Involutive scroll structures and heavenly type hierarchies

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Given a PDE system Σ, for every solution of Σ the corresponding characteristic variety defines some 'geometry' such as Einstein-Weyl geometry, self-dual geometry, involutive scroll geometry, etc.

Master-equations are PDE systems Σ whose solutions parametrise 'generic' geometries of the above type.

- Self-dual Ricci-flat geometry in 4D
  - Self-dual Ricci-flat metrics via the Plebansky heavenly equation
- Einstein-Weyl geometry in 3D
  - Einstein-Weyl structures via the Manakov-Santini system
- Self-dual geometry in 4D
  - Self-dual structures via conformal self-duality equations
- Involutive scroll geometry
  - Involutive scroll structures via the hierarchy of conformal self-duality equations

E.V. Ferapontov, B. Kruglikov, Involutive scroll structures on solutions of dispersionless integrable hierarchies, arXiv:2503.10897.

Self-dual Ricci-flat geometry in 4D

The Plebanski first heavenly equation (Plebanski, 1975) is

$$u_{13}u_{24} - u_{14}u_{23} = 1.$$

Characteristic variety (collection of characteristic covectors $p_i dx^i$) is a quadric:

$$g^{ij}p_ip_j = u_{13}p_2p_4 + u_{24}p_1p_3 - u_{14}p_2p_3 - u_{23}p_1p_4 = 0.$$

Solutions give rise to metrics

$$g = g_{ij}dx^i dx^j = u_{13}dx_1dx_3 + u_{14}dx_1dx_4 + u_{23}dx_2dx_3 + u_{24}dx_2dx_4$$

that are self-dual and Ricci-flat. All self-dual Ricci-flat metrics arise in this way!

Thus, Plebanski's equation is a master-equation governing self-dual Ricci-flat geometry. Lax pair $[X, Y] = 0$:

$$X = u_{13}\partial_4 - u_{14}\partial_3 + \lambda\partial_1, \quad Y = -u_{23}\partial_4 + u_{24}\partial_3 - \lambda\partial_2.$$

Integral surfaces of the distribution $\langle X, Y\rangle$ provide Penrose's three-parameter family of totally null surfaces, that is, $g(X, X) = g(X, Y) = g(Y, Y) = 0$, thus confirming self-duality of g.

Alternative master-equation: $u_{13} + u_{24} + u_{11}u_{22} - u_{12}^2 = 0$ (second heavenly).

Einstein-Weyl geometry in 3D

Einstein-Weyl geometry is a triple $(\mathbb{D},\ g,\ \omega)$ where $\mathbb{D}$ is a symmetric connection, g is a conformal structure and ω is a covector such that

$$\mathbb{D}_k g_{ij} = \omega_k g_{ij}, \quad R_{(ij)} = \Lambda g_{ij}.$$

Here $R_{(ij)}$ is the symmetrised Ricci tensor of $\mathbb{D}$ and Λ is some function (the first set of equations defines $\mathbb{D}$ uniquely in terms of g and ω).

Conformal invariance: $\mathbb{D} \to \mathbb{D},\ g \to \lambda g,\ \omega \to \omega + d\ln\lambda$.

Theorem (Cartan, 1941): In 3D, the triple $(\mathbb{D},\ g,\ \omega)$ satisfies the Einstein-Weyl equations if and only if there exists a **two-parameter family of surfaces** that are **totally geodesic** with respect to $\mathbb{D}$ and **null** with respect to g.

Generic Einstein-Weyl structures depend on **four arbitrary functions of two variables**.

3D Einstein-Weyl equations are integrable (Hitchin, 1980).

Einstein-Weyl structures via the Manakov-Santini system

The Manakov-Santini system (2006) is

$$P(u) + u_x^2 = 0, \quad P(v) = 0, \qquad P = \partial_x \partial_t - \partial_y^2 + u\partial_x^2 + v_x \partial_x \partial_y - v_y \partial_x^2.$$

Characteristic variety is a double quadric:

$$p_x p_t - p_y^2 + u p_x^2 + v_x p_x p_y - v_y p_x^2 = 0.$$

Solutions give rise to Einstein-Weyl structures (Dunajski, 2008):

$$g = (dy - v_x dt)^2 - 4(dx - (u - v_y)dt)dt,$$

$$\omega = -v_{xx} dy + (4u_x - 2v_{xy} + v_x v_{xx})dt.$$

All 3D Einstein-Weyl structures arise in this way! Lax pair $[X, Y] = 0$:

$$X = \partial_y + (\lambda - v_x)\partial_x + u_x \partial_\lambda, \qquad Y = \partial_t - (\lambda^2 - \lambda v_x + v_y - u)\partial_x - (\lambda u_x - u_y)\partial_\lambda.$$

Projecting integral surfaces of the distribution spanned by X, Y from (x, y, t, λ)-space to (x, y, t)-space one obtains Cartan's **two-parameter family of totally geodesic null surfaces**.

M. Dunajski, E.V. Ferapontov and B. Kruglikov, On the Einstein-Weyl and conformal self-duality equations, J. Math. Phys. **56**, *083501 (2015).*

Self-dual geometry in 4D

Let g be a conformal structure in 4D and let W be its Weyl tensor. Consider self-dual and anti-self-dual parts, $W_{\pm} = \frac{1}{2}(W \pm *W)$, where $*$ is the Hodge star operator defined as

$$*W^i_{jkl} = \frac{1}{2}\sqrt{\det g}\; g^{ia} g^{bc} \epsilon_{ajbd} W^d_{ckl}.$$

A conformal structure is said to be self-dual/anti-self-dual if either W_- or W_+ vanishes.

Theorem (Penrose, 1976): self-duality of g is equivalent to the existence of a **three-parameter family of totally null surfaces**.

It was shown by Grossman (2000) that self-dual structures locally depend on **six arbitrary functions of three variables**.

Self-dual structures via conformal self-duality equations

Conformal self-duality equations have the form

$$R(u) = u_1 w_2 - u_2 w_1, \quad R(v) = v_1 w_2 - v_2 w_1, \quad R(w) = 0,$$

where R is the second-order operator

$$R = u_1 \partial_2 \partial_3 + v_1 \partial_2 \partial_4 - u_2 \partial_1 \partial_3 - v_2 \partial_1 \partial_4.$$

Characteristic variety is a triple quadric:

$$u_1 p_2 p_3 + v_1 p_2 p_4 - u_2 p_1 p_3 - v_2 p_1 p_4 = 0.$$

Solutions give rise to self-dual conformal structures:

$$g = (u_1 dx^1 + u_2 dx^2) dx^4 - (v_1 dx^1 + v_2 dx^2) dx^3.$$

All self-dual structures in 4D arise in this way! Lax pair $[X, Y] \in span\langle X, Y \rangle$:

$$X = \lambda \partial_1 - u_1 \partial_3 - v_1 \partial_4 + \lambda w_1 \partial_\lambda, \quad Y = \lambda \partial_2 - u_2 \partial_3 - v_2 \partial_4 + \lambda w_2 \partial_\lambda.$$

Projecting integral surfaces of the distribution spanned by X, Y from $(x^1, \ldots, x^4, \lambda)$-space to $(x^1, \ldots, x^4)$-space one obtains Penrose's **three-parameter family of totally null surfaces**.

Involutive scroll geometry

In projective space $\mathbb{P}^n$ choose two complementary $\mathbb{P}^k$ and $\mathbb{P}^l$, $n = k + l + 1$. Choose two rational normal curves of degrees k and l in these subspaces, let ϕ be an isomorphism between them. The rational normal scroll $S_{k,l}$ is a smooth ruled surface of degree $n - 1$ consisting of all lines joining the corresponding points x and $\phi(x)$.

If $k < l$ then the rational normal curve of degree k is uniquely determined by the scroll and is called its directrix. Del Pezzo's theorem implies that scrolls $S_{k,l}$ are the only irreducible smooth ruled surfaces of degree $\deg(S_{k,l}) = k + l = n - 1$ spanning $\mathbb{P}^n$.

Given an $(n + 1)$-dimensional manifold M, a (rational normal) scroll structure is a distribution of rational normal scrolls $S_{k,l}$ in the projectivised cotangent bundle $\mathbb{P}^n = \mathbb{P}T^*M$. Explicitly, a scroll structure can be parametrised as $\alpha\,\omega(\lambda) + \beta\,\phi(\lambda)$ where α and β are some functions and

$$\omega(\lambda) = \omega_0 + \lambda\omega_1 + \cdots + \lambda^k\omega_k, \qquad \phi(\lambda) = \phi_0 + \lambda\phi_1 + \cdots + \lambda^l\phi_l.$$

For every $x \in M$, the equations $\omega(\lambda) = \phi(\lambda) = 0$ define a one-parameter family of α-subspaces of codimension two in T_xM parametrised by λ. A codimension two submanifold of M is said to be an *α-manifold* if all its tangent spaces are α-subspaces.

Definition. A scroll structure is said to be *involutive* if every α-subspace is tangential to some α-manifold.

Involutive $S_{1,2}$ scroll structures via heavenly hierarchy

The first three equations in the heavenly hierarchy are as follows:

$$u_{15} - u_{13}u_{44} + u_{14}u_{34} = 0, \quad u_{14}u_{23} - u_{13}u_{24} = 1, \quad u_{25} - u_{23}u_{44} + u_{24}u_{34} = 0.$$

The characteristic variety is the intersection of quadrics,

$$p_1p_5 - u_{13}p_4^2 - u_{44}p_1p_3 + u_{14}p_3p_4 + u_{34}p_1p_4 = 0,$$
$$u_{14}p_2p_3 + u_{23}p_1p_4 - u_{13}p_2p_4 - u_{24}p_1p_3 = 0,$$
$$p_2p_5 - u_{23}p_4^2 - u_{44}p_2p_3 + u_{24}p_3p_4 + u_{34}p_2p_4 = 0,$$

which specify a scroll $S_{1,2}$ parametrised as $p_i dx^i = \alpha\,\omega(\lambda) + \beta\,\phi(\lambda)$, where

$$\omega(\lambda) = -u_{14}dx^1 - u_{24}dx^2 + \lambda(dx^3 + u_{44}dx^5),$$
$$\phi(\lambda) = u_{13}dx^1 + u_{23}dx^2 + \lambda(dx^4 - u_{34}dx^5) + \lambda^2 dx^5.$$

As $\omega(\lambda)$ is linear in λ, it plays the role of directrix of the scroll. The equations $\omega(\lambda) = \phi(\lambda) = 0$ define a 3-dimensional integrable distribution

$$\langle \lambda\partial_1 + u_{14}\partial_3 - u_{13}\partial_4, \;\; \lambda\partial_2 + u_{24}\partial_3 - u_{23}\partial_4, \;\; \partial_5 - u_{44}\partial_3 + (u_{34} - \lambda)\partial_4 \rangle,$$

whose integral manifolds are α-manifolds. Thus, this $S_{1,2}$ scroll structure is involutive.

Involutive $S_{1,2}$ scroll structures in 5D

The first few flows of the hierarchy of conformal self-duality equations (Bogdanov, 2016) come from the requirement of involutivity of the distribution $\mathcal{D} = span\{X, Y, Z\}$ where

$$X = \lambda\partial_1 - u_1\partial_3 - v_1\partial_4 + \lambda w_1\partial_\lambda,$$

$$Y = \lambda\partial_2 - u_2\partial_3 - v_2\partial_4 + \lambda w_2\partial_\lambda,$$

$$Z = \partial_5 + u_4\partial_3 + (v_4 + w - \lambda)\partial_4 - \lambda w_4\partial_\lambda.$$

(9 second-order PDEs for u, v, w). Characteristic variety, which is a triple scroll $S_{1,2}$, can be parametrised as $p_i dx^i = \alpha\,\omega(\lambda) + \beta\,\phi(\lambda)$ where α, β are arbitrary functions and

$$\omega(\lambda) = u_1 dx^1 + u_2 dx^2 + \lambda(dx^3 - u_4 dx^5),$$

$$\phi(\lambda) = v_1 dx^1 + v_2 dx^2 + \lambda(dx^4 - (v_4 + w)dx^5) + \lambda^2 dx^5.$$

All involutive $S_{1,2}$ scroll structures in 5D arise in this way! Projecting integral manifolds of the distribution $\mathcal{D}$ from $(x^1, \ldots, x^5, \lambda)$-space to $(x^1, \ldots, x^5)$-space one obtains **three-parameter family of α-manifolds**.

Generic involutive $S_{1,2}$ scroll structures locally depend on 9 arbitrary functions of 3 variables.

Involutive $S_{2,2}$ scroll structures in 6D

The next few flows of the hierarchy of conformal self-duality equations come from the requirement of involutivity of the distribution $\mathcal{D} = span\{X, Y, Z, W\}$ where

$$\begin{aligned} X &= \lambda\partial_1 - u_1\partial_3 - v_1\partial_4 + \lambda w_1\partial_\lambda, \\ Y &= \lambda\partial_2 - u_2\partial_3 - v_2\partial_4 + \lambda w_2\partial_\lambda, \\ Z &= \partial_5 + u_4\partial_3 + (v_4 + w - \lambda)\partial_4 - \lambda w_4\partial_\lambda, \\ W &= \partial_6 + (u_3 + w - \lambda)\partial_3 + v_3\partial_4 - \lambda w_3\partial_\lambda. \end{aligned}$$

(18 second-order PDEs for u, v, w). Characteristic variety, which is a triple scroll $S_{2,2}$, can be parametrised as $p_i dx^i = \alpha\,\omega(\lambda) + \beta\,\phi(\lambda)$ where α, β are arbitrary functions and

$$\begin{aligned} \omega(\lambda) &= u_1 dx^1 + u_2 dx^2 + \lambda(dx^3 - u_4 dx^5 - (u_3 + w)dx^6) + \lambda^2 dx^6, \\ \phi(\lambda) &= v_1 dx^1 + v_2 dx^2 + \lambda(dx^4 - (v_4 + w)dx^5 - v_3 dx^6) + \lambda^2 dx^5. \end{aligned}$$

All involutive $S_{2,2}$ scroll structures in 6D arise in this way! Projecting integral manifolds of the distribution $\mathcal{D}$ from $(x^1, \dots, x^6, \lambda)$-space to $(x^1, \dots, x^6)$-space one obtains **three-parameter family of α-manifolds**.

Generic involutive $S_{2,2}$ scroll structures locally depend on 12 arbitrary functions of 3 variables.

Involutive scroll structures via the general heavenly hierarchy

The general heavenly hierarchy (Bogdanov, 2015) is a collection of PDEs

$$(a_k - a_j)(a_l - a_i)u_{jk}u_{il} + (a_i - a_k)(a_l - a_j)u_{ik}u_{jl} + (a_j - a_i)(a_l - a_k)u_{ij}u_{kl} = 0,$$

one equation for every quadruple of distinct indices $i, j, k, l \in \{1, \ldots, n+1\}$. The characteristic variety is the intersection of quadrics,

$$\begin{aligned}(a_k - a_j)(a_l - a_i)(u_{jk}p_ip_l + u_{il}p_jp_k) + (a_i - a_k)(a_l - a_j)(u_{ik}p_jp_l + u_{jl}p_ip_k) \\ + (a_j - a_i)(a_l - a_k)(u_{ij}p_kp_l + u_{kl}p_ip_j) &= 0.\end{aligned}$$

For $n = 3$ we obtain a scroll $S_{1,1}$ which is nothing but a quadric in $\mathbb{P}^3$.

For $n = 4$ and $n = 5$ we have the scrolls $S_{1,2}$ and $S_{2,2}$, respectively.

For $n = 2k$ and $n = 2k + 1$ we have the scrolls $S_{k-1,k}$ and $S_{k,k}$, respectively.

Question: are there integrable PDE systems in 4D whose characteristic varieties define involutive scroll structures of type $S_{k,l}$ with $|k - l| > 1$?

Happy Jubilee Sasha!