A Unified View of Sweeping Algorithms for Helmholtz

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Outline

Outline

Statement of the problem

Helmholtz equation in a bounded domain with wavenumber k :

$$\begin{cases} \left(-k^2 - \Delta\right) u = f \text{ in } \Omega \\ + \text{ appropriate boundary conditions on } \partial\Omega \end{cases}$$

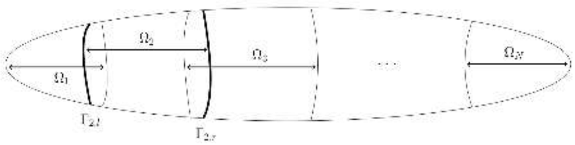


Figure: Decomposition into overlapping or non overlapping strips

Decomposition into N strips $\Rightarrow$ specialized solvers, see Martin Gander and Hui Zhang. *A class of iterative solvers for the helmholtz equation: Factorizations, sweeping preconditioners, source transfer, single layer potentials, polarized traces, and optimized schwarz methods.* Siam Review, 2019.

Unified framework for the following sweeping methods

- **1997** - Nataf, F., & Nier, F., *Convergence rate of some domain decomposition methods for overlapping and nonoverlapping subdomains*. Numerische Mathematik,

  In Gander-Zhang, this algorithm is named DOSM (Double sweep Optimized Schwarz Method).

- **2013** - Stolk, C. C., *A rapidly converging domain decomposition method for the Helmholtz equation*. Journal of Computational Physics

- **2014** - Vion, A., & Geuzaine, C., *Double sweep preconditioner for optimized Schwarz methods applied to the Helmholtz problem*. Journal of Computational Physics

All three articles present under the generic term of sweeping algorithms methods which are not the same actually!

Solve in parallel:

$$\begin{cases} \left(-k^2 - \Delta\right) u_i^{n+1} = f \text{ in } \Omega_i,\ 1 \le i \le N \\ \mathcal{B}_{i,l}\left(u_i^{n+1}\right) = \mathcal{B}_{i,l}\left(u_{i-1}^{n}\right) \text{ on } \Gamma_{i,l},\ 2 \le i \le N \\ \mathcal{B}_{i,r}\left(u_i^{n+1}\right) = \mathcal{B}_{i,r}\left(u_{i+1}^{n}\right) \text{ on } \Gamma_{i,r},\ 1 \le i \le N-1 \\ + \text{ appropriate boundary conditions on } \partial\Omega \cap \partial\Omega_i\,. \end{cases}$$

where $\mathcal{B}_{i,l}$ and $\mathcal{B}_{i,r}$ are the interface conditions.
P.L. Lions (1990) for elliptic problems and B. Després (1991) for wave propagation problems.

Interface conditions

We consider either
first-order ABC as interface conditions:

$$\begin{cases} \mathcal{B}_{i,l} = \partial_{\vec{n}_{i,l}} + Ik \\ \mathcal{B}_{i,r} = \partial_{\vec{n}_{i,r}} + Ik \end{cases}$$

where $I^2 = -1$.

OR

Perfectly Matched Layers (Beranger, 1994) which lead to significative improve of the convergence speed.

Substructuration

Why ABC as interface conditions?

We reformulate the Lions-Després method by considering only surfacic unknowns on the interfaces:

$$\begin{cases} h^n_{i,l} := \mathcal{B}_{i,l}\left(u^n_i\right), & \text{on } \Gamma_{i,l} \quad \text{for } 2 \le i \le N \\ h^n_{i,r} := \mathcal{B}_{i,r}\left(u^n_i\right), & \text{on } \Gamma_{i,r} \quad \text{for } 1 \le i \le N-1\,. \end{cases}$$

The method can be reformulated as a Jacobi algorithm on h^n:

$$h^{n+1} := \mathscr{T}(h^n) + G$$

where the global vector h^n contains the local unknowns $(h^n_{i,l})_{2\le i\le N}$ and $(h^n_{i,r})_{1\le i\le N-1}$.

Optimized Schwarz method: Substructuration

- The iteration operator $\mathscr{T}$ can be written in the form of an operator valued matrix. Therefore, we look for a vector h such that,

$$(Id - \mathscr{T})(h) = G.$$

  where G refers to the contribution of the right-hand side f.

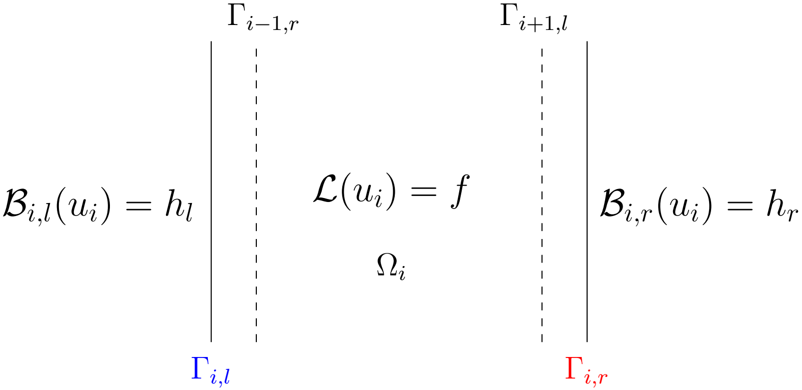
- Closer look at $\mathscr{T}$: Let's define for each subdomain the operator S_i which takes: two surfacic functions h_l and h_r and a volume function f:

$$S_i(h_l, h_r, f) := v$$

  where $v : \Omega_i \mapsto \mathbb{C}$ satisfies:

$$\begin{cases} (-k^2 - \Delta)\, v = f \ \text{ in } \Omega_i \\ \quad \mathcal{B}_{i,l}(v) = h_l \text{ on } \Gamma_{i,l} \quad (2 \le i \le N) \\ \quad \mathcal{B}_{i,r}(v) = h_r \text{ on } \Gamma_{i,r} \quad (1 \le i \le N-1) \\ \qquad + \ \text{appropriate boundary conditions on } \partial\Omega \cap \partial\Omega_i \,, \end{cases}$$

Local problem on the subdomain Ω_i



Substructured equation

In a nutshell:

$$(Id - \mathscr{T})(h) = G\,.$$

with

$$\begin{aligned} G_{i+1,l} &:= \mathcal{B}_{i+1,l}(S_i(0,0,\ f)),\ \ 1 \le i \le N-1 \\ G_{i-1,r} &:= \mathcal{B}_{i-1,r}(S_i(0,0,\ f)),\ \ 2 \le i \le N\,. \end{aligned}$$

and the substructured operator $\mathscr{T}$:

$$\begin{aligned} \mathscr{T}(h)_{i,l} &:= \mathcal{B}_{i,l}(S_{i-1}(h_{i-1,l},\ h_{i-1,r},0)),\ \ 2 \le i \le N \\ \mathscr{T}(h)_{i,r} &:= \mathcal{B}_{i,r}(S_{i+1}(h_{i+1,l},\ h_{i+1,r},0)),\ \ 1 \le i \le N-1\,. \end{aligned}$$

Nilpotency and Exact ABC

- By linearity of $(S_i)_{1\leq i\leq N}$:

$$\begin{aligned}\mathscr{T}(h)_{i+1,l} &= \mathcal{B}_{i+1,l}(S_i(h_{i,l},\ 0,\ 0)) + \mathcal{B}_{i+1,l}(S_i(0,\ h_{i,r},\ 0))\\ \mathscr{T}(h)_{i-1,r} &= \mathcal{B}_{i-1,r}(S_i(0,\ h_{i,r},\ 0)) + \mathcal{B}_{i-1,r}(S_i(h_{i,l},\ 0,\ 0))\end{aligned}$$

- For exact absorbing boundary conditions (EABC) $\mathcal{B}^{EABC}$:

$$\begin{aligned}\mathscr{T}^{EABC}(h)_{i+1,l} &= \mathcal{B}^{EABC}_{i+1,l}(S^{EABC}_i(h_{i,l},0,0))\\ \mathscr{T}^{EABC}(h)_{i-1,r} &= \mathcal{B}^{EABC}_{i-1,r}(S^{EABC}_i(0,\ h_{i,r},0))\end{aligned}$$

- $\mathscr{T}^{EABC}$ is nilpotent (order $N-1$): $(I-\mathscr{T}^{EABC})^{-1} = \sum_{i=0}^{N-2}(\mathscr{T}^{EABC})^i$.
  $\Rightarrow$ Convergence in $N-1$ steps of the Jacobi method.
  See:
  - Hagstrom et al. (1988) for $N=2$
  - Nataf, F., Rogier, F., & de Sturler, E. (1994) for arbitrary N.

$\Rightarrow$ Search for sweeping algorithms converging in one iteration for Exact ABC

Nilpotency and Exact ABC

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$\Rightarrow$ Search for sweeping algorithms converging in one iteration for Exact ABC

Outline

double sweep algorithm

Left to Right sweep

$$
\begin{aligned}
h_{i+1,l}^{n+1/2} &:= \mathcal{B}_{i+1,l}(S_i(h_{i,l}^{n+1/2},\ h_{i,r}^{n}, f))\,,\\
h_{i-1,r}^{n+1/2} &:= \mathcal{B}_{i-1,r}(S_i(h_{i,l}^{n+1/2},\ h_{i,r}^{n}, f))\,,
\end{aligned}
$$

followed by a Right to Left sweep

$$
\begin{aligned}
h_{i+1,l}^{n+1} &:= \mathcal{B}_{i+1,l}(S_i(h_{i,l}^{n+1/2},\ h_{i,r}^{n+1}, f))\,,\\
h_{i-1,r}^{n+1} &:= \mathcal{B}_{i-1,r}(S_i(h_{i,l}^{n+1/2},\ h_{i,r}^{n+1}, f))\,.
\end{aligned}
$$

Convergence in one iteration for Exact ABC as interface conditions

Double Sweep (*DS*) (N.-Nier) in a Volumic formulation

Left to Right sweep:

$$\begin{cases} \left(-k^2 - \Delta\right) u_i^{n+1/2} = f \ \text{ in } \Omega_i,\ 1 \le i \le N \\ \mathcal{B}_{i,l}\left(u_i^{n+1/2}\right) = \mathcal{B}_{i,l}\left(u_{i-1}^{n+1/2}\right) \text{ on } \Gamma_{i,l},\ 2 \le i \le N \\ \mathcal{B}_{i,r}\left(u_i^{n+1/2}\right) = \mathcal{B}_{i,r}\left(u_{i+1}^{n}\right) \text{ on } \Gamma_{i,r},\ 1 \le i \le N-1\,. \end{cases}$$

Right to Left sweep:

$$\begin{cases} \left(-k^2 - \Delta\right) u_i^{n+1} = f \ \text{ in } \Omega_i,\ 1 \le i \le N \\ \mathcal{B}_{i,l}\left(u_i^{n+1}\right) = \mathcal{B}_{i,l}\left(u_{i-1}^{n+1/2}\right) \text{ on } \Gamma_{i,l},\ 2 \le i \le N \\ \mathcal{B}_{i,r}\left(u_i^{n+1}\right) = \mathcal{B}_{i,r}\left(u_{i+1}^{n+1}\right) \text{ on } \Gamma_{i,r},\ 1 \le i \le N-1\,. \end{cases}$$

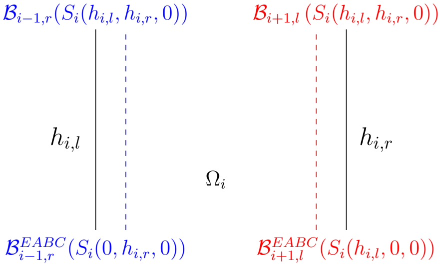
Convergence in one iteration for Exact ABC as interface conditions

Outline

In practice, the absorbing boundary conditions are non exact:

$$\begin{aligned} \mathcal{B}_{i+1,l}(S_i(0,\ h_{i,r},0)) &\neq 0 \\ \mathcal{B}_{i-1,r}(S_i(h_{i,l},0,0)) &\neq 0 \end{aligned}$$

$\Rightarrow$ Nilpotency is lost



Define a new operator $\mathscr{T}_{OSDS}$

Operator $\mathscr{T}_{OSDS}$

$$\begin{aligned} \mathscr{T}_{OSDS}(h)_{i+1,l} &:= \mathcal{B}_{i+1,l}(S_i(h_{i,l},0,0)), \ \ 1 \le i \le N-1 \\ \mathscr{T}_{OSDS}(h)_{i-1,r} &:= \mathcal{B}_{i-1,r}(S_i(0,\ h_{i,r},0)), \ \ 2 \le i \le N \end{aligned}$$

which by definition is a nilpotent operator of order $N-1$.

New algorithm:

$$(Id - \mathscr{T}_{OSDS})(h^{n+1}) = (\mathscr{T} - \mathscr{T}_{OSDS})\,(h^n) + G$$

Remark: By definition, we have 2 simultaneous sweeps occuring so the name Double sweep!

Convergence in one iteration for Exact ABC as interface conditions

Double Sweep as a preconditioner

Actually, the operator is used as a preconditioner for

$$(Id - \mathscr{T})(h) = G\,.$$

The left-preconditioned system reads:

$$(Id - \mathscr{T}_{OSDS})^{-1}(Id - \mathscr{T})(h) = (Id - \mathscr{T}_{OSDS})^{-1}G$$

Outline

Unified view with classical linear algebra

We write the substructured system with two unknowns ordering: subdomain wise numbering or left-right numbering.

Subdomain wise ordering: $(I - \mathcal{T}_{SW})(H_{SW}) = G_{SW}$

$$\begin{bmatrix} I & -\mathcal{T}_{(1,r)(2,l)} & -\mathcal{T}_{(1,r)(2,r)} & 0 & & & \\ -\mathcal{T}_{(2,l)(1,r)} & I & 0 & 0 & \ddots & & \\ 0 & 0 & I & -\mathcal{T}_{(2,r)(3,l)} & -\mathcal{T}_{(2,r)(3,r)} & 0 & \\ & \mathcal{T}_{(3,l)(2,l)} & \mathcal{T}_{(3,l)(2,r)} & I & 0 & 0 & \\ & & 0 & 0 & \ddots & \ddots & \\ & & & -\mathcal{T}_{(4,l)(3,l)} & \ddots & \ddots & \ddots \\ & & & & \ddots & \ddots & \ddots \\ & & & & & \ddots & I \end{bmatrix} \begin{bmatrix} h_{1,r} \\ h_{2,l} \\ h_{2,r} \\ h_{3,l} \\ h_{3,r} \\ h_{4,l} \\ \vdots \\ h_{N,l} \end{bmatrix} = G\,.$$

Jacobi $\Leftrightarrow$ Lions-Després algorithm
Gauss-Seidel $\Leftrightarrow$ Nataf-Nier FDA (flow directed algorithm)
Symmetric Gauss-Seidel $\Leftrightarrow$ Nataf-Nier double sweep algorithm

Left-Right numbering: $(I - \mathscr{T}_{LR})(H_{LR}) = G$

$$
\left[\begin{array}{cccc|cccc}
I & & & 0 & 0 & & & -\mathscr{T}_{(2,l)(1,r)} \\
-\mathscr{T}_{(3,l)(2,l)} & \ddots & & & & & \iddots & \\
 & \ddots & \ddots & & & \iddots & & \\
0 & & -\mathscr{T}_{(N,l)(N-1,l)} & I & -\mathscr{T}_{(N,l)(N-1,r)} & & & 0 \\
\hline
0 & & & -\mathscr{T}_{(N-1,r)(N,l)} & I & & & 0 \\
 & & \iddots & & -\mathscr{T}_{(N-2,r)(N-1,r)} & \ddots & & \\
 & \iddots & & & & \ddots & \ddots & \\
-\mathscr{T}_{(1,r)(2,l)} & & & 0 & 0 & & -\mathscr{T}_{(1,r)(2,r)} & I
\end{array}\right]
\begin{bmatrix} h_{2,l} \\ \vdots \\ \vdots \\ h_{N,l} \\ h_{N-1,r} \\ \vdots \\ \vdots \\ h_{1,r} \end{bmatrix}
= G.
$$

Block Jacobi $\Leftrightarrow$ Stolk-Vion-Geuzaine Double sweep algorithm

Block Gauss-Seidel $\Leftrightarrow$ New-1

Symmetric Block Gauss-Seidel $\Leftrightarrow$ New-2

It's all about numbering!

Left-Right numbering: $(I - \mathscr{T}_{LR})(H_{LR}) = G$

$$
\left[\begin{array}{cccc:cccc}
I & & & 0 & 0 & & & -\mathscr{T}_{(2,l)(1,r)} \\
-\mathscr{T}_{(3,l)(2,l)} & \ddots & & & & & \iddots & \\
 & \ddots & \ddots & & & \iddots & & \\
0 & & -\mathscr{T}_{(N,l)(N-1,l)} & I & -\mathscr{T}_{(N,l)(N-1,r)} & & & 0 \\
\hdashline
0 & & & -\mathscr{T}_{(N-1,r)(N,l)} & I & & & 0 \\
 & & \iddots & & -\mathscr{T}_{(N-2,r)(N-1,r)} & \ddots & & \\
 & \iddots & & & & \ddots & \ddots & \\
-\mathscr{T}_{(1,r)(2,l)} & & & 0 & 0 & & -\mathscr{T}_{(1,r)(2,r)} & I
\end{array}\right]
\begin{bmatrix} h_{2,l} \\ \vdots \\ \vdots \\ h_{N,l} \\ h_{N-1,r} \\ \vdots \\ \vdots \\ h_{1,r} \end{bmatrix}
= G\,.
$$

Block Jacobi $\Leftrightarrow$ Stolk-Vion-Geuzaine Double sweep algorithm
Block Gauss-Seidel $\Leftrightarrow$ New-1
Symmetric Block Gauss-Seidel $\Leftrightarrow$ New-2
It's all about numbering!

Outline

In order to highlight nilpotency, we introduce:
the restriction operator to the left (resp. right) boundary d.o.f R_l (resp. R_r)
four $(2N-2) \times (2N-2)$ operator valued matrices:

$$\mathscr{M}_l := R_l^T \mathscr{T} R_l \text{ (top left)}, \ \mathscr{A}_l := R_l^T \mathscr{T} R_r \text{ (top right)}$$

$$\mathscr{A}_r := R_r^T \mathscr{T} R_l \text{ (bottom left)}, \ \mathscr{M}_r := R_r^T \mathscr{T} R_r \text{ (bottom right)}$$

so that we have $I - \mathscr{T}_{LR} = I - \mathscr{M}_l - \mathscr{A}_l - \mathscr{M}_r - \mathscr{A}_r$.
It is easy to check that:

$$\boxed{\begin{array}{l} \mathscr{M}_r^{N-1} = \mathscr{M}_l^{N-1} = 0; \quad \mathscr{M}_l \mathscr{M}_r = \mathscr{M}_r \mathscr{M}_l = 0; \quad \mathscr{A}_l^2 = \mathscr{A}_r^2 = 0 \\ \mathscr{A}_l \mathscr{M}_l = \mathscr{A}_r \mathscr{M}_r = 0; \quad \mathscr{M}_l \mathscr{A}_r = \mathscr{M}_r \mathscr{A}_l = 0\,. \end{array}}$$

Convergence rates

Key operators are:

$$C_r := (I - \mathscr{M}_r)^{-1}\,\mathscr{A}_r = \sum_{i=0}^{N-2} \mathscr{M}_r^i\,\mathscr{A}_r$$

$$C_l := (I - \mathscr{M}_l)^{-1}\,\mathscr{A}_l = \sum_{i=0}^{N-2} \mathscr{M}_l^i\,\mathscr{A}_l\,.$$

Note that using the previous cancellation relations, we have

$$C_r^2 = C_l^2 = 0\,.$$

Also:

$C_r = 0 \Leftrightarrow$ Right IC are Exact ABC/PML

$C_l = 0 \Leftrightarrow$ Left IC are Exact ABC/PML

Let ρ be the spectral radius of $C_r\,C_l$.

Convergence results

Algo.	**Linear Algebra**	**Definition**
Lions-Després	Jacobi-SW	I
FDA (1997)	Gauss-Seidel-SW	$(I - \mathscr{M}_l - \mathscr{A}_l)^{-1}$
DoubleSweep (1997)	Symm. Gauss-Seidel-SW	$(I - \mathscr{M}_r - \mathscr{A}_r)^{-1}\,(I - \mathscr{M}_l - \mathscr{A}_l)^{-1}$
DoubleSweep (2013)	Block Jacobi-LR	$(I - \mathscr{M}_r - \mathscr{M}_l)^{-1}$
New-1	Block GS-LR	$(I - \mathscr{M}_r - \mathscr{M}_l - \mathscr{A}_r)^{-1}$
New-2	Symm. Block GS-LR	$(I - \mathscr{M}_r - \mathscr{M}_l - \mathscr{A}_l)^{-1} \times (I - \mathscr{M}_r - \mathscr{M}_l) \times (I - \mathscr{M}_r - \mathscr{M}_l - \mathscr{A}_r)^{-1}$

Table: Algorithms and their substructured formula

Algo.	**Ampl. Error**	**Spectral Radius**
Lions-Després	$\mathscr{M}_r + \mathscr{M}_l + \mathscr{A}_r + \mathscr{A}_l$	$\rho^{1/N}$
FDA (1997)	$(I + C_l)(\mathscr{M}_r + \mathscr{A}_r)$	$\rho^{2/N}$
DoubleSweep (1997)	$(I + C_r)\,C_l\,(\mathscr{M}_r + \mathscr{A}_r)$	ρ
DoubleSweep (2013)	$C_r + C_l$	$2\rho^{1/2}$
New-1	$(I + C_r)\,C_l$	ρ
New-2	$(I + C_l)\,C_r\,C_l$	ρ

Table: Algorithms and their convergence properties

Outline

Framework

In this section, we present numerical results when solving the substructured equation

$$(Id - \mathscr{T})(h) = G$$

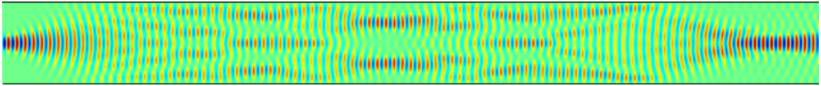
with the GMRES algorithm right preconditioned.

- Volumic version is implemented which allows for approximate subdomain solves (not tested here)
- Recall that these sweeping algorithms are of interest
  - if you have only one or two cores (rare event these days)
  - or if you have multiple right hand sides since it enables pipelining (see Stolk 2013 and Vion-Geuzaine 2014)

Homogoneous waveguide

$$\begin{cases} \left(-k^2 - \Delta\right) u = f \ \text{ in } \Omega \\ \quad (\partial_{\vec{n}} + Ik)\, u = 0 \ \text{ on } \{x = N\} \times [0, 1] \\ \quad (\partial_{\vec{n}} + Ik)\, u = u_g \ \text{ on } \{x = 0\} \times [0, 1] \\ \qquad\qquad u = 0 \ \text{ on } [0, N] \times \{y = 0, y = 1\} \end{cases}$$

where $u_g = e^{-120(y-0.5)^2)} sin(\pi y)$, (Dirichlet conditions in black).



Homogeneous waveguide ($k = 20\pi$)

Homogeneous waveguide $k = 20$

N	Jacobi	DS-2013	DS-1997	New-1	New-2
5	29 (18)	9 (5)	5 (3)	5 (3)	5 (3)
10	62 (39)	12 (7)	7 (4)	6 (4)	6 (4)
20	135 (81)	18 (10)	10 (6)	9 (6)	9 (6)
40	283 (163)	26 (12)	14 (8)	14 (7)	14 (8)
80	744 (329)	42 (20)	23 (12)	22 (12)	23 (12)

Table: Volumic preconditioner with ABC0 interface conditions, $k = 20$, $\delta = 4h$, TOL=$10^{-6}(10^{-3})$, nppwl = 24 , P1

N	Jacobi	DS-2013	DS-1997	New-1	New-2
5	33 (21)	9 (5)	5 (2)	4 (2)	5 (3)
10	71 (40)	11 (6)	6 (3)	5 (3)	6 (3)
20	150 (80)	13 (7)	7 (4)	6 (3)	7 (4)
40	293 (143)	17 (9)	9 (5)	8 (5)	9 (5)
80	690 (276)	22 (14)	12 (7)	12 (7)	12 (7)

Table: Volumic preconditioner with PML interface conditions, homogeneous waveguide, $k = 20$, $\delta = 4h$, TOL=$10^{-6}(10^{-3})$, nppwl = 24 , P1

Homogoneous waveguide $k = 20\pi$

N	Jacobi	DS-2013	DS-1997	New-1	New-2
5	35	15	10	9	9
10	72	22	14	13	14
20	150	34	22	23	23
40	335	58	38	45	42

Table: Volumic preconditioner with ABC0 interface conditions, $k = 20\pi$, $\delta = 4h$, TOL=10^{-6}, nppwl = 24 , P1

N	Jacobi	DS-2013	DS-1997	New-1	New-2
5	38	10	6	5	6
10	78	11	7	5	6
20	162	17	9	7	9
40	304	23	12	10	12

Table: Volumic preconditioner with PML interface conditions, homogeneous waveguide, $k = 20\pi$, $\delta = 4h$, TOL=10^{-6}, nppwl = 24 , P1

Semi open cavity vs. Waveguide

N	DS-2013	DS-1997	New-1	New-2
5	12 (9)	6 (5)	6 (4)	7 (5)
10	14 (11)	7 (6)	7 (5)	7 (6)
20	16 (13)	9 (7)	8 (6)	8 (7)
40	19 (17)	10 (9)	10 (8)	10 (9)

Table: Volumic preconditioner with PML interface conditions, semi open cavity (waveguide), $k = 3.18$

N	DS-2013	DS-1997	New-1	New-2
5	18 (10)	9 (6)	9 (5)	10 (6)
10	19 (11)	10 (7)	9 (5)	11 (6)
20	26 (17)	14 (9)	12 (7)	14 (9)
40	34 (23)	18 (12)	16 (10)	19 (12)

Table: Volumic preconditioner with PML interface conditions, semi open cavity (waveguide), $k = 10$.

Overlap influence?

N	Jacobi	DS-2013	DS-1997	New-1	New-2
2	136	19	11	10	10
4	135	18	10	9	9
8	137	17	9	8	9
16	149	21	11	11	11

Table: Volumic preconditioner, homogeneous waveguide with first order ABC,, $k = 20$, δ varies, TOL=10^{-6}, nppwl = 24 , P1

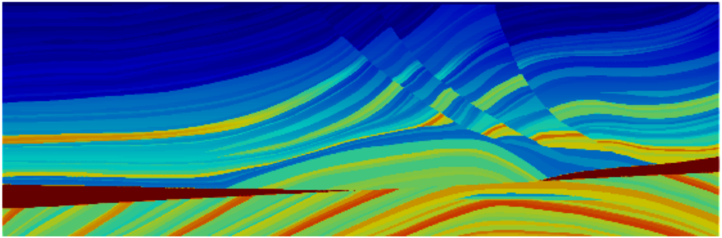


Figure: Velocity model of the Marmousi test case

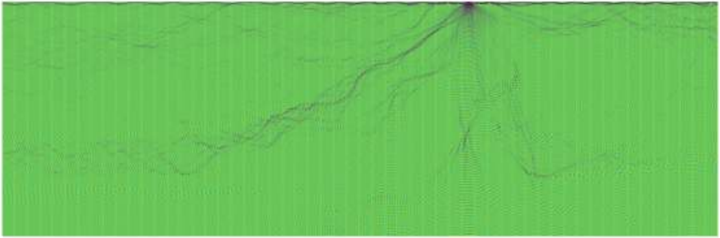


Figure: Real part of the solution for $f = 100$ Hz for the Marmousi test

Marmousi test case – FreeFem DSL

	25 Hz					50 Hz				
N	Jacobi	2013	1997	New-1	New-2	Jacobi	2013	1997	New-1	New-2
3	22	11	10	7	7					
7	46	17	13	10	11	XXX	19	16	13	14
14	94	25	18	19	20	98	28	22	21	21
28	185	41	27	30	30	195	49	38	60	47
56	382	98	65	101	70	426	123	90	X	143
112						1505	> 400	> 400	> 400	> 400

Table: Volumic preconditioner with first-order ABC, Marmousi test case, $\delta = 4h$, TOL=10^{-6}, nppwl = 8 , P2

	25 Hz					50 Hz				
N	Jacobi	2013	1997	New-1	New-2	Jacobi	2013	1997	New-1	New-2
3	14	7	6	4	5					
7	33	10	8	5	6	XXX	10	9	6	6
14	64	11	8	6	7	69	12	10	7	8
28	126	13	9	7	8	133	14	11	7	8
56	247	18	12	11	12	260	18	13	10	11
112						531	32	22	20	21

Table: Volumic preconditioner with PML interface conditions, Marmousi test case, $\delta = 4h$, TOL=10^{-6}, nppwl = 8 , P2

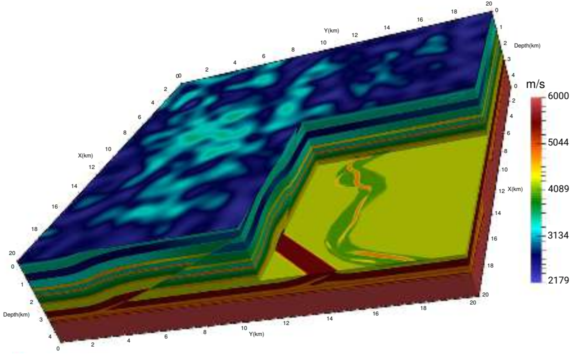


Figure: Velocity model of the 3D Overthrust benchmark

Figure: Real part of the solution for $f = 4$ Hz for the 3D Overthrust benchmark

N	1 Hz Jacobi	1 Hz DS-1997	1 Hz DS-2013	2 Hz Jacobi	2 Hz DS-1997	2 Hz DS-2013	4 Hz Jacobi	4 Hz DS-1997	4 Hz DS-2013
3	14	?	7						
7	32	?	9	34	?	10			
14	94	?	18	68	?	15			
28				163	?	30	163	?	32
56							366	?	87

Table: Volumic preconditioner with first-order ABC, Overthrust test case, $\delta = 4h$, TOL=10^{-6}, nppwl = 10 , P1

N	1 Hz Jacobi	1 Hz DS-1997	1 Hz DS-2013	2 Hz Jacobi	2 Hz DS-1997	2 Hz DS-2013	4 Hz Jacobi	4 Hz DS-1997	4 Hz DS-2013
3	11 (7)	?	5 (3)						
7	28 (15)	?	7 (4)	30 (16)	?	8 (4)			
14	59 (31)	?	9 (5)	64 (32)	?	11 (6)			
28				123 (61)	?	16 (8)	121 (63)	?	18 (9)
56							224 (122)	?	28 (15)

Table: Volumic preconditioner with PML interface conditions, Overthrust test case, $\delta = 4h$, TOL=$10^{-6}(10^{-3})$, nppwl = 10 , P1

? means waiting for the account on the new HPC facility.

Outline

Conclusion

Summary

- Clarification of ”Double sweep algorithms”
- Convergence rates: ρ for DS-1997 vs. $2\rho^{1/2}$ DS-2013
- DS-2013 allows for a parallel implementation that alleviates the differences in convergence rates
- Natural volumic implementation for DS-1997

Not considered here

- Inexact subdomain solves
- Multidirectional sweeping preconditioners, L-Sweeps, Modave et al., , Demanet et al., Gander et al., ...

THANK YOU FOR YOUR ATTENTION !