# $L_p$ theory for parabolic equations with local and non-local time derivatives

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# Outlines

- Parabolic equations in divergence form in Sobolev spaces: Lower-order coefficients are not necessarily bounded.
  - Joint work with Seungjin Ryu and Kwan Woo.
- Parabolic equations with fractional time derivatives in Sobolev spaces.
  - Joint work with Hongjie Dong.
- Also introduce some of Krylov's work.

# Elliptic and Parabolic equations

- Elliptic equations

$$a^{ij}D_{ij}u + b^iD_iu + cu = f$$

$$D_i(a^{ij}D_ju + a^iu) + b^iD_iu + cu = D_ig_i + f$$

- Parabolic equations

$$-u_t + a^{ij}D_{ij}u + b^iD_iu + cu = f$$

$$-u_t + D_i(a^{ij}D_ju + a^iu) + b^iD_iu + cu = D_ig_i + f$$

- Solution spaces: Sobolev spaces

$$W_p^2(\Omega) = \{u, Du, D^2u \in L_p(\Omega)\}.$$

# Assumptions and Question

$$-u_t + D_i(a^{ij}D_j u + a^i u) + b^i D_i u + cu = D_i g_i + f$$

- Assumptions on $a^{ij}$: strong ellipticity and boundedness

$$a^{ij}\xi_i\xi_j \geq \delta|\xi|^2,\ |a^{ij}| \leq \delta^{-1} \text{ for } \delta \in (0,1).$$

- Regularity or summability assumptions
  - Some regularity assumptions are needed for $a^{ij}$ for $p \neq 2$.
  - If $a^i$, $b^i$, and $c$ are bounded, no other conditions are needed for $a^i$, $b^i$, and $c$.
- If $a^i$, $b^i$, and $c$ are not bounded, what assumptions do we need?

# Equations in divergence form with unbounded lower-order coefficients

- Ladyzenskaja, Solonnikov, Ural'tseva, 1967 for $p = 2$.

$$-u_t + D_i(a^{ij}D_ju + a^iu) + b^iD_iu + cu = D_ig_i + f$$

- Assumptions: for $n \geq 3$

$$a^i, b^i, |c|^{1/2} \in L_{q,r}(\Omega_T),\ g_i \in L_2(\Omega_T),\ f \in L_{q_1,r_1}(\Omega_T),$$

$$\frac{n}{q} + \frac{2}{r} \leq 1,\ q \in [n, \infty],\ r \in [2,\infty],$$

$$\frac{n}{q_1} + \frac{2}{r_1} \leq 2 + \frac{n}{2},\ q_1 \in \left[\frac{2n}{n+2}, 2\right],\ r_1 \in [1,2].$$

An additional smallness assumption on $a^i, b^i, c$ when $r = \infty$.

# Mixed $L_{p,q}(\Omega_T)$ norm

$$\Omega_T = (0,T) \times \Omega, \quad \Omega \subset \mathbb{R}^n$$

$$f \in L_{p,q}(\Omega_T)$$

$$\|f\|_{p,q} = \|f\|_{L_{p,q}(\Omega_T)} = \left( \int_0^T \left( \int_\Omega |f(t,x)|^p \, dx \right)^{q/p} \right)^{1/q} < \infty.$$

# Observation

**A simple case:** $a^i = c = 0$

$$-u_t + D_i(a^{ij}D_j u) + b^i D^i u = D_i g_i + f,$$

$$u, Du \in L_{p,q}(\Omega_T)$$

➡ $-u_t + D_i(a^{ij}D_j u) = D_i g_i + F$, where $F = f - b^i D_i u$.

- However, it is not possible to have $F \in L_{p,q}(\Omega_T)$ unless $b^i \in L_\infty(\Omega_T)$.
- Thus, we first need to solve equations

$$-u_t + D_i(a^{ij}D_j u) = D_i g_i + f,$$

where $u, Du \in L_{p,q}(\Omega_T)\, f \in L_{p_1,q_1}(\Omega_T)$, $p_1 \le p$, $q_1 \le q$.

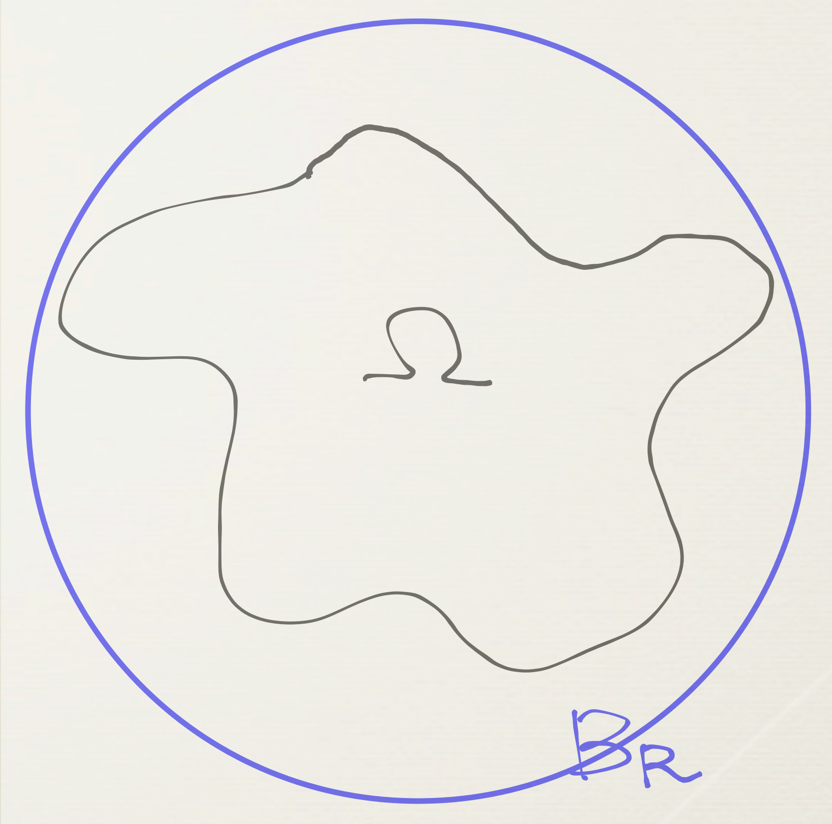
# Elliptic equations with data having a lower Summability

$$D_i(a^{ij}D_ju) = D_ig_i + f, \quad u|_{\partial\Omega} = 0$$

where $u, Du, g_i \in L_p(\Omega), f \in L_{p_1}(\Omega), p_1 \le p \in (1,\infty)$.

- Find $w$ satisfying $\Delta w = fI_\Omega$ in $B_R \supset \Omega$ and set $G_i = D_iw$ so that

$$D_iG_i = f \quad \text{in } \Omega.$$




- $f \in L_{p_1}(\Omega)$ for $p_1 \ge \dfrac{np}{n+p}$, then

$$w \in W_{p_1}(\Omega) \quad \Rightarrow \quad G_i \in L_p(\Omega).$$

➡ $D_i(a^{ij}D_ju) = D_ig_i + f \iff D_i(a^{ij}D_ju) = D_ig_i + D_iG_i$

# Elliptic case

$$D_i(a^{ij}D_ju + a^iu) + b^iD^iu = D_ig_i + f \text{ in } W_p^1(\Omega)$$

➡ $D_i(a^{ij}D_ju) = D_i(g_i - a^iu) + f - b^iD^iu,$

where we need $b^iD^iu \in L_{p_1}(\Omega)$ for $p_1 \geq \dfrac{np}{n+p}$.

➡ Because $D_iu \in L_p(\Omega)$, we need $b^i \in L_r(\Omega)$,

where $r \in [n, \infty)$ and $r > p_1 = \dfrac{np}{n+p}$, $p \in (r/(r-1), \infty)$.

In particular, $r \geq 2$ if $n = 2$.

# Some reference for elliptic equations in divergence form

- Stampacchia 1965, Ladyzhenskaya, Ural'tseva 1968: $p = 2$, $a^i, b^i \in L_r(\Omega)$, $c \in L_{r/2}(\Omega)$, where $r \in [n, \infty)$ for $n \geq 3$ and $r \in (2,\infty)$ for $n = 2$, $c$ has an additional assumption.
- Trudinger, 1973: General degenerate divergence form operators, $p = 2$, $D_i a^i + c \leq 0$. Weak maximum principle.
- H. Kim and Y.-H. Kim, 2015: $a^i = c = 0$, Laplace operator, $W^1_p$ estimate.

  $$b = (b^1, \ldots, b^n) = b_1 + b_2, \operatorname{div} b_1 = 0, b_1 \in L_n,$$

  $$b_2 \in L_r, \text{ where } r \geq n \text{ for } n \geq 3 \text{ and } r > 2 \text{ for } n = 2.$$

- Hyunwoo Kwon, 2020. $n = r = 2$ and $p > 2$ ($a^i = 0$) for an elliptic operator which can be converted to a non-divergence type equation.

# Aleksandrov's estimate

- Krylov 2021: Aleksandrov's estimate $u(x) \le N\|f_-\|_{L_{n_0}(\Omega)}$ for $u$ satisfying

$$a^{ij}D_{ij}u + b^iD_iu + cu = f, \quad u|_{\partial\Omega} = 0$$

where $b_n \in L_n$, $c \le 0$, $f \in L_{n_0}$ with $n_0 < n$. (Previously, $f \in L_n$)

- Krylov 2021: Elliptic case, $b \in L_n(\Omega)$, $c \in L_q(\Omega)$, $f \in L_p(\Omega)$ with $1 < p < n$, more precisely,

$$n/2 < q < n, \quad p \le q$$

or

$$q = n/2, \quad 1 < p < n/2, \quad n \ge 3.$$

- Krylov also considered Morrey classes for $b^i$.

# N. V. Krylov's maximum principle
# (Parabolic version of Alekandrov's estimate)

$$-u_t + a^{ij}D_{ij}u + b^i D_i u = f$$

- Krylov 1976: $b \in L_\infty, f \in L_{n+1}$
- Nazarov, Ural′tseva 1985, K. Tso 1985:

$$b \in L_{n+1}, \ \ f \in L_{n+1}.$$

- Krylov 2021: $b \in L_{n+1}$ $(b \in L_{n_0+1}), f \in L_{n_0+1}$,

  where $n_0 < n$ with an additional condition on $b$.

  Mixed norms.

# Some reference for parabolic equations with unbounded coefficients in divergence form

- Ladyzenskaja, Solonnikov, Uralceva 1967: $p = 2$.

- Nazarov and Uraltseva 2011: Qualitative properties (the maximum principle, the Harnack inequality, and the Liouville theorem) of solutions of parabolic (and elliptic) equations with unbounded lower-order coefficients.

- There are some results by Krylov (2021) for non-divergence type parabolic equations with unbounded coefficients, where the lower-order terms are not treated as perturbation terms.

# Main results

$$-u_t + D_i(a^{ij}D_j u + a^i u) + b^i D^i u + cu = D_i g_i + \sum_{k=1}^{m} f_k$$

- $p, q \in (1,\infty)$, $u, Du \in L_{p,q}(\Omega_T)$
- $a^i \in L_{\ell_1,r_1}(\Omega_T)$, $b^i \in L_{\ell_2,r_2}(\Omega_T)$, $c \in L_{\ell_3,r_3}(\Omega_T)$
- $g_i \in L_{p,q}(\Omega_T)$, $f_k \in L_{p_k,q_k}(\Omega_T)$,

Then, the above equation is solvable whenever

$$-u_t + D_i(a^{ij}D_j u) = D_i g_i + f$$

and its dual problem is solvable when $g_i, f \in L_{p,q}(\Omega_T)$ so that $u, Du \in L_{p,q}(\Omega_T)$ ($a^{ij}$ need an appropriate regularity condition).

- Boundary condition: Dirichlet or conormal derivative condition.

# Conditions on the coefficients

For simplicity, assume that $a^i = c = 0$.

- $b^i \in L_{\ell_2,r_2}(\Omega_T)$, where $\ell_2 \in (n,\infty]$ and $r_2 \in [2,\infty)$ such that

$$\frac{n}{\ell_2} + \frac{2}{r_2} \le 1.$$

- $$\begin{cases} \dfrac{p}{p-1} \le \ell_2 \quad \text{and} \quad \dfrac{q}{q-1} \le r_2, \\ \dfrac{p}{p-1} \le \ell_2 \quad \text{and} \quad \dfrac{q}{q-1} < r_2 \quad \text{if} \quad \dfrac{n}{\ell_2} + \dfrac{2}{r_2} = 1. \end{cases}$$

- $\ell_2 = n$, then $r_2 = \infty$. In this case $\|b^i\|_{L_{\ell_2,r_2}}$ needs to be small.

# Solution to parabolic equations in divergence form

$u$ is said to satisfy

$$-u_t + D_i(a^{ij}D_j u + a^i u) + b_i D_i u + cu = D_i g_i + f$$

if for any $\varphi \in C_0^\infty((0,T)\times\Omega)$, we have

$$\int_{\Omega_T} \left( u\varphi_t - a^{ij}D_j u D_i\varphi - a^i u D_i\varphi + b^i D_i u\varphi + cu\varphi \right)\, dx\, dt = \int_{\Omega_T} \left( f\varphi - g_i D_i\varphi \right)\, dx\, dt$$

- $u \in \mathscr{H}_p^1(\Omega_T) = \{u, Du \in L_p(\Omega_T), u_t \in \mathbb{H}_p^{-1}(\Omega_T)\}$ for $g_i, f \in L_p(\Omega_T)$.
  - $v \in \mathbb{H}_p^{-1}(\Omega_T)$ if there exist $g_i, f \in L_p(\Omega_T)$ such that $v = D_i g_i + f$ in the distribution sense.
- $u \in H_p^1(0,T; H_p^{-1}(\Omega)) \cap L_p(0,T; H_p^1(\Omega))$

# To prove main results

## A simple case: $a^i = c = 0$

$$-u_t + D_i(a^{ij} D_j u) + b^i D^i u = D_i g_i + f$$

➡ $-u_t + D_i(a^{ij} D_j u) = D_i g_i + F$, where $F = f - b^i D_i u$.

- However, it is not possible to have $F \in L_{p,q}(\Omega_T)$ unless $b^i \in L_\infty(\Omega_T)$.
- Thus, we first solve equations

$$-u_t + D_i(a^{ij} D_j u) = D_i g_i + \sum_{k=1}^{m} f_k,$$

where $u, Du \in L_{p,q}(\Omega_T)\, f_k \in L_{p_k,q_k}(\Omega_T)$, $p_k \le p,\ q_k \le q$.

# Equations with the right-hand side having lower summability

$$-u_t + D_i(a^{ij}D_j u) = D_i g_i + \sum_{k=1}^{m} f_k,$$

$$u, Du \in L_{p,q}(\Omega_T),\ g \in L_{p,q}(\Omega_T),\ f_k \in L_{p_k,q_k}(\Omega_T),\ p_k < p,\ q_k < q.$$

Not as easy as the elliptic case!

- For each $t \in [0,T]$, find $\operatorname{div}\Phi(t,x) = f(t,x)$: no increase in time summability! Thus, not applicable if $q_k < q$.
- Find $w \in W^{1,2}_{p_k,q_k}(\Omega_T)$ such that $f = \Delta w - w_t$ in $\Omega_T$. Then write

$$(u-w)_t + D_i(a^{ij}D_j(u-w)) = D_i(g_i + D_i w - a^{ij}D_j w)$$

and solve for $u - w$: If the domain is not good enough, then one cannot solve the non-divergence type equation $f = \Delta w - w_t$.

# Equations with the right-hand side having lower summability

$$-u_t + D_i(a^{ij}D_ju) = D_ig_i + \sum_{k=1}^{m} f_k,$$

where $p, q \in (1,\infty)$, $u, Du \in L_{p,q}(\Omega_T)$, $f_k \in L_{p_k,q_k}(\Omega_T)$, $p_k \leq p$, $q_k \leq q$.

- What are possible $(p_k, q_k)$?

  Non-divergence $-u_t + \Delta u = D_ig_i + f$, where $D_ig_i, f \in L_{p_k,q_k}(\Omega_T)$.

➡ $u \in W^{1,2}_{p_k,q_k}(\Omega_T) = \{u, Du, D^2u, u_t \in L_{p_k,q_k}(\Omega_T)\}$.

➡ To have $Du \in L_{p,q}(\Omega_T)$, we need $1 + \frac{n}{p} + \frac{2}{q} \geq \frac{n}{p_k} + \frac{2}{q_k}$.

# Equations with the right-hand side having lower summability

$$-u_t + D_i(a^{ij} D_j u) = D_i g_i + \sum_{k=1}^{m} f_k$$

in $\Omega_T$ with the Dirichlet or conormal derivative boundary condition using the duality argument and a parabolic Sobolev embedding.

- $1 + \frac{n}{p} + \frac{2}{q} \geq \frac{n}{p_k} + \frac{2}{q_k}, \, p_k \in [1,p], \, q_k \in [1,q]$
  - $(p_k, q_k, p) \neq (1, q, n/(n-1))$ for $n \geq 2$.
  - $q_k > 1 \quad$ if $\quad 1 + \frac{n}{p} + \frac{2}{q} = \frac{n}{p_k} + \frac{2}{q_k}.$

# Parabolic Sobolev embedding

$$-u_t + D_i(a^{ij}D_ju) = D_ig_i + \sum_{k=1}^{m} f_k$$

➡ $u, Du \in L_{p,q}(\Omega_T)$,

$$u_t = D_i(a^{ij}D_ju - g_i) - \sum_{k=1}^{m} f_k = D_iG_i + F_k,$$

That is, we need to have a Sobolev embedding for $u$ such that

$$u, Du \in L_{p,q}(\Omega_T),\ u_t = D_iG_i + F_k,$$

where $G_i \in L_{p,q}(\Omega_T)$, $F_k \in L_{p_k,q_k}(\Omega_T)$.

# Parabolic Sobolev embedding

$u, Du \in L_{p,q}(\Omega_T)$, $u_t = D_i G_i + F_k$, $G_i \in L_{p,q}(\Omega_T)$, $F_k \in L_{p_k,q_k}(\Omega_T)$.

Then, for $(p_0, q_0)$, we have

$$\|u\|_{p_0,q_0} \le N\left( \| |u| + |Du| + |g| \|_{p,q} + \sum_{k=1}^{m} \|f_k\|_{p_k,q_k} \right),$$

where

- If $q_0 = q$, then $\dfrac{n}{p} \le 1 + \dfrac{n}{p_0}, \quad (p, p_0) \neq (n(\ge 2), \infty)$.
  - $(p_k, q_k)$ are arbitrary.

# Parabolic Sobolev embedding

$$\|u\|_{p_0,q_0} \le N\left( \| |u| + |Du| + |g| \|_{p,q} + \sum_{k=1}^{m} \|f_k\|_{p_k,q_k} \right)$$

- If $q_0 > q$, then $\frac{n}{p} + \frac{2}{q} \le 1 + \frac{n}{p_0} + \frac{2}{q_0}$ and

  $1 < q < q_0 < \infty$ if $\frac{n}{p} + \frac{2}{q} = 1 + \frac{d}{p_0} + \frac{2}{q_0}$

  - $(p_k, q_k)$ are real numbers such that

    $p_k \in [1,p_0]$, $q_k \in [1,q_0]$, $\frac{n}{p_k} + \frac{2}{q_k} \le 2 + \frac{n}{p_0} + \frac{2}{q_0}$,

    $1 = q_k < q_0 = \infty$ or $1 < q_k < q_0 < \infty$ if $\frac{n}{p_k} + \frac{2}{q_k} = 2 + \frac{n}{p_0} + \frac{2}{q_0}$.

# Remark

- $H_p^1(0,T;H_q^{-1}(\Omega)) \cap L_p(0,T;H_q^1(\Omega))$, but if $u_t = D_i g_i + \sum_{k=1}^m f_k$, then $u \in H_p^1(0,T;H_q^{-1}(\Omega)) \cap L_p(0,T;H_q^1(\Omega))$?
- If is easy if $\Omega = \mathbb{R}^n$ because if $u, Du \in L_p(\Omega_T)$ and $u_t = D_i g_i + f$, where $g_i, f \in L_p(\Omega_T)$, then

$$(1-\Delta)^{-1/2} u \in W_p^{1,2}(\Omega_T)$$

  - When the Dirichlet boundary condition is considered, I just extended $u$ to be zero outside the domain and used the embedding for $\Omega = \mathbb{R}^n$, just as in the elliptic case $u \in W_p^1(\Omega)$ with $u|_{\partial\Omega} = 0$, then one can use embeddings for the zero extension of $u$ in $\mathbb{R}^n$. However, it is wrong!

# Remark

- In the elliptic case, if $u \in W_p^1(\Omega)$ and $u\,|_{\partial\Omega} = 0$, then $\bar{u} \in W_p^1(\mathbb{R}^n)$, where $\bar{u} = \begin{cases} u & x \in \Omega, \\ 0 & x \in \Omega^c . \end{cases}$

- If $u \in \mathscr{H}_p^1(\Omega_T) = \{u \in L_p(\Omega_T), u, Du, u_t \in \mathbb{H}_p^{-1}(\Omega_T)\}$ with zero lateral boundary condition, then the zero extension $\bar{u}$, where

$$\bar{u}(t,x) = \begin{cases} u(t,x) & \text{for } (t,x) \in (0,T) \times \Omega, \\ 0 & \text{for } (t,x) \in (0,T) \times \Omega^c, \end{cases}$$

is in the same class of functions? No!

# Parabolic Sobolev embedding

$$\|u\|_{p_0,q_0} \leq N\left( \| |u| + |Du| + |g| \|_{p,q} + \sum_{k=1}^{m} \|f_k\|_{p_k,q_k} \right)$$

- Mollifications
- Reifenberg flat domain
- Remark about an endpoint case ($q = 1$) needs additional restrictions!
    - For $q_0 = \infty$, see Alkhutov and Gordeev 2008, Agresti, Lindemulder, Veraar 2021.

# Proof of the main results

$$-u_t + D_i(a^{ij}D_ju) + b^iD^iu = D_ig_i + f$$

- Move the lower-order terms to the right-hand side.
- Use the solvability of equations without lower-order terms, but with the right-hands side having lower summability.
- Parabolic Sobolev embedding to write

$$\|b_iD_u\|_{p_k,q_k} \leq \|b^i\|_{\ell_2,r_2}\|Du\|_{p,q},$$

  then absorb to the left-hand side of $\|b^i\|_{\ell_2,r_2}$ is small.

- Divide the time interval into short intervals. This is why we do not deal with $\|b^i\|_{n,\infty}$ unless this is sufficiently small.

# Parabolic equations with non-local time derivative

- Parabolic equations with local time derivative
  - $-u_t + a^{ij}D_{ij}u = f,$
  - $-u_t + D_i(a^{ij}D_ju) = D_ig_i + f$
- Parabolic equations with non-local time derivative
  - $-\partial_t^\alpha u + a^{ij}D_{ij}u = f,$
  - $-\partial_t^\alpha u + D_i(a^{ij}D_ju) = D_ig_i + f$
- Bounded lower-order coefficients or they are zero.
- Anomalous diffusions

# Fractional time derivative

- $\alpha$-th integral of $u$

$$I^\alpha u(t,x) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}u(s,x)\,ds$$

- Caputo fractional derivative of order $\alpha \in (0,1)$

$$\begin{aligned}\partial_t^\alpha u &= \partial_t I_0^{1-\alpha}(u(\,\cdot\,,x) - u(0,x)) \\ &= \frac{1}{\Gamma(1-\alpha)}\partial_t \int_0^t (t-s)^{-\alpha}[u(s,x) - u(0,x)]\,ds\end{aligned}$$

  - 0 integral then 1 derivative = 1 derivative.
  - $1-\alpha$ integral then 1 derivative = $\alpha$ derivative.

# Objective

For the usual elliptic and parabolic equations

- $a^{ij} D_{ij} u = f, \quad D_i(a^{ij} D_j u) = D_i g_i + f$
- $-u_t + a^{ij} D_{ij} u = f, \quad -u_t + D_i(a^{ij} D_j u) = D_i g_i + f$

there are many unique solvability results in Sobolev spaces when the coefficients $a^{ij}$ are very irregular.

We would like to deal with irregular coefficients as those for the usual parabolic equations in Sobolev spaces for time fractional parabolic equations.

- $a^{ij} = a^{ij}(t, x_1, x')$ is measurable in $(t, x_1)$ and has small mean oscillations in $x'$

# Equations with rough $a^{ij}$

## Counterexamples

$$a^{ij}D_{ij}u = f, \qquad -u_t + a^{ij}D_{ij}u = f$$

- There are counterexamples when $a^{ij}(x)$ have no regularity assumptions.
  - Meyers 1963, Piccinini and Spagnolo 1972, divergence, $p \neq 2$
  - Ladyzhenskaya and N. N. Ural′ceva, 1973, non-divergence
  - Dong and K. 2014
  - Parabolic case: Krylov 2016. $n = 1$, $a^{ij} = a^{ij}(t, x)$. $p \in (1,3/2) \cup (3,\infty)$

# Brief history

$$a^{ij}D_{ij}u = f, \qquad -u_t + a^{ij}D_{ij}u = f$$

- $a^{ij}$ have vanishing mean oscillations (VMO coefficients)

$$⨍_{B_r(x)} \left| a^{ij}(y) - ⨍_{B_r(x)} a^{ij}(y) \right| dx \to 0 \text{ as } r \searrow 0$$

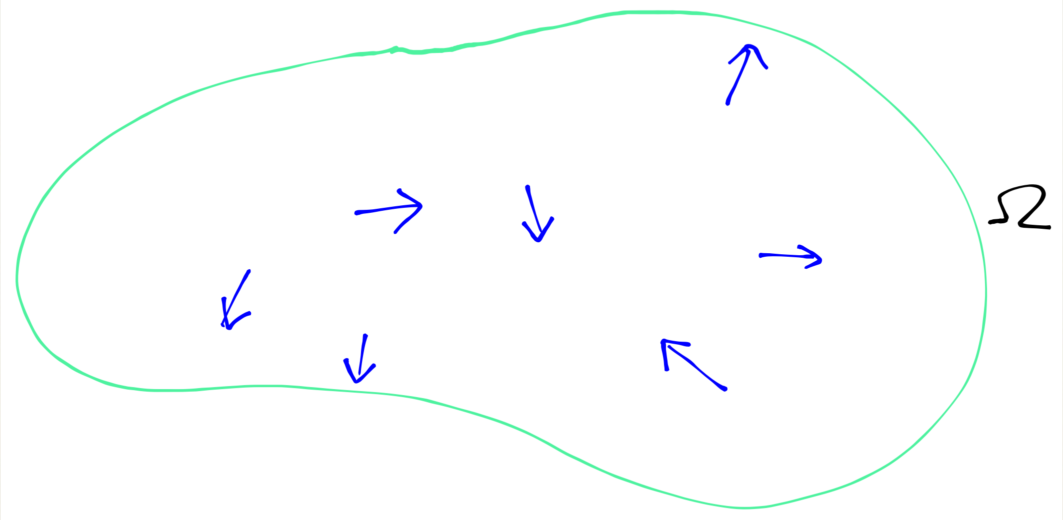
- F. Chiarenza, M. Frasca, and P. Longo 1991, elliptic.
- M. Bramanti and M. C. Cerutti 1993, parabolic.
- Di Fazio 1996
- Byun and Wang 2004, BMO (small bounded mean oscillation)

# Brief history

- Krylov 2007, $a^{ij} = a^{ij}(t, x)$: measurable in time and VMO in $x$.
  - Mean oscillation estimates for perturbation.
- Krylov and K. 2007, $a^{ij}(x_1, x')$ is measurable in $x_1$ and VMO in $x' \in \mathbb{R}^{n-1}$. Elliptic.
  - G. Chiti 1976, $p = 2$
- There are many results for equations with rough coefficients. For instance, $a^{ij}(t, x_1, x')$: measurable in $(t, x_1)$ and VMO in $x'$, where $x_1 \in \mathbb{R}$ and $x' \in \mathbb{R}^{d-1}$.

# Variably partially VMO coefficients

- Krylov 2009, variably partially VMO elliptic, non-divergence type, $p > 2$.
  - $a^{ij}(x)$ can be measurable in one direction (measurable direction), but the direction can differ depending on $x$.




  - There appeared many results for equations with such coefficients (also called, for instance, $(\sigma, R)$-vanishing of codimension 1), but Krylov's coefficients are more general (the involved map is not necessarily linear).

# Some references for fractional equations

- Zacher 2005, 2009, 2013: Non-divergence type fractional parabolic and Voltera type equation

$$u(t,x) + \int_0^t a(t-s)Au(s,x)\,ds = f(t,x),$$

where $A$ is a time independent operator. Divergence type fraction parabolic equations in the Hilbert space setting, De Giorgi–Nash type theorem

- M. Allen, L. Caffarelli, and A. Vasseur, 2016: De Giorgi-Nash-Moser type Holder estimates for parabolic equations with fractional operators in both $t$ and $x$.

- I. Kim, K.-H. Kim, S. Lim, 2017: $a^{ij}(t,x)$ are continuous in $x$ and piecewise continuous in $t$. $\alpha \in (0,2)$. $L_q(L_p)$-estimates.

# Main results

- Dong and K. 2019

  $a^{ij}(t,x)$ is measurable in $t$ and has small mean oscillations in $x$, then, for $f \in L_p(\mathbb{R}^n_T)$ there exists a unique solution $u$ satisfying

$$-\partial_t^\alpha u + a^{ij} D_{ij} u = f$$

  in $\mathbb{R}^n_T = (0,T) \times \mathbb{R}^n$.

  The initial condition is zero.

- There are other results for coefficients measurable in one variable as well as for divergence type equations. Weights are also considered.

# Solution spaces as vector valued functions

For non-divergence time fractional parabolic equations ($\alpha \in (0,1)$), for instance, Zacher 2006 considers

$$H_p^\alpha((0,T), L_p(\mathbb{R}^n)) \cap L_p((0,T), H_p^2(\mathbb{R}^n))$$

- When the initial trace is zero, ${}_0H_p^\alpha((0,T), L_p(\mathbb{R}))$ in place of $H_p^\alpha((0,T), L_p(\mathbb{R}^n))$

- For $\alpha \in (0,1/p)$, the initial trace does not make sense. In particular, $H_p^\alpha((0,T), L_p(\mathbb{R}^n)) = {}_0H_p^\alpha((0,T), L_p(\mathbb{R}^n))$.

- For $\alpha > 1/p$, the initial trace is well defined.

- When the initial is non-zero, Zacher 2006 considers $H_p^\alpha((0,T), L_p(\mathbb{R}^n)) \cap L_p((0,T), H_p^2(\mathbb{R}^n))$ except $\alpha = 1/p$.

# Solution spaces

What is an appropriate space for fractional parabolic equations including the case $\alpha = 1/p$?

- When $p = 2$, $\alpha = 1/2$, Kubica, Ryszewska, Yamamoto 2020 (if functions have only the time variable) has
  - $H_2^\alpha(0,T)$ for $\alpha \in (0,1/2)$
  - $\left\{ u : u \in H_2^\alpha(0,T), \int_0^T |u(t)|^2 t^{-1}\, dt < \infty \right\}$ for $\alpha = 1/2$
  - $\{u \in H_2^\alpha(0,T) : u(0) = 0\}$ for $\alpha \in (1/2,1)$

  as solution spaces with zero initial conditions.

# Solution spaces $\mathbb{H}_p^{\alpha,0}(\Omega_T)$, $\mathscr{H}_p^{\alpha,-1}(\Omega_T)$

- $u \in \mathbb{H}_{p,0}^{\alpha,0}(\Omega_T)$ if there exists $f \in L_p(\Omega_T)$ such that

$$\int_{\Omega_T} I^{1-\alpha} u \, \varphi_t \, dx \, dt = - \int_{\Omega_T} f \varphi \, dx \, dt$$

for all $\varphi \in C_0^\infty\left([0,T) \times \Omega\right)$. That is, $\partial_t I^{1-\alpha} u = f$.

If $u \in \mathbb{H}_{p,0}^{\alpha,0}(\Omega_T)$, then in some sense $u(0,x) = 0$.

- $u \in \mathbb{H}_p^{\alpha,0}(\Omega_T)$ if there exists $u_0 = u_0(x) \in L_p(\Omega)$ such that

$$u - u_0 \in \mathbb{H}_{p,0}^{\alpha,0}(\Omega_T).$$

# Solution spaces $\mathbb{H}_p^{\alpha,0}(\Omega_T)$, $\mathscr{H}_p^{\alpha,-1}(\Omega_T)$

- $u \in \mathbb{H}_p^{\alpha,0}(\Omega_T)$ if there exists $u_0 = u_0(x) \in L_p(\Omega)$ such that

$$u - u_0 \in \mathbb{H}_{p,0}^{\alpha,0}(\Omega_T).$$

  - For $\alpha \in (0,1/p)$, $\mathbb{H}_{p,0}^{\alpha,0}(\Omega_T) = \mathbb{H}_p^{\alpha,0}(\Omega_T)$.
  - For $\alpha \in [1/p,1)$, if $u \in \mathbb{H}_p^{\alpha,0}(\Omega_T)$, then there exists a unique $u_0(x) \in L_p(\Omega_T)$ such that $u - u_0 \in \mathbb{H}_{p,0}^{\alpha,0}(\Omega_T)$.

- Similarly, we define $\mathscr{H}_p^{\alpha,-1}(\Omega_T)$, but using $H_p^{-1}(\Omega)$ instead of $L_p(\Omega)$.

- If $\alpha \in (1/p,1)$, then the initial traces belong to Besov type spaces as in Agresti, Lindemulder, Veraar 2021.

# Solution spaces for fractional parabolic equations

- Non-divergence case
  - $\mathbb{H}_p^{\alpha,2}(\Omega_T) = \{u \in L_p(\Omega_T) : u \in \mathbb{H}_p^{\alpha,0}(\Omega_T), Du, D^2u \in L_p(\Omega_T)\}$
  - $\mathbb{H}_{p,0}^{\alpha,2}(\Omega_T) = \{u \in L_p(\Omega_T) : u \in \mathbb{H}_{p,0}^{\alpha,0}(\Omega_T), Du, D^2u \in L_p(\Omega_T)\}$
- Divergence case
  - $\mathscr{H}_p^{\alpha,1}(\Omega_T) = \{u \in L_p(\Omega_T) : u \in \mathscr{H}_p^{\alpha,-1}(\Omega_T), Du \in L_p(\Omega_T)\}$
  - $\mathscr{H}_{p,0}^{\alpha,1}(\Omega_T) = \{u \in L_p(\Omega_T) : u \in \mathscr{H}_{p,0}^{\alpha,-1}(\Omega_T), Du \in L_p(\Omega_T)\}$

# Approach

$$-\partial_t^\alpha u + a^{ij} D_{ij} u = f$$

- We try to obtain mean oscillation estimates for $D^2 u$,

$$⨍_{Q_r} |D^2 u - (D^2 u)_{Q_r}| \, dx \, dt,$$

  where $Q_r(t_0, x_0) = (t_0 - r^{2/\alpha}, t_0) \times B_r(x_0)$ and $(f)_{Q_r}$ is the average of $f$ over $Q_r$.

- $u = w + v$, where $v$ satisfies $-\partial_t^\alpha v + a^{ij} D_{ij} v = 0$ in $Q_r$.

- If $\alpha = 1$, we have a sufficiently regularity for $D^2 v$.

- However, for $\alpha \in (0,1)$, it is only possible to obtain some increase of summability of $D^2 v$.

# Approach

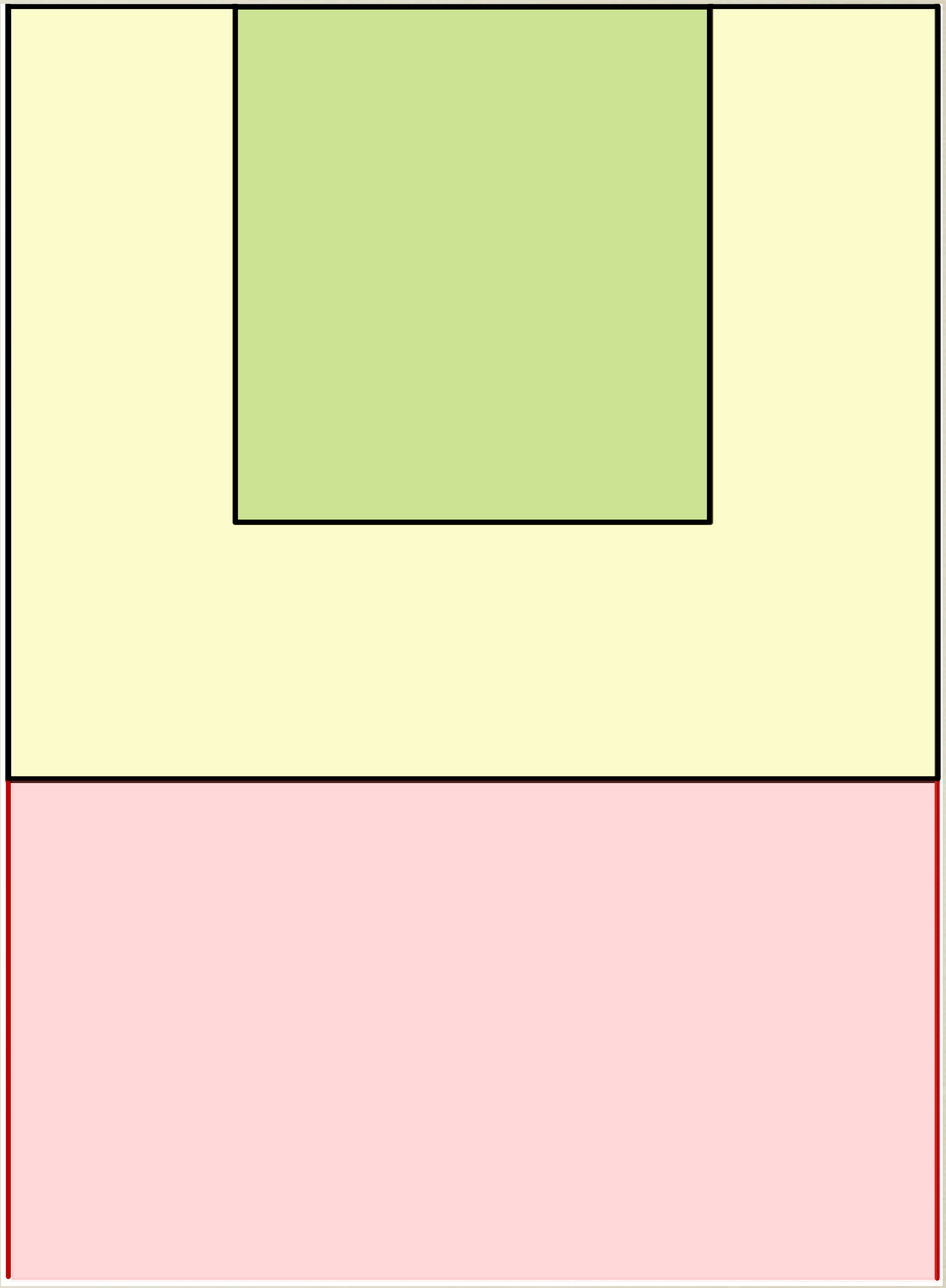
- Due to the non-local time derivative

  $$\partial_t^\alpha v \sim \partial_t \int_0^t (t-s)^{-\alpha} v(s,x)\, ds$$

  for $v(0,x) = 0$,

  it is not possible to obtain a sufficient regularity for solutions to homogeneous equations.

- Consider homogenous equations not on $Q_r(t_0, x_0)$, but on infinite cylinder $(-\infty, t_0) \times B_r(x_0)$.



# New decomposition

$$-\partial_t^\alpha u + a^{ij} D_{ij} u = f$$

- $u = w + v$, not on a cylinder $(t_0 - r^2, t_0) \times B_r(x_0)$, but $(-\infty, t_0) \times B_r(x_0)$.

- More precisely, since the time fractional equation is dealt with on $\Omega_T = (0,T) \times \Omega$, we consider $w, v$ on $(0,t_0) \times B_r(x_0)$ for $t_0 \in [0,T]$ and extend to be zero for $(-\infty,0) \times B_r(x_0)$.

- $\left( |D^2 v|^{p_1} \right)^{1/p_1}_{Q_{r/2}(t_1,0)} \leq N \sum_{j=1}^{\infty} j^{-(1+\alpha)} \left( |D^2 v|^{p_0} \right)^{1/p_0}_{Q_r(t_1-(j-1)r^{2/\alpha},0)}.$

# Modified mean oscillation estimates

- Then we deal with $w$ as well on $(-\infty, t_0) \times B_r(x_0)$.
- $\left( |D^2 w|^{p_0} \right)^{1/p_0}_{Q_{1/2}(t_1,0)} \le \sum_{k=0}^{\infty} c_k (|f|^{p_0})^{1/p_0}_{(s_{k+1},s_k)\times B_1}$.
- Combining the above two estimates, we have

$$\left( |D^2 u - (D^2 u)_{Q_{\kappa r}(t_0,x_0)}| \right)_{Q_r(t_0,x_0)} \le N\kappa^{-\sigma} (\mathcal{S}\mathcal{M}|D^2 u|^{p_0})^{1/p_0}(t_0, x_0)$$

$$+N\kappa^{(d+2/\alpha)/p_0} \sum_{k=0}^{\infty} c_k \left( |f|^{p_0} \right)^{1/p_0}_{\left(t_0-(2^{k+2}-2)(\kappa r)^{2/\alpha}, t_0\right)\times B_{\kappa r}(x_0)}$$

- This approach is also applicable to the usual parabolic equation.

# Thank you