Matrix factorizations for some discriminants

(jt w. C. Ingalls, S. May, M. Talarico)

I Motivation

disc. of a polyn. (deg 3) $\quad t^3+at^2+bt+c$

$a=0$, $\Delta = 4b^3+27c^2$

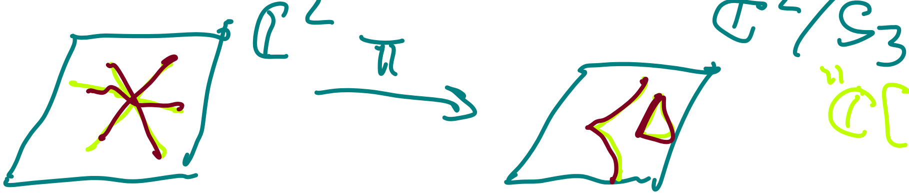
roots: x_1, x_2, x_3 $\quad b=(x_1x_2+x_1x_3+x_2x_3)$, $c = x_1x_2x_3$

(restrict to $x_1+x_2+x_3=0$)

$\rightsquigarrow$ 2 roots coincide means, that

$$V((x_1-x_2)(x_2-x_3)(x_1-x_3)) = \mathcal{A}(S_3)$$
$$(x_1-x_2)(2x_2+x_1)(2x_1+x_2) \subseteq \mathbb{C}^2$$
$$\Updownarrow$$
$$\Delta = 0.$$

$S_3 \circlearrowright \mathbb{C}^2 : \mathbb{C}[x_1,x_2]$



$\overset{\shortparallel}{\mathbb{C}[b,c]}$

$V(\Delta) \subseteq \mathbb{C}^2/S_3$

Generalize to finite cx. refl. groups:

$G \circlearrowright \mathbb{C}^n$, $\quad S := \mathbb{C}[x_1,\dots,x_n]$

quotient $\mathbb{C}^n/G$ is smooth $\quad R := S^G \cong \mathbb{C}[\underbrace{f_1,\dots,f_n}_{\text{basic inv.}}]$

$\qquad\qquad\quad\ \ \cong \mathbb{C}^n$

$\overset{\text{mirrors}}{\mathcal{A}(G)} \subseteq \mathbb{C}^n \qquad \pi(\mathcal{A}(G)) = V(\Delta) \subseteq \mathbb{C}^n/G$

non-normal hypersurface
[Arnold], [Saito]: free divisors
[Orlik-Terao] $\rightsquigarrow$ stratification

Homological properties of $V(\Delta) \rightsquigarrow$ sing. category $\underline{MCM}(R/(\Delta))$

In part: find $M_1, \ldots, M_k \in MCM(R/\Delta)$
s.t. $\mathrm{End}_{R/\Delta}(\underbrace{\bigoplus_{i=1}^{k} M_i}_{\text{faithful}})$ is a $\underline{NCR}$ $\hookrightarrow$ gldim $< \infty$

II Matrix factorizations

A = polyn. (graded) ring, $f \neq 0 \in A$
$A/(f)$

Def: A MF of f is a pair of $k \times k$ matrices (M, N), s.t. $M \cdot N = N \cdot M = f \cdot \mathbb{1}_k$.

every MF gives us an MCM-module:

$$0 \to A^k \xrightarrow{N} A^k \xrightarrow{M} A^k \to \boxed{\mathrm{coker}(M)} \to 0$$

$\in MCM(A/(f))$

Thm [Eisenbud 1980]:

$$\underline{MCM}(A/(f)) \simeq \underline{RMF}(f) \qquad A \xrightarrow{f} A \xrightarrow{1} A$$

$= MF(f) / \{(1, f), (f, 1)\}$

III MF's for discriminants

ex: $G = S_3 \qquad \Delta = 4s_3^2 + s_2^3 \qquad s_i = x_1^i + x_2^i + x_3^i$

$$R = \underbrace{\mathbb{C}[s_1, s_2, s_3]}_{A}$$

Jacobian:

$$\underbrace{\begin{pmatrix} 1 & 1 & 1 \\ 2x_1 & 2x_2 & 2x_3 \\ 3x_1^2 & 3x_2^2 & 3x_3^2 \end{pmatrix}}_{Jac} \underbrace{\begin{pmatrix} 1 & 2x_1 & 3x_1^2 \\ 1 & 2x_2 & 3x_2^2 \\ 1 & 2x_3 & 3x_3^2 \end{pmatrix}}_{Jac^T} = \underbrace{\begin{pmatrix} 3 & 2s_1 & 3s_2 \\ 2s_1 & 4s_2 & 6s_3 \\ 3s_2 & 6s_3 & 9s_4 \end{pmatrix}}_{M}$$

express in $s_{1,2,3}$

$\det(M) = \Delta$ ← up to constant, and $N = \frac{1}{\det M} \cdot M^{adj}$.

Use rep. theory of G to get MF's + NCR of Δ:

G = a. refl. grp. $\curvearrowright \mathbb{C}^n$, $S = \mathbb{C}[x_1, \dots, x_n]$

$R = S^G = \mathbb{C}[u_1, \dots, u_n]$

$$A(G) = \text{ref. arr.} = V\Big(\underbrace{\left|\frac{\partial u_i}{\partial x_j}\right|}_{\substack{Jac= \\ \| \\ z}}\Big)$$

(if $\underline{\text{ord of gen. refl.} = 2}$), ($z = Jac \Leftarrow$)

$V(\Delta) = V(z \cdot Jac)$

Going back to S:

$$\begin{array}{ccccc} S & \xrightarrow{Jac} & S & \xrightarrow{z} & S \\ \wr\| & & \wr\| & & \wr\| \\ R^{|G|} & & R^{|G|} & & R^{|G|} \end{array}$$

$\rightsquigarrow$ MF of Δ: $\operatorname{coker}(z) = S/z$ over R/Δ.

If G = gen. by order 2 reflections

skew grp ring

Thm [Buchweitz-F-Ingalls] Let $A = S * G$,
$\mathrm{End}_{R/\Delta}(S/z) \cong A/AeA$

$e = \frac{1}{|G|} \sum_{g \in G} g$

$\Rightarrow \mathrm{gldim}(\mathrm{End}_{R/\Delta}(S/z)) < \infty \Rightarrow$ NCR of R/Δ.

Moreover (McKay correspondence)
$\{$irred nontrivial G-reps$\} \xleftrightarrow{1-1} \{$indec. direct summands of $S/z\}$ (R/Δ-)

If $\mathrm{rank}(G) = 2$: Δ = ADE - curve sing.
finite MCM-type [Greuel-Knörrer]

$\mathrm{MCM}(R/\Delta) \simeq \mathrm{add}(S/z)$

Rmk: This is based on Auslander's theorem:
$G \subseteq GL(n, \mathbb{C})$, not containing any pseudoreflection, then $\mathrm{End}_R(S) \cong A$

$\dim S = 2$, $G \subseteq SL(2, \mathbb{C})$: classical McKay corr.
$\mathrm{Spec}(R) = \mathbb{C}^2/G$ = ADE - surf.

Q: How to find MF's of the isotypical comp of S/z?

V Isotypical comp. of S/z (G = $\mathbb{C}$. ref. group)

S

By [Chevalley]: $S \underset{G\text{-rep}}{\simeq} S/\underbrace{(R_+)}_{\substack{\text{id. gen.} \\ \text{by basic inv.}}} \otimes R \simeq R \otimes \mathbb{C}G$

⇝ Get isot. decomp. of S/I: $\{V_0, V_1, \dots, V_d\}$

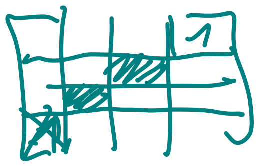
V_0: triv.

V_i: $\dim n_i$

$S/I \underset{R/\Delta\text{-mods}}{\simeq} \bigoplus_{i=1}^{d} M_i \otimes V_i$ ← get n_i copies of M_i

Ex: $G = S_3$ $\quad S \xrightarrow{\partial c} S \xrightarrow{z} S$

$R^6 \longrightarrow R^6 \longrightarrow R^6$



But: for S_n and $G(r,p,n)$

$S_n \cong G(1,1,n)$ $\quad G(r,1,n) = \mu_r \wr S_n$

have a nice basis: higher Specht polynomials [Ariki-Terasoma-Yamada]

For $G = S_n$ ⇝ $S \simeq R \otimes \left(\bigoplus_{\lambda \vdash n} V_\lambda^{\dim V_\lambda}\right) \simeq \left(\bigoplus_{\lambda \vdash n} \bigoplus_{V,S \in ST(\lambda)} \mathbb{C}F_S^V\right)$

⇝ need a variant of F_T^V ⇝ H_T^V

Thm [FIMT] $G = S_n$, the MF given by (z,z) $\quad S \xrightarrow{z} S \xrightarrow{z} S$, can be decomposed

$$(z,z) = \bigoplus_{\lambda \vdash n} \bigoplus_{T \in ST(\lambda)} \left(\langle H_T^V : V \in ST(\lambda)\rangle, \langle F_{T^{conj}}^V : V \in ST(\lambda)\rangle\right)$$