# ON THE ARITHMETICITY OF THE MAPPING CLASS GROUP

**“Towards a combinatorial approach to arithmetic and geometry”**

**(Joint ongoing project with S. Mochizuki and S. Tsujimura)**


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“Recent advances in anabelian geometry and related topics”

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- $2g-2+r>0, \quad r>0, \quad \Pi_{g,r}$ profinite **surface group**
- $m \geq 1, \quad X_m$ : $m$-th **configuration space** of a hyperbolic curve $X$ of type $(g,r)$ over a field $k=\bar{k}$ with $\mathrm{char}(k)=0$

$$\Pi_m^{(g,r)} \stackrel{\mathrm{def}}{=} \pi_1(X_m)$$

- $m \geq 4$

$$\mathrm{Out}^{\mathrm{F}}(\Pi_\infty^{(g,r)}) \stackrel{\mathrm{def}}{=} \mathrm{Out}^{\mathrm{F}}(\Pi_m^{(g,r)}) = \mathrm{Out}^{\mathrm{FC}}(\Pi_m^{(g,r)}) \qquad \mathrm{Out}(\Pi_m^{(0,3)}) = \mathrm{GT} \times S_{m+3}$$

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \Pi_{(\mathcal{M}_{g,r})_{\overline{\mathbb{Q}}}} & \longrightarrow & \Pi_{(\mathcal{M}_{g,r})_{\mathbb{Q}}} & \longrightarrow & G_{\mathbb{Q}} & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \mathrm{Out}^{\mathrm{F}}(\Pi_\infty^{(g,r)})^{\mathrm{geo}} & \longrightarrow & \mathrm{Out}^{\mathrm{F}}(\Pi_\infty^{(g,r)}) & \xrightarrow{\mathrm{Trp}} & \mathrm{GT} & \longrightarrow & 1
\end{array}$$

$$\mathrm{Out}^{\mathrm{F}}_{(g,r)} \stackrel{\mathrm{def}}{=} \mathrm{Out}^{\mathrm{F}}(\Pi_\infty^{(g,r)}) \times_{\mathrm{GT}} G_{\mathbb{Q}}$$

**Aim:** to prove

**Theorem A :** The map $\Pi_{(\mathcal{M}_{g,r})_{\mathbb{Q}}} \to \mathrm{Out}^{\mathrm{F}}_{(g,r)}$ is **surjective**

**Corollary B :** The map $\Pi_{(\mathcal{M}_{g,r})_{\overline{\mathbb{Q}}}} \to \mathrm{Out}^{\mathrm{F}}(\Pi_\infty^{(g,r)})^{\mathrm{geo}}$ is **surjective**

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \Pi_{g,r} & \longrightarrow & \Pi_{g,r} \rtimes^{\mathrm{out}} \Pi_{(\mathcal{M}_{g,r})_{\mathbb{Q}}} & \longrightarrow & \Pi_{(\mathcal{M}_{g,r})_{\mathbb{Q}}} & \longrightarrow & 1 \\
 & & \Big\downarrow{\scriptstyle \mathrm{id}} & & \Big\downarrow & & \Big\downarrow & & \\
1 & \longrightarrow & \Pi_{g,r} & \longrightarrow & \Pi_{g,r} \rtimes^{\mathrm{out}} \mathrm{Out}^{\mathrm{F}}_{(g,r)} & \longrightarrow & \mathrm{Out}^{\mathrm{F}}_{(g,r)} & \longrightarrow & 1
\end{array}$$

**Show:**

(**scheme**) $\Pi_{g,r} \rtimes^{\mathrm{out}} \Pi_{(\mathcal{M}_{g,r})_{\mathbb{Q}}} \to \Pi_{g,r} \rtimes^{\mathrm{out}} \mathrm{Out}^{\mathrm{F}}_{(g,r)}$ (**combinatorial**)

is **surjective**

$\Pi_{g,r} \twoheadrightarrow Q$ **finite, characteristic, centre free (cofinal)**

**Show:**

(**scheme**) $Q \rtimes^{\mathrm{out}} \Pi_{(\mathcal{M}_{g,r})_{\mathbb{Q}}} \to Q \rtimes^{\mathrm{out}} \mathrm{Out}^{\mathrm{F}}_{(g,r)}$ (**combinatorial**)

is **surjective**

## 1. Anabelian construction of $\mathrm{Out}^{\mathrm{F}}_{(g,r)}$-labelled coverings

- $[X] \in (\mathcal{M}_{g,r})^{\mathrm{cl}}_{\mathbb{Q}} \quad \rightsquigarrow \quad s : G_F \to \Pi_{(\mathcal{M}_{g,r})_{\mathbb{Q}}} \quad F/\mathbb{Q}$ finite

$$1 \to \Pi_{g,r} \to \Pi_X \to G_F \to 1$$

- $\sigma \in \mathrm{Out}^{\mathrm{F}}_{(g,r)} \quad \rightsquigarrow \quad G_F^\sigma \stackrel{\mathrm{def}}{=} \sigma s(G_F)\sigma^{-1} \quad \rightsquigarrow \quad \Pi_{X^\sigma} \stackrel{\sim}{\to} \Pi_X$

$$1 \to \Pi_{g,r} \to \Pi_{X^\sigma} \to G_F^\sigma \to 1$$

$$\Pi_{X^\sigma} \quad \rightsquigarrow \quad (F_\sigma,\ \overline{F_\sigma},\ G_F^\sigma = \mathrm{Aut}(\overline{F}_\sigma/F_\sigma),\ X^\sigma,\ \mathcal{O}(X^\sigma))$$

- $\Pi_{g,r} \twoheadrightarrow Q$ **finite, characteristic, centre-free (cofinal)**
- $\sigma \in Q \rtimes^{\mathrm{out}} \mathrm{Out}^{\mathrm{F}}_{(g,r)}$ :

$$\Pi_{Y^\sigma} \subseteq_{\mathrm{open}} \Pi_{X^\sigma} \quad \rightsquigarrow \quad (F_{Y^\sigma},\ \overline{F_{Y^\sigma}} = \overline{F_\sigma},\ Y^\sigma,\ \mathcal{O}(Y^\sigma),\ Y^\sigma \to X^\sigma \textbf{ geo. Gal.})$$

**Function space approach:**

$$\begin{array}{ccc}
\mathcal{O}(Y^\sigma) & \longrightarrow & \mathrm{Fct}[(Y^\sigma)^{\mathrm{cl}}(\overline{F_\sigma}), \overline{F_\sigma}] \\
\uparrow & & \uparrow \\
\mathcal{O}(X^\sigma) & \longrightarrow & \mathrm{Fct}[(X^\sigma)^{\mathrm{cl}}(\overline{F_\sigma}), \overline{F_\sigma}]
\end{array}$$

- $\sigma, \tau \in Q \rtimes^{\mathrm{out}} \mathrm{Out}^{\mathrm{F}}_{(g,r)} \ : \ Y^\sigma \to X^\sigma \ \ \rightsquigarrow \ \ Y^\tau \to X^\tau$ **combinatorial !**

**dilatation:**

$\tau\sigma^{-1} \in \mathrm{Out}^{\mathrm{F}}_{(g,r)}$ **a priori combinatorial, non-geometric !**

**meaningless comparison**:

$$\begin{array}{ccc} \mathrm{Fct}[(Y^\sigma)^{\mathrm{cl}}(\overline{F_\sigma}), \overline{F_\sigma}] & \longrightarrow & \mathrm{Fct}[(Y^\tau)^{\mathrm{cl}}(\overline{F_\tau}), \overline{F_\tau}] \\ \uparrow & & \uparrow \\ \mathcal{O}(Y^\sigma) & \longrightarrow & \mathcal{O}(Y^\tau) \end{array}$$

- **Aim**: Construct a **single (non labelled)** ($\mathrm{Out}^{\mathrm{F}}_{(g,r)}$-)function space

$$\mathrm{Fct}[Y^{\mathrm{act}}_{\overline{\mathbb{Q}}}, \overline{\mathbb{Q}}]$$

$$\mathcal{O}(Y^\sigma) \ \hookrightarrow \ \mathrm{Fct}[Y^{\mathrm{act}}_{\overline{\mathbb{Q}}}, \overline{\mathbb{Q}}] \ \hookleftarrow \ \mathcal{O}(Y^\tau)$$

**want a** $Q \rtimes^{\mathrm{out}} \mathrm{Out}^{\mathrm{F}}_{(g,r)}$ **action** on $\mathrm{Fct}[Y^{\mathrm{act}}_{\overline{\mathbb{Q}}}, \overline{\mathbb{Q}}]$

## 2. Synchronisation of labelled objects over $\overline{\mathbb{Q}}$

- **Synchronisation of labelled copies of $\overline{\mathbb{Q}}$ and $\mathbb{P}^1(\overline{\mathbb{Q}})$**

$$G_{\mathbb{Q}} \hookrightarrow \mathrm{GT} \hookrightarrow \mathrm{Out}(\Pi_{0,3})$$

$\rightsquigarrow$

$$1 \to \Pi_{0,3} \to \Pi_{\mathrm{Trp}} \to G_{\mathbb{Q}} \to 1$$

$\rightsquigarrow$

$$(\overline{F_\sigma} \overset{\sim}{\to}) \quad M \overset{\mathrm{def}}{=} \overline{\mathbb{Q}}_{\mathrm{comb}} \qquad \mathrm{Trp}(M) \qquad \mathrm{Cusp}(M) = \{0, 1, \infty\}$$

$\rightsquigarrow$ **Tripod labelling:** $\quad \mathbb{P}^1(M) = \{x_1, x_2, \ldots, x_n, x_{n+1}, \ldots\}$

- **Synchronisation of $(X^\sigma)^{\mathrm{cl}}(\overline{\mathbb{Q}})$, $(Y^\sigma)^{\mathrm{cl}}(\overline{\mathbb{Q}})$, via simple coverings**

$[X] \in (\mathcal{M}_{g,r})^{\mathrm{cl}}_{\mathbb{Q}}$, **fix** $d \geq g+1$, $f : X^{\mathrm{cp}}_M \to \mathbb{P}^1_M$ simple degree d covering

**choose section:** $\quad \mathbb{P}^1(M) \xrightarrow{\delta_f} X^{\mathrm{cp}}(M) \twoheadrightarrow \mathbb{P}^1(M)$

$\rightsquigarrow$ **Tripod labelling:**

$$\mathrm{Sect}_{X_M, f, \delta_f} = \{x_{\alpha_i}\}_{i \geq 1} \subset X^{\mathrm{cl}}(M)$$

$$\mathrm{Sect}_{Y_M, f, \delta_f} = \{[y_{\alpha_i}]\}_{i \geq 1} \subset Y^{\mathrm{cl}}(M)$$

$\sigma \in \mathrm{Out}^{\mathrm{F}}_{g,r}$ : $\quad \mathrm{Sect}_{X^\sigma_M, f^\sigma, \delta_{f^\sigma}} = \{x^\sigma_{\alpha_i}\}_{i \geq 1} \subset (X^\sigma)^{\mathrm{cl}}(M)$

$\sigma \in Q \rtimes^{\mathrm{out}} \mathrm{Out}^{\mathrm{F}}_{g,r}$ : $\quad \mathrm{Sect}_{Y^\sigma_M, f^\sigma, \delta_{f^\sigma}} = \{[y^\sigma_{\alpha_i}]\}_{i \geq 1} \subset (Y^\sigma)^{\mathrm{cl}}(M)$

**Hilbertian points**

$$\mathrm{Sect}^{\mathrm{Hilb}}_{X^\sigma_M, f^\sigma, \delta_{f^\sigma}} \subseteq_{\mathrm{infinite}} \mathrm{Sect}_{X^\sigma_M, f^\sigma, \delta_{f^\sigma}}$$

$$\mathrm{Sect}^{\mathrm{Hilb}}_{Y^\sigma_M, f^\sigma, \delta_{f^\sigma}} \subseteq_{\mathrm{infinite}} \mathrm{Sect}_{Y^\sigma_M, f^\sigma, \delta_{f^\sigma}}$$

**Hurwitz datum of type $(g, r, d)$:**

$$([X], f : X_M \to \mathbb{P}^1_M, \delta_f) : \quad \text{will ultimately vary}$$

## 3. Group-theoretic construction of $\mathrm{Out}^{\mathrm{F}}$-active points

$m \geq 1$ $\qquad \mathrm{Out}^{\mathrm{F}}(\Pi_{\infty}^{(g,r+|m|)}) \subset \mathrm{Out}^{\mathrm{F}}(\Pi_{\infty}^{(g,r+m)})$

$$(*) \qquad 1 \to \Pi_m^{(g,r)} \to \mathrm{Out}^{\mathrm{F}}(\Pi_{\infty}^{(g,r+|m|)}) \to \mathrm{Out}^{\mathrm{F}}(\Pi_{\infty}^{(g,r)}) \to 1$$

$$\mathrm{Out}^{\mathrm{F}}_{(g,r+|m|)} \stackrel{\mathrm{def}}{=} \mathrm{Out}^{\mathrm{F}}(\Pi_{\infty}^{(g,r+|m|)}) \times_{\mathrm{GT}} G_{\mathbb{Q}}$$

$$(**) \qquad 1 \to \Pi_{g,r+m} \to \Pi_{g,r+m} \rtimes^{\mathrm{out}} \mathrm{Out}^{\mathrm{F}}_{(g,r+|m|)} \to \mathrm{Out}^{\mathrm{F}}_{(g,r+|m|)} \to 1$$

$$(***) \qquad 1 \to \Pi_{g,r+\infty} \to \Pi_{g,r+\infty} \rtimes^{\mathrm{out}} \mathrm{Out}^{\mathrm{F}}_{(g,r+|\infty|)} \to \mathrm{Out}^{\mathrm{F}}_{(g,r+|\infty|)} \to 1$$

$$x_\alpha \in \mathrm{Sect}^{\mathrm{Hilb}}_{X_M,f,\delta_f} \qquad \longleftrightarrow \qquad x_\alpha^{\mathrm{act}} = [I_\alpha]_{\Pi_{g,r+\infty}}$$

$I_\alpha \subset \Pi_{g,r+\infty}$ **cuspidal inertia** $\longleftrightarrow$ $x_\alpha$

$$\{y_\alpha\} \in \mathrm{Sect}^{\mathrm{Hilb}}_{Y_M,f,\delta_f} \qquad \rightsquigarrow \qquad \{y_\alpha^{\mathrm{act}}\} = \{[I_\alpha]_{r\Pi^{Q}_{g,r+\infty}}\}$$

$$X_M^{\mathrm{act}} \qquad Y_M^{\mathrm{act}} \qquad (X_M^{\sigma})^{\mathrm{act}} \qquad (Y_M^{\sigma})^{\mathrm{act}}$$

$$\Pi_{g,r+\infty} \rtimes^{\mathrm{out}} \mathrm{Out}^{\mathrm{F}}_{(g,r+|\infty|)} \twoheadrightarrow Q \rtimes^{\mathrm{out}} \mathrm{Out}^{\mathrm{F}}_{(g,r+|\infty|)} \twoheadrightarrow \mathrm{Out}^{\mathrm{F}}_{(g,r)}$$

$$\mathrm{Fct}[(Y_M^{\sigma})^{\mathrm{cl}}, M] \quad \twoheadrightarrow \quad \mathrm{Fct}[(Y_M^{\sigma})^{\mathrm{act}}, M] \quad \stackrel{\sim}{\to} \quad \mathrm{Fct}[Y_M^{\mathrm{act}}, M]$$

$\mathrm{Fct}[Y_M^{\mathrm{act}}, M]$ **common container** to the various $\mathcal{O}(Y^{\sigma})_{\sigma}$ and $\mathcal{O}(X^{\sigma})_{\sigma}$

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \Pi_{g,r+\infty} & \longrightarrow & \Pi_{g,r+\infty} \rtimes^{\text{out}} \mathrm{Out}^{\mathrm{F}}_{(g,r+|\infty|)} & \longrightarrow & \mathrm{Out}^{\mathrm{F}}_{(g,r+|\infty|)} & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & Q & \longrightarrow & Q \rtimes^{\text{out}} \mathrm{Out}^{\mathrm{F}}_{(g,r+|\infty|)} & \longrightarrow & \mathrm{Out}^{\mathrm{F}}_{(g,r+|\infty|)} & \longrightarrow & 1
\end{array}$$

$\exists$ a **finitely generated** subgroup:

$$R \subset Q \rtimes^{\text{out}} \mathrm{Out}^{\mathrm{F}}_{(g,r+|\infty|)}$$

and a **finite** sub-quotient (of $Q \rtimes^{\text{out}} \mathrm{Out}^{\mathrm{F}}_{(g,r+|\infty|)}$):

$$R \twoheadrightarrow {}_Q N_Q$$

(**) $$1 \to Q \to {}_Q N_Q \to N_Q \to 1$$

**acts** on $Y_M^{\text{act}}$ $$\mathrm{Im}(\mathrm{Out}^{\mathrm{F}}_{(g,r)}) = \mathrm{Im}(N_Q) \subset \mathrm{Out}(Q)$$

$$\bar{\sigma} \in N_Q : \qquad \mathcal{O}(X^{\text{mod}}) \ = \ \mathrm{Im}[\mathcal{O}(X^{\bar{\sigma}})]$$

$$\{\mathrm{Im}[\mathcal{O}(Y^{\bar{\sigma}})]\}_{\bar{\sigma} \in {}_Q N_Q} \quad \subset \quad \mathrm{Fct}[Y_M^{\text{act}}, M]$$

**Fixed Combinatorial base point** $x_0$:

$$1 \to \Pi_{g,r+1} \to \Pi_{g,r+1} \rtimes^{\text{out}} \mathrm{Out}^{\mathrm{F}}_{(g,r+|1|)} \to \mathrm{Out}^{\mathrm{F}}_{(g,r+|1|)} \to 1$$

$$x_0 = [I_0]_{\Pi_{g,r+|1|}}$$

$$\{x_0, x^{\mathrm{act}}_{\alpha_1}, x^{\mathrm{act}}_{\alpha_2}, \dots, x^{\mathrm{act}}_{\alpha_n}, \dots\}$$

$$\{\{y_{0,j}\}_{1\le j\le t},\ \{y^{\mathrm{act}}_{\alpha_{1,j}}\}_{1\le j\le t},\ \{y^{\mathrm{act}}_{\alpha_{2,j}}\}_{1\le j\le t}, \dots,\ \{y^{\mathrm{act}}_{\alpha_{n,j}}\}_{1\le j\le t}, \dots\}$$

$$\mathrm{Out}^{\mathrm{F}}_{(g,r+|1|)} \xrightarrow{s_{x_0}} \Pi_{g,r} \rtimes^{\mathrm{out}} \mathrm{Out}^{\mathrm{F}}_{(g,r+|1|)} \twoheadrightarrow \mathrm{Out}^{\mathrm{F}}_{(g,r+|1|)}$$

**Splitting** of $(**)$: $\quad 1 \to Q \to {}_QN_Q \to N_Q \to 1$

$$s_{x_0} : N_Q \to {}_QN_Q$$

$\rightsquigarrow$

$$y_0 \in \{y^{\mathrm{act}}_{0,j}\}_{1\le j\le t}$$

**Over moduli:**

$$\begin{array}{ccccc}
\mathcal{Y}_{\mathrm{mod},m} & \longrightarrow & \mathcal{X}_{\mathrm{mod},m} & \longrightarrow & \mathcal{M}_m = \mathcal{M}_{g,r+|m|} \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{Y}_{\mathrm{mod},1} & \xrightarrow{\text{fét}} & \mathcal{X}_{\mathrm{mod},1} & \longrightarrow & \mathcal{M}_1 = \mathcal{M}_{g,r+|1|} \\
 & & \downarrow & & \downarrow \\
 & & \mathcal{X}_{\mathrm{mod}} & \longrightarrow & \mathcal{M} = \mathcal{M}_{g,r}
\end{array}$$

$$1 \to \Pi_{g,r} \to \Pi_{\mathcal{X}_{\mathrm{mod},1}} \to \Pi_{\mathcal{M}_1} \to 1$$

$$\{\mathcal{Y}_{\mathrm{mod},1} \xrightarrow{\text{fét}} \mathcal{X}_{\mathrm{mod},1}\} \quad \longleftrightarrow \quad \{\Pi^Q_{g,r}.s(\Pi_{\mathcal{M}_1}) \subset \Pi_{\mathcal{X}_{\mathrm{mod},1}}\}$$

$$\Pi^Q_{g,r} \stackrel{\text{def}}{=} \text{Ker } (\Pi_{g,r} \twoheadrightarrow Q)$$

**Limit: (moduli problem)**

$$\mathcal{Y}_{\mathrm{mod},\infty} \to \mathcal{X}_{\mathrm{mod},\infty} \to \mathcal{M}_\infty = \mathcal{M}_{g,r+|\infty|} \to \mathcal{M} = \mathcal{M}_{g,r}$$

$$\mathcal{O}(\mathcal{X}_{\mathrm{mod},\infty}) \hookrightarrow \prod \mathcal{O}(X^{\mathrm{mod}}) \subset \prod_{([X],f,\delta_f)\in\mathcal{H}\mathrm{urw}} \mathrm{Fct}[X^{\mathrm{act}}_M, M \cup \{\infty\}]$$

$$\alpha \stackrel{\text{def}}{=} \bar{\sigma} \in {}_QN_Q :$$

$$\rho_\alpha : \mathcal{O}(\mathcal{Y}_{\mathrm{mod},\infty}) \hookrightarrow \prod \mathrm{Im}_{\bar{\sigma}}(\mathcal{O}(Y^{\bar{\sigma}})) \subset \prod_{\mathcal{H}\mathrm{urw}} \mathrm{Fct}[Y^{\mathrm{act}}_M, M \cup \{\infty\}]$$

$$\mathcal{O}(\mathcal{X}^{\text{ét}}_{\text{mod},\infty}) \stackrel{\text{def}}{=} \text{Im}[\mathcal{O}(\mathcal{X}_{\text{mod},\infty})] \qquad \mathcal{O}(\mathcal{Y}^{\text{ét}}_{\text{mod},\infty})^{\alpha} \stackrel{\text{def}}{=} \rho_{\alpha}([\mathcal{O}(\mathcal{Y}_{\text{mod},\infty})])$$

$$\mathcal{R}_{\mathcal{X},\text{mod}} \stackrel{\text{def}}{=} \mathcal{O}(\mathcal{X}^{\text{ét}}_{\text{mod},\infty}) \subset \prod_{\mathcal{H}\text{urw}} \text{Fct}[X^{\text{act}}_M, M \cup \{\infty\}]$$

$$\mathcal{R}_{\mathcal{Y},\text{mod}} \stackrel{\text{def}}{=} < \mathcal{O}(\mathcal{Y}^{\text{ét}}_{\text{mod},\infty})^{\alpha} >_{\alpha \in_Q N_Q} \subset \prod_{\mathcal{H}\text{urw}} \text{Fct}[Y^{\text{act}}_M, M \cup \{\infty\}]$$

$\mathcal{R}_{\mathcal{Y},\text{mod}}/\mathcal{R}_{\mathcal{X},\text{mod}}$ **finite étale** (after possibly shrinking the base $\mathcal{M}$)

**5. Analysis of idempotents** (study the idempotents of $\mathcal{R}_{\mathcal{Y},\text{mod}}$)

**project onto a single Hurwitz datum:**

Hur $= ([X], f, \delta_f) \in \mathcal{H}$urw, a **fixed** Hurwitz datum of type $(g, r, d)$

$$\phi_{\text{Hur}} : \prod_{([X],f,\delta_f)\in\mathcal{H}\text{urw}} (\text{Fct}[Y^{\text{act}}_M, M \cup \{\infty\}]) \twoheadrightarrow \text{Fct}[Y^{\text{act}}_M, M \cup \{\infty\}]$$

$$R_X \stackrel{\text{def}}{=} \mathcal{O}(X)$$

$$R_Y \stackrel{\text{def}}{=} < \mathcal{O}(Y)^{\alpha} >_{\alpha \in_Q N_Q} \quad \textbf{finite étale} \text{ over } \quad R_X$$

$$\mathcal{R}_{\mathcal{Y},\text{mod}} \to R_Y$$

$\mathcal{E}_{\text{mod}} = \{\xi_i\}_{i\in I}$ **max. orth. idemp. of** $\mathcal{R}_{\mathcal{Y},\text{mod}}$

$$\sum_{i\in I} \xi_i = 1$$

$Q$ **acts** on $\mathcal{E}_{\text{mod}}$

$$E_{\text{Hur}} \subset R_Y \qquad \text{(non-zero) image of } \mathcal{E}_{\text{mod}}$$

$$E_{\text{Hur}} = \{e_j\}_{j\in J} \ \text{ orth. idemp. } \sum_{j\in J} e_j = 1$$

**Lemma 1.** $\exists$ **infinite** subsets

$$X_M^{\text{act},\dagger} \subseteq X_M^{\text{act}}$$

$$Y_M^{\text{act},\dagger} \stackrel{\text{def}}{=} X_M^{\text{act},\dagger} \times_{X_M^{\text{act}}} Y_M^{\text{act}} \subseteq Y_M^{\text{act}}$$

such that if

$$R_X^\dagger,\ R_Y^\dagger,\ E_{\text{Hur}}^\dagger \subset \text{Fct}[Y_M^{\text{act},\dagger}, M\cup\{\infty\}]$$

arise from $R_X$, $R_Y$, $E_{\text{Hur}}$, and

$$E_{\text{Hur}}^{\dagger,\neq 0} = \{\epsilon \in E_{\text{Hur}}^\dagger : \epsilon \neq 0\},$$

then

**(i)** $E_{\text{Hur}}^{\dagger,\neq 0}$ is a set of **orth. idem.**

**(ii)** the action of $Q$ on $E_{\text{Hur}}^{\dagger,\neq 0}$ is **transitive**

**Corollary 1.** Assume $\mathbb{Q} \subset K_0 \subset K \subset \overline{\mathbb{Q}}$; $K/K_0$ Galois (possibly infinite); $Y/K$; $X = X_0 \times_{K_0} K$ such that $Y/X_0$ Galois;

$$1 \to Q \to \mathrm{Gal}(Y/X_0) \to \mathrm{Gal}(X/X_0) \to 1.$$

Then, there **exists** a homomorphism

$$\rho_{X,Q} : N_Q \to \mathrm{Gal}(X/X_0)$$

$\rightsquigarrow$ **"New mysterious"** action of $N_Q$ on $Q$ via $\rho_{X,Q}$

**Proof.** Recall $R_Y \stackrel{\text{def}}{=} < \mathcal{O}(Y)^\alpha >_{\alpha \in_Q N_Q}$. After restricting to $(\dagger)$ :

$$\mathrm{Spec}(R_Y) = \bigsqcup_i Y_i \ \to \ X \ \to \ X_0$$

$D_i$ = Decp. gp. of an idempotent $\epsilon_i \ \longleftrightarrow$ a connected component $Y_i$ :

$$1 \to D_i \cap Q \to D_i \ (\subset {}_Q N_Q) \ \to \ N_Q \to 1$$

$\rightsquigarrow$

$$\begin{array}{ccc} D_i \cap Q & \longrightarrow & \mathrm{Gal}(Y_i/X) \\ \downarrow & & \downarrow \\ D_i & \longrightarrow & \mathrm{Gal}(Y_i/X_0) \end{array}$$

$\rightsquigarrow$

$$D_i/(D_i \cap Q) \ = \ N_Q \ \to \ \mathrm{Gal}(Y_i/X_0)/Q$$

**Proposition 1.** After restriction to $Y_M^{\mathrm{act},\dagger}$ and $(\mathcal{H}\mathrm{urw})^\dagger$ :

$$\mathcal{R}^\dagger_{\mathcal{Y},\mathrm{mod}} \stackrel{\text{def}}{=} \mathrm{Im}(\mathcal{R}_{\mathcal{Y},\mathrm{mod}}) \subset \prod_{([X],f,\delta_f)\in(\mathcal{H}\mathrm{urw})^\dagger} \mathrm{Fct}[Y_M^{\mathrm{act},\dagger}, M \cup \{\infty\}];$$

the action of $Q$ on $\mathcal{E}^\dagger_{\mathrm{mod}}$ (= non zero image of $\mathcal{E}_{\mathrm{mod}}$) is **transitive**

$$\begin{array}{ccccccccc} 1 & \longrightarrow & Q & \longrightarrow & ({}_Q N_Q)^{\text{alg}} & \longrightarrow & (N_Q)^{\text{alg}} \overset{\text{def}}{=} \text{Im}(\Pi_{(\mathcal{M}_{g,r})_{\mathbb{Q}}}) & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \text{inj}\downarrow & & \\ 1 & \longrightarrow & Q & \longrightarrow & {}_Q N_Q & \longrightarrow & N_Q & \longrightarrow & 1 \end{array}$$

**Aim:** Show that

$$(N_Q)^{\text{alg}} = N_Q$$

**Proposition 2.** There exists a map

$$\rho_Q : N_Q \to (N_Q)^{\text{alg}}$$

$\rightsquigarrow$ **"New mysterious"** action of $N_Q$ on $Q$ via $\rho_Q$.

**Proof.** Similar to the proof of Corollary 1.

**Proposition 3.** The map $\rho_Q : N_Q \to (N_Q)^{\text{alg}}$ satisfies the following

**(i)** The composite map

$$(N_Q)^{\text{alg}} \subseteq N_Q \xrightarrow{\rho_Q} (N_Q)^{\text{alg}}$$

is **identity**

**(ii)** $\rho_Q$ is **injective**

**Proof:** **(i)** $\rho_Q$ restricted to decomposition groups ($\subset N_Q^{\text{alg}}$) is identity. Use **Hurwitz spaces** of simple coverings, and the **Hilbertian** property.

**(ii)** Passage from $(g, r)$ to $(g, r+1)$

Use **Hilbertian** property:

- Need 2-types of Hilbertain points
- Make sure in the deletion process, keep both types of points