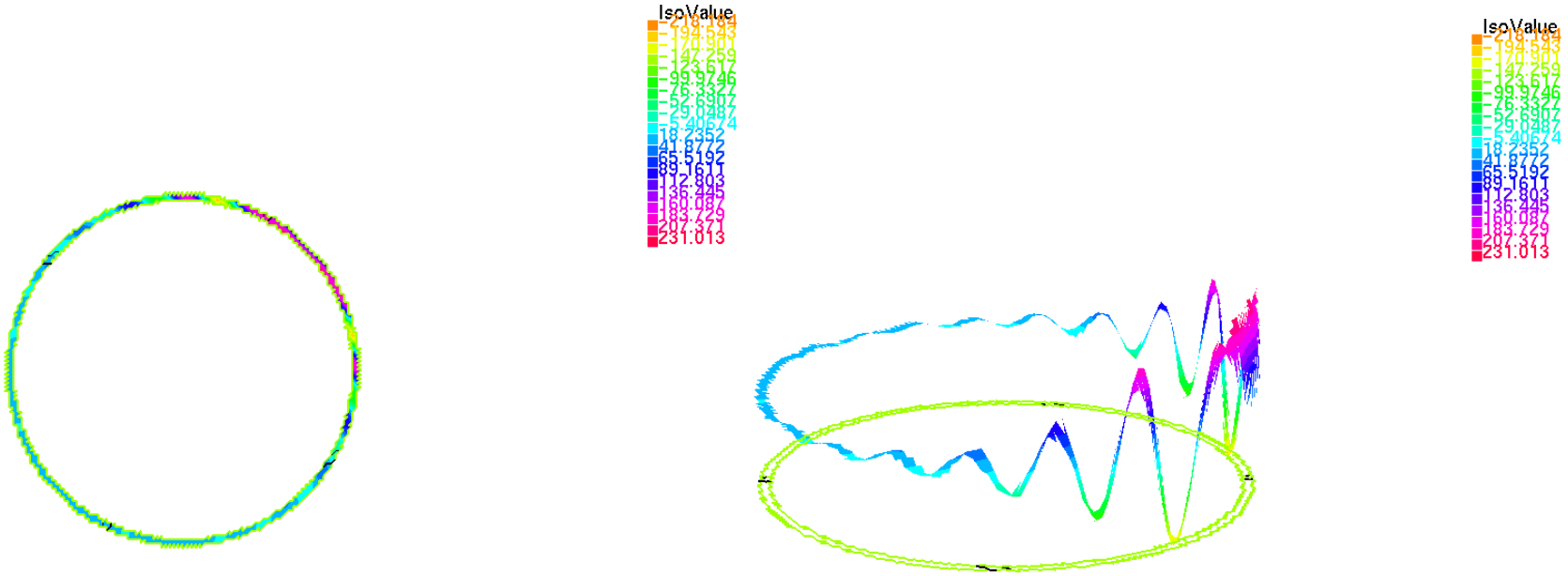
Grid 160×160, $\beta = 10^{-5}$, P_1-FE, matrix length $\approx 50K$.



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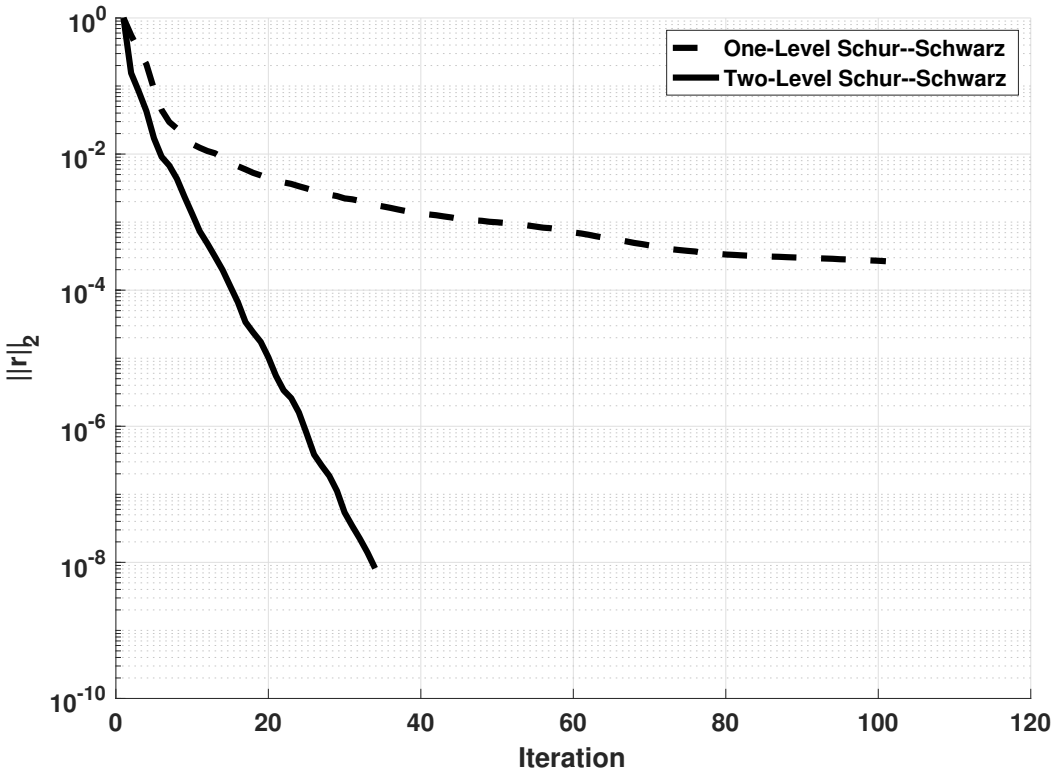

Figure: Residual history

Summary

- Local HPSD matrices provide a simple way to construct preconditioners that are effective, efficient, black-box and easy to set up
- Provable: Diagonally weighted normal equations matrix (Schur complement); HPD; Diagonally dominant HPD
- All preconditioner are accessible in PETSc PCHPDDM

Thank you for your attention!

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