Convergence on the finite-depth fluid equation in the shallow water and deep water limits

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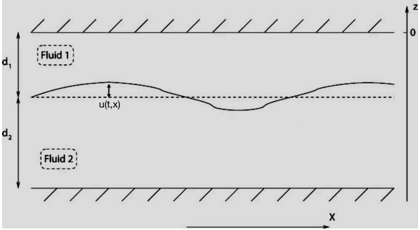
Finite-depth fluid equation



Finite-depth fluid equation (FDF): on $\mathcal{M} = \mathbb{R}$ or $\mathbb{T} = (\mathbb{R}/2\pi\mathbb{Z})$

$$(\text{FDF}_\delta) \qquad \partial_t u - \mathcal{L}_\delta \circ \partial_x u + \partial_x(u^2) = 0$$

- $\delta = d_1 + d_2$ = fluid depth $(0 < \delta < \infty)$
- $\mathcal{L}_\delta = -\partial_x \coth(\delta\partial_x) + \frac{1}{\delta}$
- $\mathcal{L}_\delta \circ \partial_x$ = dispersive (not parabolic - ***no*** smoothing at the linear level)

Q: Study the convergence property to FDF in

- deep water limit: $\delta \to \infty$
- shallow water limit: $\delta \to 0$

Deep water limit: $\delta \to \infty$

(FDF) $$\partial_t u - \mathcal{L}_\delta \partial_x u + \partial_x(u^2) = 0$$

$\mathcal{L}_\delta$ in the integral form:

$$\mathcal{L}_\delta \partial_x u(x) = \frac{1}{2\delta} \text{ p.v.} \int_{-\infty}^{\infty} \coth\left(\frac{\pi(x-y)}{2\delta}\right) \partial_y^2 u(y) dy$$

- As $\delta \to \infty$, $\frac{1}{2\delta} \coth(\frac{\pi(x-y)}{2\delta}) \longrightarrow \frac{1}{2\delta} \frac{2\delta}{\pi(x-y)} = \frac{1}{\pi(x-y)}$

  $\Longrightarrow$ by taking $\delta \to \infty$,

$$\mathcal{L}_\delta \partial_x u(x) \longrightarrow \frac{1}{\pi} \text{p.v.} \int_{-\infty}^{\infty} \frac{\partial_y^2 u(y)}{x-y} dy = \mathcal{H} \partial_x^2 u(x)$$

Hence, we expect **FDF** to converge to the **Benjamin-Ono equation:**

(BO) $$\partial_t u - \mathcal{H} \partial_x^2 u + \partial_x(u^2) = 0$$

Goal: Mathematically justify this formal convergence

Shallow water limit: $\delta \to 0$

- Scaled amplitude: $v(t,x) = \frac{3}{\delta} u(\frac{3}{\delta} t, x)$

Scaled FDF: $\partial_t v - \frac{3}{\delta} \mathcal{L}_\delta \partial_x v + \partial_x (v^2) = 0$

- Write $\mathcal{L}_\delta$ as a formal power series in δ:

$$\mathcal{L}_\delta = -\partial_x \left(\underbrace{\frac{e^{\delta\partial_x} + e^{-\delta\partial_x}}{e^{\delta\partial_x} - e^{-\delta\partial_x}}}_{=\coth(\delta\partial_x)} - \frac{1}{\delta}\partial_x^{-1} \right) = -\partial_x \left(\frac{\delta\partial_x}{3} + \frac{\delta^3\partial_x^3}{45} + O(\delta^5) \right)$$

- Take $\delta \to 0$: $-\frac{3}{\delta}\mathcal{L}_\delta \partial_x = \partial_x^3 - \frac{\delta^2\partial_x^5}{15} + O(\delta^5) \longrightarrow \partial_x^3$

Hence, we expect **scaled FDF** to converge to the **Korteweg-de Vries equation:**

(KdV) $$\partial_t v + \partial_x^3 v + \partial_x (v^2) = 0$$

Goal: Mathematically justify this formal convergence

Physical interests:

- Oceanography, atmospheric science, quantum hall effect, etc.
  Lipovskiy '86 , Berntson-Langmann-Lenells '20

Previous results:

- Solitary wave convergence: Joseph '77
- Soliton convergence and numerical study: Kubota-Ko-Dobbs '78
- Convergence of FDF in $C_t H_x^s$ on $\mathbb{R}$ or $\mathbb{T}$ (Abdelouhab-Bona-Felland-Saut '89):
  (i) deep water limit for $s > \frac{3}{2}$
  (ii) shallow water limit for $s \geq 2$
  $\Longleftarrow$ classical energy method

Part 1: Deterministic approach

- for lower values of s
- for a general nonlinearity (mathematical interest)

Part 2: Probabilistic approach

- convergence of the statistical ensemble

Part 1: Deterministic approach
microscopic viewpoint

Main result

$\mathcal{M} = \mathbb{R}$ or $\mathbb{T}$

Theorem 1: Li '22

(i) **Deep water limit ($\delta \to \infty$):** Given $s \geq \frac{3}{4}$, fix $u_0 \in H^s(\mathcal{M})$. Given $\delta \gg 1$, let

- u_δ be the solution to **FDF** with $u_\delta|_{t=0} = u_0$
- u be the solution to **BO** with $u|_{t=0} = u_0$.

Then, u_δ converges to u in $C_T H^s_x$, where $T = T(\|u_0\|_{H^s})$ = local existence time

(ii) **Shallow water limit ($\delta \to 0$):** Given $s \geq \frac{2}{3}$, fix $v_0 \in H^s(\mathcal{M})$. Given $\delta > 0$, let

- v_δ be the solution to **scaled FDF** with $v_\delta|_{t=0} = v_0$
- v be the solution to **KdV** with $v|_{t=0} = v_0$.

Then, v_δ converges to v in $C_T H^s_x$, where $T = T(\|v_0\|_{H^s})$ = local existence time

- improved regularities for FDF on $\mathcal{M} = \mathbb{R}$ and $\mathbb{T}$
- Our argument (robust) works for a general analytic nonlinearity $f(u)$
- ***First convergence result on*** $\mathbb{T}$ for u^k, $k \geq 3$
  Some previous results on $\mathbb{R}$ for u^k, $k \geq 3$: Guo-Wang '08, Han-Wang '08

Idea of the proof: (deep-water limit on $\mathbb{T}$)

- On $\mathbb{T}$, the problem is more difficult due to the lack of the local smoothing estimate

Main goals:

- "Uniform" local well-posedness of FDF_δ in $C_T H^s_x$, $\delta \gg 1$
- Convergence property of $\{u_\delta\}_{\delta \gg 1}$ as $\delta \to \infty$

Strategy: Write FDF_δ as

$$\underbrace{\partial_t u_\delta - \mathcal{H}\partial_x^2 u_\delta + \partial_x(u_\delta^2)}_{=\mathrm{BO}} + (\mathcal{H}\partial_x - \mathcal{L}_\delta)\partial_x u_\delta = 0$$

- $\mathcal{F}\{(\mathcal{H}\partial_x - \mathcal{L}_\delta) f\}(n) \approx \frac{2}{\delta}\widehat{f}(n)$

- Given $\gamma > \delta \gg 1$, let u_γ, u_δ solve FDF with the same initial data u_0. Then, by integration by parts (= typical tool in hyperbolic quasilinear equtions),

$$\frac{d}{dt}\|u_\delta - u_\gamma\|^2_{H^{s-1}} \lesssim \left(\frac{1}{\delta}\right)^2 \|u_\delta\|^2_{H^s} + \|u_\delta - u_\gamma\|^2_{H^{s-1}}$$

Proposition:

(i) **A priori bound:**

$$\|u_\delta\|_{L^\infty_T H^s_x} \leq \|u_0\|_{H^s} + T^\theta C(\|u_\delta\|_{L^\infty_T H^s_x})\|u_\delta\|_{L^\infty_T H^s_x}$$

(ii) **Difference estimate:**

$$\begin{aligned}\|u_\delta^{(1)} - u_\gamma^{(2)}\|_{L^\infty_T H^{s-1}_x} &\leq \|u_0^{(1)} - u_0^{(2)}\|_{H^{s-1}} \\ &\quad + T^\theta C(\|u_0^{(1)}\|_{H^s}, \|u_0^{(2)}\|_{H^s})\|u_\delta^{(1)} - u_\gamma^{(2)}\|_{L^\infty_T H^{s-1}_x}\end{aligned}$$

Difficulty on $\mathbb{T}$: ***No*** local smoothing estimate (which is available on $\mathbb{R}$)

Main tools:

- Fourier restriction norm method: Bourgain '93
  - $X^{s,b}$-spaces: $\|u\|_{X^{s,b}} = \|\langle n\rangle^s \langle \tau - h_\delta(n)\rangle^b \widehat{u}(\tau, n)\|_{\ell^2_n L^2_\tau}$
    $\Longleftarrow$ space-time Sobolev space adapted to dispersive equations
- As for (i), multiply FDF_δ by u_δ and integrate in t
  $\Longrightarrow$ Littlewood-Paley decomposition &
  separately estimate **resonant** and **non-resonant** interactions

(a) Resonant interaction:

- ***short-time Strichartz estimate***: Koch-Tzvetkov '03

  $\Longleftarrow$ over time intervals of length $\sim N^{-\theta}$, $N =$ size of spatial freq.

- new ingredient: ***uniform in $\delta \gg 1$***

$$cn^2 \leq \big(\coth(\delta n) - \tfrac{1}{\delta n}\big)n^2 \leq Cn^2$$

(b) Non-resonant interaction:

- Gain of derivative from "*multilinear dispersion*"

  $\Longleftarrow$ manifested in multilinear analysis via the Fourier restriction norm method

Part 2: Probabilistic approach
macroscopic viewpoint

Defocusing Gibbs measures

Hamiltonian structures for generalized FDF (with u^k for $k =$odd):

- Deep-water ($\delta \gg 1$, unscaled):

$$H_\delta(u) = \frac{1}{2}\int_{\mathbb{T}} u\mathcal{L}_\delta u\, dx + \frac{1}{k+1}\int_{\mathbb{T}} u^{k+1} dx$$

$\Longrightarrow$ **Gibbs measure:** $d\rho_\delta(u) = Z_\delta^{-1} \exp\big(-H_\delta(u)\big) du$

- Shallow-water ($\delta \ll 1$, scaled):

$$\widetilde{H}_\delta(v) := \frac{3}{2\delta}\int_{\mathbb{T}} v\mathcal{L}_\delta v\, dx + \frac{1}{k+1}\int_{\mathbb{T}} v^{k+1} dx$$

$\Longrightarrow$ **Gibbs measure:** $d\widetilde{\rho}_\delta(v) = \widetilde{Z}_\delta^{-1} \exp\big(-\widetilde{H}_\delta(v)\big) dv$

Q1: Fix $\delta > 0$. Can we construct invariant Gibbs dynamics for (scaled) gFDF?

Q2: Can we study the deep water limit ($\delta \to \infty$) and shallow water limit ($\delta \to 0$) with the Gibbs measure initial data?

Main result

Theorem 2: Li-Oh-Zheng '22

Let $k \geq 3$ be an odd integer. Then,

(i) **Deep water limit ($\delta \to \infty$):**

- (i.a) **measure convergence:** Gibbs measure ρ_δ converges to ρ_{gBO} in total variation
- (i.b) **dynamics convergence:** $u_\delta \stackrel{\mathcal{D}}{\to} u_{\text{gBO}}$ in $C(\mathbb{R}; H^{-\varepsilon}(\mathbb{T}))$

(ii) **Shallow water limit ($\delta \to 0$):**

- (ii.a) **measure convergence:** Gibbs measure $\widetilde{\rho}_\delta$ converges to ρ_{gKdV} **weakly**
- (ii.b) **dynamics convergence:** $v_\delta \stackrel{\mathcal{D}}{\to} v_{\text{gKdV}}$ in $C(\mathbb{R}; H^{-\varepsilon}(\mathbb{T}))$

- When $\delta \to 0$,
  $\widetilde{\rho}_\delta$ is supported on $H^{-\varepsilon}(\mathbb{T})$, while ρ_{gKdV} is supported on $H^{\frac{1}{2}-\varepsilon}(\mathbb{T})$
- global existence and convergence of solutions *without* uniqueness

Ideas of proof:

- based on a modification of the argument by Albeverio-Cruzeiro '90, Da Prato-Debussche '02, Burq-Thomann-Tzvetkov '18, Oh-Thomann '18
  - compactness argument (at the level of measures on space-time distributions)
  - almost sure global existence without uniqueness