Some attacks on algebraic lattice problems

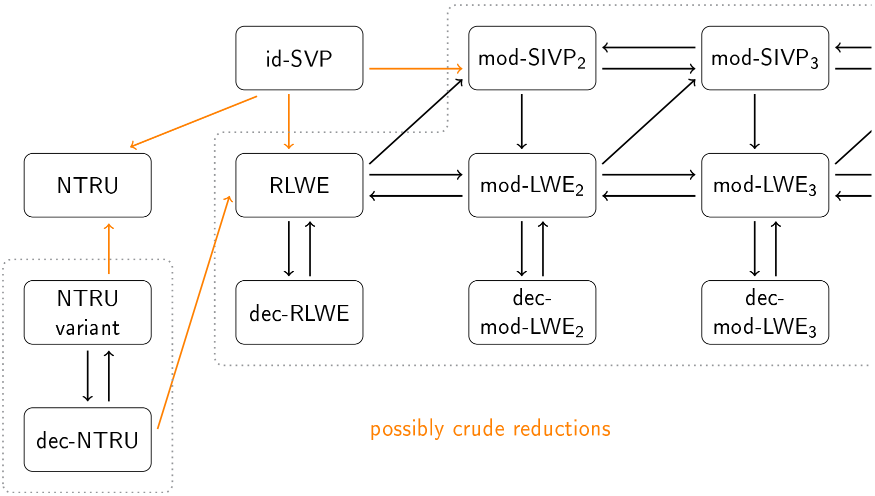
Alice Pellet--Mary

CNRS and university of Bordeaux, France

Fundations and applications of lattice-based cryptography workshop

25-28 July 2022, Edinburgh

Two algebraic attacks



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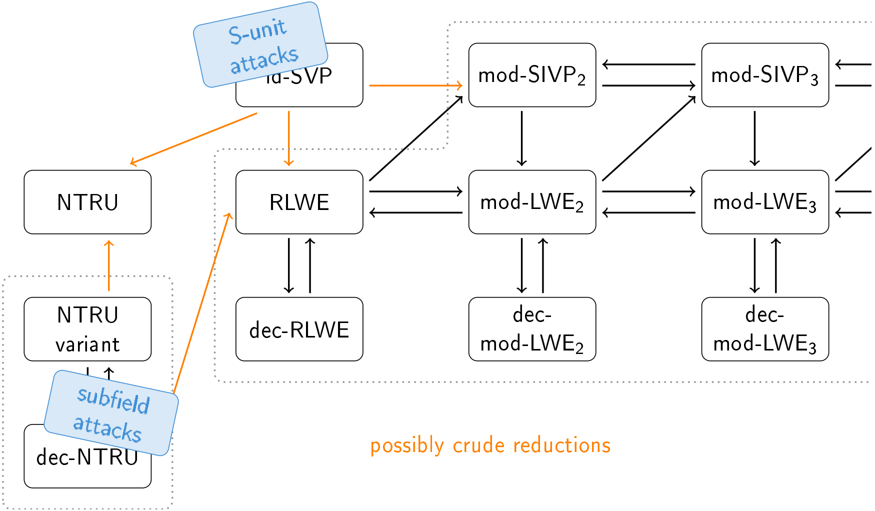

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Outline of the talk

1. S-unit attacks on id-SVP

2. subfield attacks on dec-NTRU

Attacks on id-SVP

Brief history:

- using units [CGS14,CDPR16] $\rightsquigarrow$ cyclotomics, only principal ideals

[CGS14] Campbell, Groves, and Shepherd. Soliloquy: A cautionary tale. ETSI 2nd Quantum-Safe Crypto Workshop
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- using S-units [PHS19,BR20] $\rightsquigarrow$ all ideals, different trade-offs

[PHS19] Pellet-Mary, Hanrot, Stehlé. Approx-SVP in ideal lattices with pre-processing. Eurocrypt.

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- can also break some exotic cryptographic primitives/assumptions

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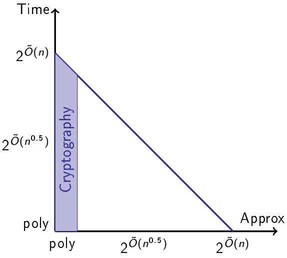
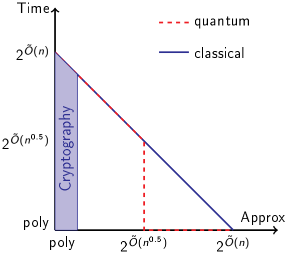
id-SVP vs SVP



SVP and mod-SVP$_k$

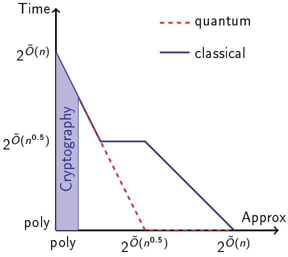
$(k \geq 2)$



id-SVP [CDW21]

(in cyclotomic fields)

units + Stickelberger



id-SVP [PHS19]

(with $2^{O(n)}$ pre-processing)

S-units

[CDW21] Cramer, Ducas, Wesolowski. Mildly short vectors in cyclotomic ideal lattices in quantum polynomial time. Journal of the ACM.

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Number theoretical reminders

From now on:

- $K = \mathbb{Q}[X]/(X^d + 1)$ $\quad (d = 2^\ell)$
- $\mathcal{O}_K = \mathbb{Z}[X]/(X^d + 1)$

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Units: $O_K^\times = \{a \in O_K \mid \exists b \in O_K, ab = 1\}$

Principal ideals: $\langle g \rangle = \{gr \mid r \in O_K\}$

- g is a generator of $\langle g \rangle$
- $\{$ generators of $\langle g \rangle$ $\} = \{gu \,|\, u \in O_K^\times\}$

Dimension of ideal lattices: $n = d$

The Log function

$$\begin{aligned} \mathrm{Log} : K &\to \mathbb{R}^d \\ y &\mapsto (\log|y(\alpha_1)|, \cdots, \log|y(\alpha_d)|) \end{aligned}$$

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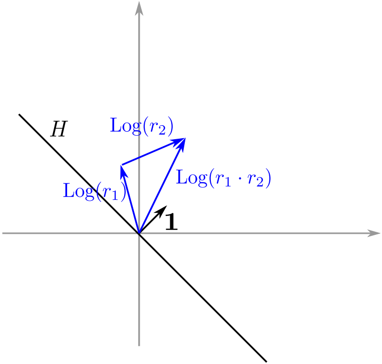
Let $\mathbf{1} = (1, \cdots, 1)$ and $H = \mathbf{1}^\perp$.

Properties ($r \in O_K$)

$\text{Log}\, r = h + a \cdot \mathbf{1}$, with $h \in H$

- $\text{Log}(r_1 \cdot r_2) = \text{Log}(r_1) + \text{Log}(r_2)$



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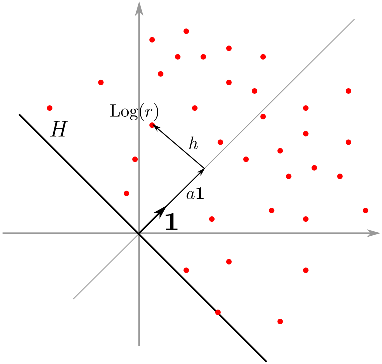

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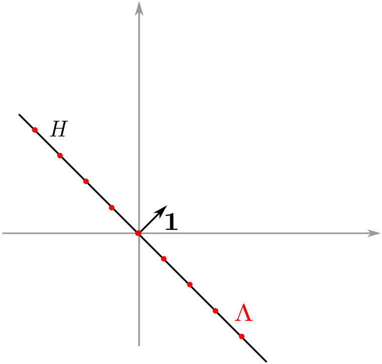

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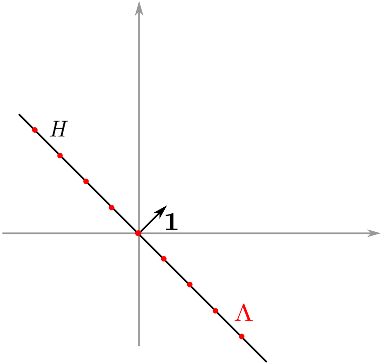

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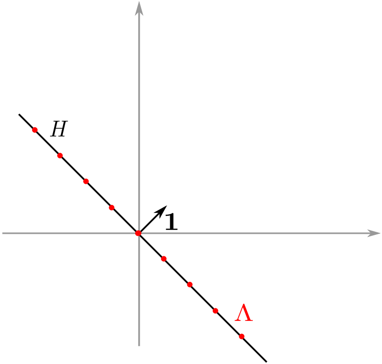
- $\|r\| \simeq \exp(\|\text{Log}\, r\|_\infty)$



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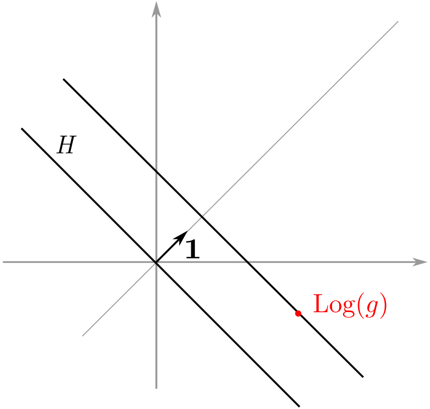
What does $\text{Log}\langle g \rangle$ look like?



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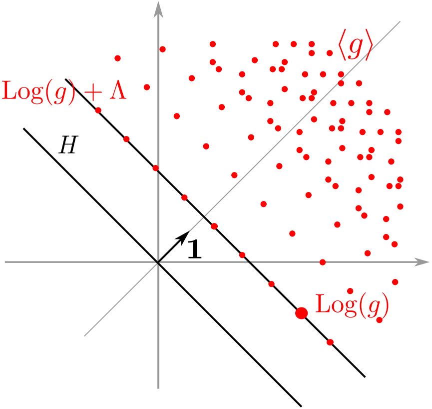

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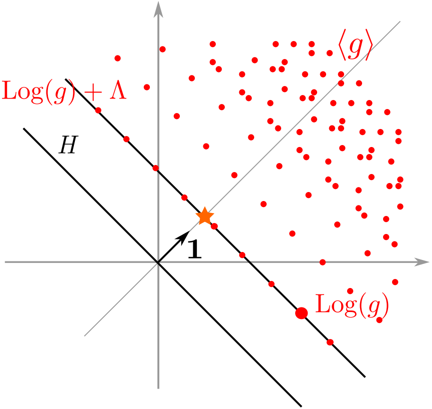

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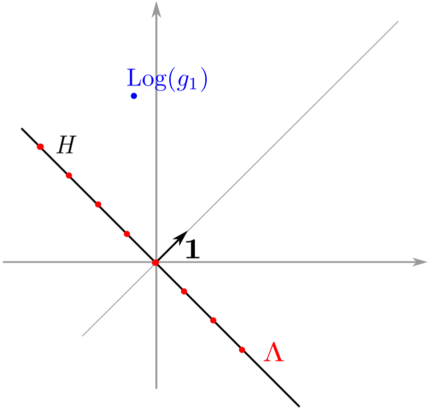

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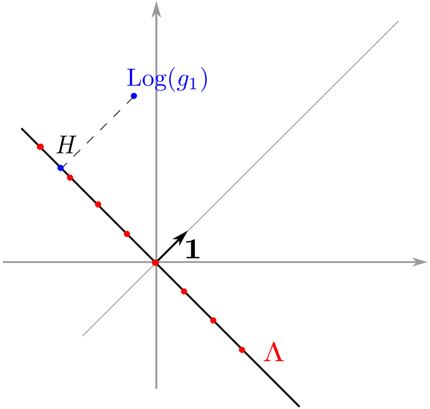

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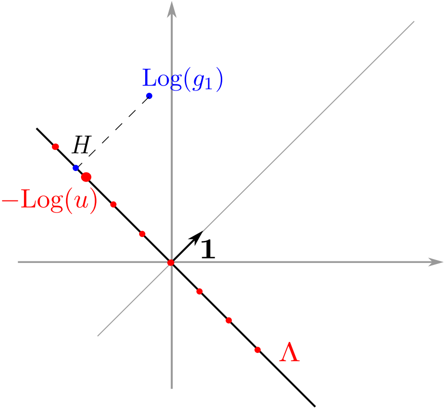

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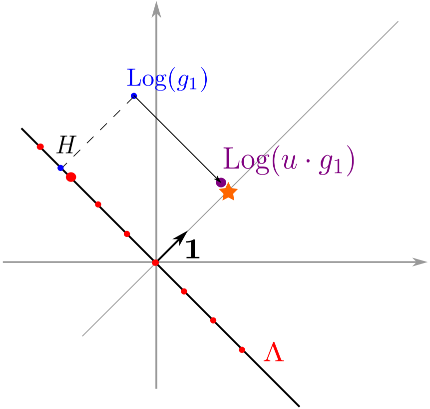

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    $\Rightarrow$ CVP in poly time
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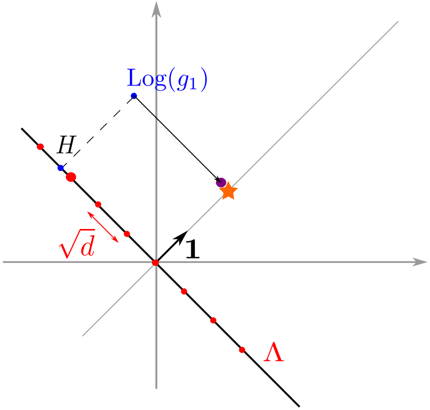

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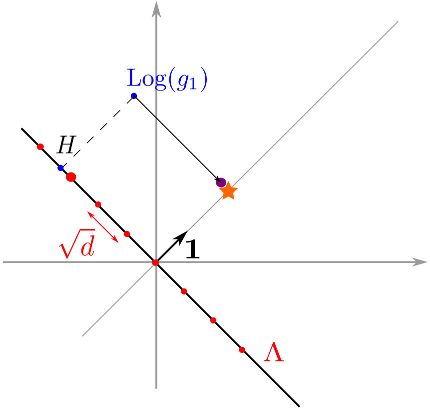

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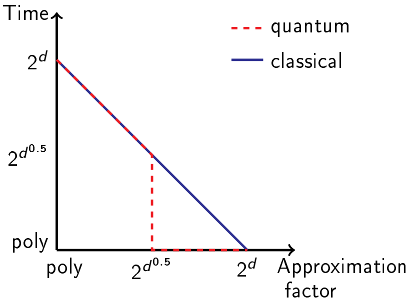

- Heuristic
- Cyclotomic fields

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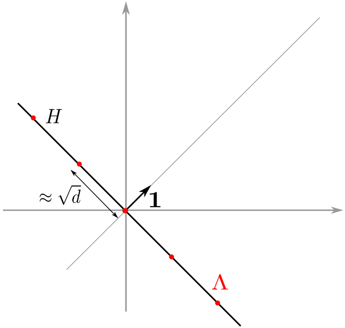

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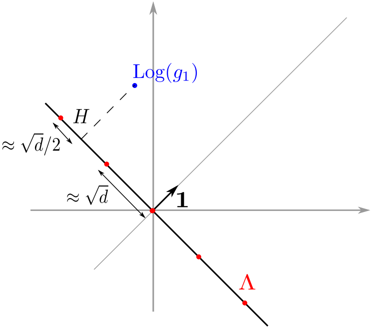

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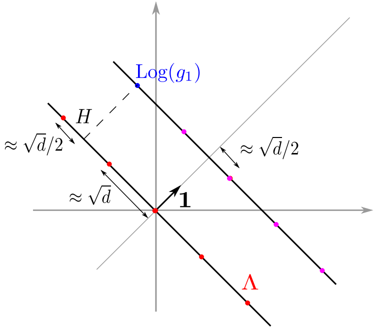

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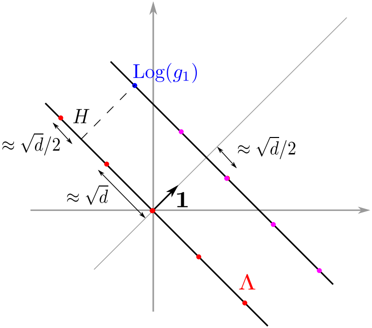

The covering radius of Λ is $\approx \sqrt{d}$

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− we don't know a good basis of the Log-S-unit lattice

- need to pre-compute it (time $2^{O(d)}$)
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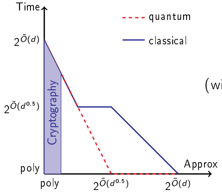

(with $2^{O(d)}$ pre-processing)

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- [CDW21] beats BKZ-300 for fields of degree $\gtrsim 16,000$

[DPW19] Ducas, Plançon, Wesolowski. On the shortness of vectors to be found by the ideal-SVP quantum algorithm. Crypto.

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- main limitation so far: computation of the S-units
  - [BR19] computes S-units up to degree 70
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  - [BR19] computes S-units up to degree 70
  - [BLNR21] computes a sublattice of the S-units up to degree 210
- no CVP with pre-processing so far (BKZ then Babai nearest plane)

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Outline of the talk

1. S-unit attacks on id-SVP
2. subfield attacks on dec-NTRU

Reminder: NTRU [HPS98]

dec-NTRU

Parameters: $q \geq B > 1$ and ψ distribution over $\mathcal{O}_K$ outputting elements $\leq B$

Objective: distinguish between h as above and u, where

- u is uniform in $\mathcal{O}_K/(q\mathcal{O}_K)$
- $f, g \leftarrow \psi$ conditioned on g invertible modulo q
- $h = f \cdot g^{-1} \bmod q$

[HPS98] Hoffstein, Pipher, and Silverman. NTRU: a ring based public key cryptosystem. ANTS.

Attacks on dec-NTRU

Brief history:

- subfield attacks [ABD16,CJL16]
  - solves dec-NTRU in time $\approx \exp\left(\frac{d \cdot \log B}{(\log q)^2}\right)$
    $\rightsquigarrow$ e.g., for $B = O(1)$, poly time attack when $q \gtrsim 2^{\sqrt{d}}$

[ABD16] Albrecht, Bai, and Ducas. A subfield lattice attack on overstretched NTRU assumptions. Crypto.

[CJL16] Cheon, Jeong, and Lee. An algorithm for NTRU problems. LMS J Comput Math.

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[ABD16] Albrecht, Bai, and Ducas. A subfield lattice attack on overstretched NTRU assumptions. Crypto.

[CJL16] Cheon, Jeong, and Lee. An algorithm for NTRU problems. LMS J Comput Math.

Attacks on dec-NTRU

Brief history:

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Impact: few schemes use such large q's, but some of them did
(e.g., some FHE schemes or multilinear maps)

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Example:

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In this slide $K = \mathbb{Q}[X]/(X^d + 1)$
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Subfields: If L subfield of K, there exist $S_L \subseteq \{1, \cdots, d\}$ s.t.

- $|S_L| = [K : L] - 1$
- for all $f \in K$,

$$\mathcal{N}_{K/L}(f) := f \cdot \prod_{i \in S_L} \sigma_i(f) \in L$$

A subfield attack on dec-NTRU [ABD16]

Objective: distinguish between

- $h = f/g \bmod q$ with $\|f\|, \|g\| \leq B$
- h uniform mod q

Attack: runs in time $\approx \exp\left(\frac{d \cdot \log B}{(\log q)^2}\right)$

On the board

[ABD16] Albrecht, Bai, and Ducas. A subfield lattice attack on overstretched NTRU assumptions. Crypto.

Other algorithms using subfields

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[BFHP22] Biasse, Fieker, Hofmann, and Page. Norm relations and computational problems in number fields. Journal of the London Mathematical Society.

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[PXWC21] Pan, Xu, Wadleigh, and Cheng. On the ideal shortest vector problem over random rational primes. Eurocrypt.

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Two algebraic attacks:

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Thank you