Uniqueness of the solution of the filtering equations

in Honor of N. Krylov, at the occasion of his 80th Birthday

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joint work with Dan Crisan, Imperial College, London

ICMS Harmonic Analysis, Stochastics and PDEs

The filtering problem

- Assume that $\{(X_t, Y_t),\ t \geq 0\}$ is given as

$$X_t = X_0 + \int_0^t f(s, X_s)ds + \int_0^y g(s, X_s)dV_s + \int_0^t \bar{g}(s, X_s)dW_s,$$
$$Y_t = \int_0^t h(s, X_s)ds + W_t,$$

  where X_t takes its values in $\mathbb{R}^d$, Y_t in $\mathbb{R}^m$, V_t and W_t are mutually independent Brownian motions, resp. k and m dimensional. This is defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, equipped with a filtration $\mathcal{F}_t$.
- Let $\mathcal{Y}_t := \sigma\{Y_s,\ 0 \leq s \leq t\}$. We wish to "compute" the conditional law of X_t, given $\mathcal{Y}_t$, for all $t \geq 0$.
- Let

$$Z_t = \exp\left(\int_0^t (h(s, X_s), dW_s) - \frac{1}{2}\int_0^t |h(s, x_s)|^2 ds\right),$$

$$\tilde{\mathbb{P}} \text{ defined by } \frac{d\tilde{\mathbb{P}}}{d\mathbb{P}}|_{\mathcal{F}_t} = Z_t,\ \tilde{Z}_t := Z_t^{-1} = \frac{d\mathbb{P}}{d\tilde{\mathbb{P}}}|_{\mathcal{F}_t}.$$

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The Zakai equation 1

- The so-called Kallianpur–Striebel formula is easy to verify :

$$\mathbb{E}\left[\varphi(X_t)|\mathcal{Y}_t\right] = \frac{\tilde{\mathbb{E}}\left[\tilde{Z}_t\varphi(X_t)|\mathcal{Y}_t\right]}{\tilde{\mathbb{E}}\left[\tilde{Z}_t|\mathcal{Y}_t\right]}\,.$$

- Under $\tilde{\mathbb{P}}$, $\{Y_t,\ t \ge 0\}$ is a Brownian motion independent of V_t and the measure–valued process π_t defined by $\pi_t(\varphi) := \tilde{\mathbb{E}}\left[\tilde{Z}_t\varphi(X_t)|\mathcal{Y}_t\right]$ solves

$$\pi_t(\varphi) = \pi_0(\varphi) + \int_0^t \pi_s(A_s\varphi)ds + \sum_{j=1}^m \pi_s(B_s^j\varphi)dY_s^j,$$

- where with $a = gg^T + \bar{g}\bar{g}^T$,

$$(A_s\varphi)(x) = \frac{1}{2}\sum_{i,j} a_{ij}(s,x)\partial^2_{x_i,x_j}(x) + \sum_i f_i(s,x)\partial_{x_i}\varphi(x),$$

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The Zakai equation is easily obtained as follows (with φ smooth enough) :

- First develop $\tilde{Z}_t\varphi(X_t)$ using Itô's formula.
- Note that
  1. For ξ_t $\mathcal{F}_t$ measurable, $\tilde{\mathbb{E}}[\xi_t|\mathcal{Y}_{t+s}] = \tilde{\mathbb{E}}[\xi_t|\mathcal{Y}_t]$, since $\mathcal{Y}_t$ is the filtration of a Brownian motion under $\tilde{\mathbb{P}}$.
  2. $\tilde{\mathbb{E}}[\int_0^t \tilde{Z}_s\psi(X_s)ds|\mathcal{Y}_t] = \int_0^t \pi_s(\psi)ds$.
  3. $\tilde{\mathbb{E}}[\int_0^t \tilde{Z}_s\psi(X_s)dY_s^j|\mathcal{Y}_t] = \int_0^t \pi_s(\psi)dY_s^j$.
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The Zakai equation 3

- π_t is called the "unnormalized conditional law of X_t, given $\mathcal{Y}_t$". Indeed

$$\mathbb{E}[\varphi(X_t)|\mathcal{Y}_t] = \frac{\pi_t(\varphi)}{\pi_t(1)} \text{ (see the K–S formula)}$$

- With a smooth enough test function $u(t,x)$, the Zakai equation becomes

$$\pi_t(u_t) = \pi_0(u_0) + \int_0^t \pi_s(\partial_s u_s + A_s u_s)ds + \sum_{j=1}^m \int_0^t \pi_s(B_s^j u_s)dY_s^j,$$

- Of course, in order to "compute" the conditional law, we would have to solve an SPDE. There are by now well established "particle filtering algorithms" which give reasonably good approximations of the conditional law.
- In the case where X_0 is Gaussian, f and h are linear in x, a does not depend upon x, then (X_t, Y_t) is a Gaussian process and the conditional law $\bar{\pi}_t$ is Gaussian, see the well–known Kalman–Bucy filter.

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Uniqueness of the Zakai equation 0

- Given $r \in L^\infty(0,T;\mathbb{R}^m)$, we consider the complex valued process

$$\theta_t = \exp\left(i\int_0^t (r_s, dY_s) + \frac{1}{2}\int_0^t |r_s|^2 ds\right),$$

so that $\theta_t = 1 + i\int_0^t \theta_s (r_s, dY_s)$.

- Consider the set of r.v.'s $S_T = \{\theta_T,\ r \in L^\infty(0,T;\mathbb{R}^m)\}$. If $X \in L^1(\Omega, \mathcal{Y}_T, \tilde{\mathbb{P}})$ is such that $\tilde{\mathbb{E}}[\theta_T X] = 0$ for all $\theta_T \in S_T$, then $X = 0$ a.s.

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- From Itô's formula,

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- If

$$\partial_t u_t + A_t u_t + i\sum_j r_t^j B_t^j u_t = 0,\ 0 \le t \le T,$$

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$$\tilde{\mathbb{E}}\left(\sqrt{\int_0^T \theta_t^2[\pi_t(B_t^j u_t) + r_t^j \pi_t(u_t)]^2 dt}\right) < \infty,\ 1 \le j \le m,\ (*)$$

- then $\theta_t\pi_t(u_t))$ is a $\tilde{\mathbb{P}}$ martingale, and $\tilde{\mathbb{E}}[\theta_T\pi_T(\varphi)] = \pi_0(u_0)$.

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Uniqueness of the Zakai equation 2

- Suppose that for any $T > 0$, $r \in L^\infty(0, T; \mathbb{R}^m)$ and φ in a dense subset of $C_b(\mathbb{R}^d)$, the above backward parabolic PDE has a smooth enough solution which satisfies $(*)$. Then we have uniqueness of the solution of the Zakai equation in the space of measure valued processes satisfying some condition to insure $(*)$.
- If all coefficients are bounded, as well as the solution of the backward PDE and its first order derivatives, then we have uniqueness in the set of measure valued processes satisfying $\mathbb{E}[\sup_{0\le t\le T}\pi_t(1)] < \infty$.
- Such a result has been obtained by A. Bensoussan in his book *Stochastic Control of Partially Observable Systems* with no ellipticity assumption, allowing the coefficients f and h to have linear growth, provided a, f and h have bounded derivatives of order 1 and 2 w.r.t. the spatial variables.
- This uniqueness result is obtained via a duality argument (well–known in Probability and in PDE).

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A more general filtering problem

- It is quite natural to generalize the above filtering problem as follows :

$$X_t = X_0 + \int_0^t f(s, X_s, Y_s)ds + \int_0^t g(s, X_s, Y_s)dV_s + \int_0^t \bar{g}(s, X_s, Y_s)dW_s$$

$$Y_t = \int_0^t h_1(s, Y_s)ds + \int_0^t k(s, Y_s)[h_2(s, X_s, Y_s)ds + dW_s],$$

where the matrix k need not be invertible for all (s, y).

- In this case, the Zakai equation takes the form

$$\pi_t(\varphi) = \pi_0(\varphi) + \int_0^t \pi_s(A_s\varphi)ds + \sum_{j=1}^m \int_0^t \pi_s(B_s^j\varphi)k^{-1}(s, Y_s)(dY_s^j - h_1(s, Y_s)ds)$$

where here B_s^j is as above, but with h replaced by h_2.

- For this Zakai equation, the above uniqueness argument will not work ! Indeed, the solution of the backward PDE would be at each time $0 < t < T$ a function of all the future of the process Y_s for $t \leq s \leq T$. So we cannot write an Itô formula for such $\theta_t \pi_t(u_t)$.

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What is a Backward SDE ?

- For a minute, replace our backward parabolic PDE by an ODE (and assume all processes are one–dimensional). Given the process Y_t (assumed to be a BM), consider the backward ODE

$$z'(t) = -f(Y_t, z(t)), z(T) = z\,.$$

Of course, at each time $0 \leq t < T$, $z(t)$ will be a function of $\{Y_s,\ t \leq s \leq T\}$.

- Can we transform that backward ODE into a backward SDE whose solution would be at each time t adapted to the past of Y ?
- The answer is yes. This is the content of the theory of BSDE. Find a *pair of adapted processes* $(z(t), v(t))$ such that

$$z(t) = z + \int_t^T f(Y_s, z(s))ds + \int_t^T v(s)dY_s\,.$$

The idea is that adding the stochastic integral term $\int_t^T v(s)dY_s$ and allowing to choose freely the process v permits to force the solution to be adapted to the past of Y.

What is a Backward SDE ?

- For a minute, replace our backward parabolic PDE by an ODE (and assume all processes are one–dimensional). Given the process Y_t (assumed to be a BM), consider the backward ODE

$$z'(t) = -f(Y_t, z(t)), z(T) = z\,.$$

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Another example of a BSDE

- Again, all processes are 1–dimensional, and Y_t is a Brownian motion. Let $\xi := \Phi(Y_{[0,T]})$ be square integrable. f being Lipschitz, we are looking for a pair of progressively measurable processes (x_t, v_t) such that
$$x_t = \xi + \int_t^T f(x_s)ds - \int_t^T v_s dY_s,\ 0 \le t \le T\,.$$
- There exists a unique such solution (x_t, v_t) which satisfies
$$\mathbb{E}\left[\sup_{0\le t\le T}|x_t|^2 + \int_0^T |v_t|^2 dt\right] < \infty\,.$$
- Note that x_T is a function of $\{Y_t,\ 0 \le t \le T\}$. For $0 < t < T$, x_t is a function of $\{Y_s,\ 0 \le s \le t\}$, and x_0 is deterministic. In the case $f \equiv 0$, we have that $x_t = \mathbb{E}[\xi|\mathcal{Y}_t]$.

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Backward SPDE

- Assume that $k = I$ $h_1 = 0$, $h_2 = h$. We go back to the original Zakai equation, but with coefficients depending upon Y_t.
- We consider the BSPDE

$$du_t + (A_t u_t + \sum_j [B_t^j v_t^j + i r_t^j B_t^j u_t + i r_t^j v_t^j])dt = \sum_j v_t^j dY_t^j, \ u_T = \varphi\,.$$

which is equivalent to the system of real–valued BSPDEs

$$du_t^1 + (A_t u_t^1 + \sum_j [B_t^j v_t^{1,j} - r_t^j B_t^j u_t^2 - r_t^j v_t^{2,j}])dt = v_t^{1,j} dY_t^j, \ u_T^1 = \varphi;$$

$$du_t^2 + (A_t u_t^2 + \sum_j [B_t^j v_t^{2,j} + r_t^j B_t^j u_t^1 + r_t^j v_t^{1,j}])dt = v_t^{2,j} dY_t^j, \ u_T^2 = 0\,.$$

- Adapting to this system known results for BSPDEs, we can show that if all our coefficients are bounded, together with their derivatives up to order n in x, and φ is smooth, the above system of BSPDEs has a solution such that for $i = 1, 2$, wih $\|\cdot\|_n$ denoting the norm in the Sobolev space H^n, $\mathbb{E}[\sup_{0\le t\le T} \|u_t^i\|_n^2 + \int_0^T \|v^i\|_n^2 dt] < \infty$.

Backward SPDE

- Assume that $k = l\ h_1 = 0$, $h_2 = h$. We go back to the original Zakai equation, but with coefficients depending upon Y_t.
- We consider the BSPDE

$$du_t + (A_t u_t + \sum_j [B^j_t v^j_t + ir^j_t B^j_t u_t + ir^j_t v^j_t])dt = \sum_j v^j_t dY^j_t,\ u_T = \varphi\,.$$

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An ad hoc Itô formula

- From the Zakai equation written in weak form, which gives the semimartingale decomposition of $\pi_t(\varphi)$, we have deduced the semimartingale decomposition of $\pi_t(u_t)$ in case $u \in C^{1,2}$.
- Now we need to develop $\pi_t(u_t)$ in case

$$u(t,x) = u(0,x) + \int_0^t \Sigma(s,x)ds + \sum_j \int_0^t \Lambda^j(s,x)dY_s^j,\ 0 \le t \le T$$

such that the processes $A_t u_t + \Sigma_t + \sum_j B_t^j \Lambda_t^j$ and $B_t^j u_t + \Lambda_t^j$ are $C_b(\mathbb{R}^d)$ valued.
- We have the formula

$$\pi_t(u_t) = \pi_0(u_0) + \int_0^t \pi_s(A_s u_s + \Sigma_s + \sum_j B_s^j \Lambda_s^j)ds$$

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Uniqueness of the Zakai equation using a duality argument with BSPDEs

- We assume that the above assumptions hold for some $n > 2 + d/2$. Then we can show that if u is a solution of the above BSPDE, then

$$d\theta_t \pi_t(u_t) = \theta_t \sum_j \pi_t(B_t^j u_t + v_t^j + i r_t^j u_t) dY_t^j$$

and provided that $\mathbb{E}\left[\sup_{0\le t\le T} \pi_t(1)^2\right] < \infty$, $\{\theta_t \pi_t(u_t),\ 0 \le t \le T\}$ is a martingale

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Theorem

If the coefficients a, f and h are of class C_b^n as functions of x for some $n > 2 + d/2$, then the Zakai equation has a unique solution in the class of $\mathcal{Y}_t$–adapted measure valued processes satisfying for any $T > 0$

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THANK YOU FOR

YOUR ATTENTION !

HAPPY BIRTHDAY

NIKOLAI !