A probabilistic formula for the gradient of solutions of some Hypoelliptic Dirichlet problems

Giuseppe Da Prato (*Scuola Normale Superiore)*
and
Luciano Tubaro *(Università di Trento)*

Edinburgh, June 23, 2022

Introduction and setting of the problem

Consider the following problem in $H = \mathbb{R}^d$,

$$\begin{cases} D_t u(t,x) = \frac{1}{2}\,\mathrm{Tr}\,[C D_x^2 u(t,x)] + \langle Ax, D_x u(t,x)\rangle, \quad t \geq 0,\ x \in \mathbb{R}^d, \\ u(0,x) = \varphi(x) \in \mathbb{R}^d \end{cases} \tag{1}$$

where A and C are $d \times d$ matrices and C is semi–definite positive. We shall assume that

$$\det\left(\int_0^T e^{tA}\, C\, e^{tA^*}\, dt\right) > 0, \quad \forall\, T > 0. \tag{2}$$

Therefore the operator above is hypoelliptic.
We are mainly interested in the case when, besides (2), we have $\det C = 0$; in this case the problem is degenerate.

It is well known that, under assumption (2), problem (1) has a unique C^∞ strict solution, given by

$$u(t,x) = \mathbb{E}[\varphi(X(t,x))], \quad t > 0,\ x \in \mathbb{R}^d, \qquad (3),$$

where $X(t,x)$ is the Ornstein–Uhlenbeck process

$$X(t,x) = e^{tA}x + \int_0^t e^{(t-s)A}\sqrt{C}\,dW(s) =: e^{tA}x + W_A(t),$$

and W is an $\mathbb{R}^d$–valued Wiener process on a probability space

$$(\Omega, \mathscr{F}, \mathbb{P}).$$

This talk is devoted to the regularity of the solution to the following Cauchy–Dirichlet problem,

$$\begin{cases} D_t(t,x) = \frac{1}{2}\operatorname{Tr}[CD_x^2 u(t,x)] + \langle Ax, D_x u(t,x)\rangle, \quad t>0,\ x\in\mathscr{O}_r, \\ u(t,x)=0, \quad t>0,\ x\in\partial\mathscr{O}_r, \\ u(0,x)=\varphi(x), \quad x\in\mathscr{O}_r, \end{cases} \tag{5}$$

where $\mathscr{O}_r$ is an open, bounded subset of $\mathbb{R}^d$ of the form $\mathscr{O}_r = \{g < r\}$, g being regular and convex, and $\varphi \in B_b(\mathbb{R}^d)$ is bounded and Borel.

A solution of problem (5) is provided by the probabilistic formula

$$u(T,x) = \mathbb{E}\left[\varphi(X(T,x))\,\mathbb{1}_{T\le\tau_x}\right], \quad T>0,\ x\in\overline{\mathscr{O}_r},$$

where τ_x is the exit time of $X(\cdot,x)$ from $\overline{\mathscr{O}_r}$,

$$\tau_x = \inf\{s\in[0,T]:\ e^{sA}x + W_A(s) \in \overline{\mathscr{O}_r}^c\}.$$

In other words,

$$u(T,x) := \mathbb{R}_T^{\mathscr{O}_r}\varphi(x) = \qquad (5)$$

$$= \int_{\{g(e^{sA}x+W_A(s))\le r,\ \forall\, 0\le s\le T\}} \varphi(e^{TA}x + W_A(T))\, d\mathbb{P}.$$

By this formula it is hard to show the existence of the gradient of $u(T,x)$ up to the boundary of $\mathscr{O}_r$.

By using suitable transformations of identity (5) we shall prove, however, the existence of

$$D_x R_T^{\mathscr{O}_r}\varphi$$

up to the boundary as well as an explicit formula when φ is just Borel and bounded on $\overline{\mathscr{O}_r}$

Some references

Regularity of $u(T,x)$ at the interior of $\mathscr{O}_r$ was proved by

Dynkin, Markov processees, Springer 1965.

Regularity up to the boundary for general hypoelliptic operators has been proved by

Cattiaux, Bull. Sci. Math. (I) and (II), 90-91

using Malliavin calculus; but his results do not cover problem (5), because he requires $\det C > 0$.

We believe that our method should work for more general equations of the form

$$D_t u(t,x) = \frac{1}{2}\,\mathrm{Tr}\,[C D_x^2 u(t,x)] + \langle Ax + b(x), D_x u(t,x)\rangle.$$

where $b : \mathbb{R}^d \to \mathbb{R}^d$ is nonlinear and regular.

Example

Let $d = 2$, $x = (x_1, x_2)$, $\mathscr{O} = B_1 = \{x \in \mathbb{R}^2 : |x| < 1\}$.

$$\begin{cases} D_t u(t, x_1, x_2) = \frac{1}{2} D_{x_1}^2 u(t, x_1, x_2) + x_1 D_{x_2} u(t, x_1, x_2) \\ u(t, x) = 0, \quad |x| = 1, \\ u(0, x) = \varphi(x), \quad x = (x_1, x_2) \in \mathbb{R}^2. \end{cases}$$

Then $A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $C = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. *Consequently*

$$Q_t = \int_0^t e^{sA} C e^{sA^*} ds = \int_0^t \begin{pmatrix} 1 & s \\ s & s^2 \end{pmatrix} ds = \begin{pmatrix} t & t^2/2 \\ t^2/2 & t^3/3 \end{pmatrix}.$$

Therefore $\det C = 0$ *but* $\det Q_t > 0$ *for any* $t > 0$.

The main result

We fix $T > 0$ and set $u(T, x) = R_T^{\mathscr{O}_r}\varphi(x)$. Then we start by the previous identity

$$R_T^{\mathscr{O}_r}\varphi(x) = \int_{\{g(e^{sA}x+W_A(s))\le r,\,\forall\, 0\le s\le T\}} \varphi(e^{TA}x + W_A(T))\, d\mathbb{P}. \tag{5}$$

We make the change of variables

$$\Omega \to X = L^2(0, T; \mathbb{R}^d), \quad \omega \to W_A(\cdot)(\omega),$$

$$R_T^{\mathscr{O}_r}\varphi(x) = \int_{\{g(e^{sA}x+h(s))\le r,\,\forall\, 0\le s\le T\}} \varphi(e^{TA}x + h(T))\, N_{\mathbb{Q}_T}(dh), \tag{6}$$

where $N_{\mathbb{Q}_T}$ is the law of $W_A(\cdot)$ in X, which is Gaussian as well known.

To show that $R_T^{\mathscr{O}_r}\varphi$ is differentiable, we shall first eliminate $e^{TA}x$ from the argument of φ using the Cameron–Martin formula.

The obvious translation $h \to h - e^{\cdot A}x$ does not work, because the measures $N_{e^{\cdot A}x,\mathbb{Q}_T}$ and $N_{\mathbb{Q}_T}$ are singular for $x \neq 0$.

For this reason we construct another translation $h \to h - a(x, \cdot)$ such that $a(x, \cdot)$ belongs to $\mathbb{Q}_T(X)$ for all $x \in H$ and:

$$a(x, T) = e^{TA}x, \quad \forall\, x \in H.$$

With such a translation $h \to h - a(x, \cdot)$, we shall find

$$\varphi(e^{TA}x + h(T)) \to \varphi(h(T))$$

so that x will disappear from φ.

$a(x,\cdot)$ is provided by the following simple lemma.

Lemma

For all $T>0$ *the matrix*

$$V := \int_0^T r\, e^{rA} C e^{rA^*}\, dr$$

is non singular. Moreover, setting

$$v(x,t) := e^{(T-t)A^*} V^{-1} e^{TA} x, \quad t \in [0,T],\ x \in X.$$

and

$$a(x,\cdot) := \mathbb{Q}_T v(x,\cdot), \quad x \in H,$$

it results

$$a(x,T) = e^{TA} x, \quad \forall\, x \in H,$$

as required.

Since measures $N_{a(x,\cdot),\mathbb{Q}_T}$ and $N_{\mathbb{Q}_T}$ are equivalent, by the Cameron–Martin Theorem we have

$$\frac{dN_{a(x,\cdot),\mathbb{Q}_T}}{dN_{\mathbb{Q}_T}}(h) = \exp\left\{-\frac{1}{2}|\mathbb{Q}_T^{-1/2}a(x,\cdot)|_X^2 + \langle \mathbb{Q}_T^{-1/2}a(x,\cdot), \mathbb{Q}_T^{-1/2}h\rangle_X\right\},$$

After some simple computations, we can write

$$\frac{dN_{a(x,\cdot),\mathbb{Q}_T}}{dN_{\mathbb{Q}_T}}(h) = \exp\left\{-\frac{1}{2}F(x) + G(x,h)\right\}, \quad x \in H,\ h \in X,$$

where

$$F(x) := \langle v(x,t), a(x,\cdot)\rangle_X$$

and,

$$G(x,h) = \langle v(x,t), h\rangle_X.$$

Note that both F and G are regular.

Now our identity becomes

$$R_T^{\mathscr{O}_r}\varphi(x) = \int_{\{M(h+d(x,\cdot)\le r\}} \varphi(h(T)) \exp\left\{-\tfrac{1}{2}\, F(x) + G(x,h)\right\} N_{\mathbb{Q}_T}(dh), \tag{7}$$

where for all $x \in \overline{\mathscr{O}_r}$, we have:

$$d(x,t) = e^{tA}x - a(x,t), \quad t \in [0,T],$$

and

$$M(h+d(x,\cdot)) = \sup_{t\in[0,T]} \{g(h(t)+d(x,t))\}, \quad h \in X = L^2(0,T];\mathbb{R}^d).$$

Note that the variable x does not appear anymore in the argument of φ but only in the domain of integration of (7).

Remark

One can show that the measure $N_{\mathbb{Q}_T}$ is concentrated on $E := C([0,T];\mathbb{R}^d)$.

Moreover, its Cameron–Martin space is still given by $\mathbb{Q}_T^{1/2}(X)$.

As a consequence, integrations with respect to $N_{\mathbb{Q}_T}$ can be performed equivalently both in E and in X, giving the same results.

Since the mapping $x \to M(h + d(x, \cdot))$ is continuous on E the semigroup $R_T^{\mathscr{O}_r}$, $T > 0$, is strong Feller that is

$$\varphi \in B_b(\overline{\mathscr{O}_r}) \Rightarrow R_T^{\mathscr{O}_r} \varphi \in C_b(\overline{\mathscr{O}_r}), \quad \forall\, T > 0.$$

More difficult is to show that $R_T^{\mathscr{O}_r}\varphi$ is x–differentiable for all $\varphi \in B_b(\overline{\mathscr{O}_r})$ and all $T > 0$.

As it is expected, differentiation of $R_T^{\mathscr{O}_r}\varphi$ will produce infinite dimensional surface integrals which, unfortunately, cannot be handled by the classical theory of

Airault–Malliavin, Bull. Sci. Math. 88

.

To overcome this difficulty we shall introduce an approximation $R^{\mathscr{O}_r}_{T,n}\varphi$ of $R^{\mathscr{O}_r}_T\varphi$ defined by finite dimensional integrals.

More precisely, we shall replace any function h from E by a suitable step function.

Let

$$\sigma_n := \{t_j = j\,T\,2^{-n}, \quad j = 0, 1, \dots, 2^n\}, \quad n \in \mathbb{N},$$

be a decomposition of $[0, T]$; then the linear mapping

$$E = C([0, T]; H) \to H^{2^n}, \quad h \to (h(t_1), h(t_2), \cdots, h(t_{2^n})),$$

has a Gaussian law, say $N_{\mathbb{Q}_{T,n}}$.

Now we define

$$R^{\mathscr{O}_r}_{T,n}\varphi(x)=\int_{\{M_n(\xi+d(x,\cdot))\le r\}}\varphi(\xi_{2^n})\exp\{-\tfrac{1}{2}F(x)+G^n(x,\xi)\}N_{\mathbb{Q}_{T,n}}(d\xi),$$

where

$$M_n(\xi+d(x,\cdot))=\max\left\{g(\xi_j+d(x,t_j)),\ t_j=j\,T\,2^{-n},\ j=0,1,\dots,2^n\right\}$$

and

$$G^n(x,\xi)=\sum_{j=1}^{2^n}(v(x,t_j)\cdot\xi_j)\,(t_j-t_{j-1}),\quad \xi\in H^{2^n},$$

recall that $G(x,h)=\langle v(x,t),h\rangle_X$.

Now we can differentiate $R^{\mathscr{O}_r}_{T,n}\varphi(x)$ in any direction $y \in \mathbb{R}^d$.

$$D_x R^{\mathscr{O}_r}_{T,n}\varphi(x) \cdot y =: I_1(n,x,y) + I_2(n,x,y),$$

where

$$I_1(n,x,y) = \int_{\{M_n(h+d(x,\cdot))\le r\}} \varphi(h(T)) \exp\left\{-\tfrac{1}{2}F(x) + G^n(x,h)\right\}$$

$$\times \left(-\tfrac{1}{2}F_x(x)y + G^n_x(x,h)y - G^n_h(x,h)\cdot(d_x(x,\cdot)y)\right) N_{\mathbb{Q}_T}(dh). \tag{8}$$

$$I_2(n,x,y) = \int_{\{M_n(h+d(x,\cdot))\le r\}} \varphi(h(T)) \exp\left\{-\tfrac{1}{2}F(x) + G^n(x,h)\right\}$$

$$\times \langle \mathbb{Q}_{T,n}^{-1/2}(d_x(x,\cdot)y), \mathbb{Q}_{T,n}^{-1/2} h\rangle_{H^{2n}} N_{\mathbb{Q}_T}(dh). \tag{9}$$

Letting $n \to \infty$ in $I_1(n,x,y)$ is easy, just by using the Dominated Convergence Theorem, whereas for $I_2(n,x,y)$ there is a problem due to the factor

$$\langle \mathbb{Q}_{T,n}^{-1/2}(d_x(x,\cdot)y), \mathbb{Q}_{T,n}^{-1/2}\, h\rangle_{H^{2^n}}, \qquad \text{(bad term)}$$

that will converge as $n \to \infty$ to

$$\langle \mathbb{Q}_{T}^{-1/2}(d_x(x,\cdot)y), \mathbb{Q}_{T}^{-1/2}\, h\rangle_{X}.$$

But this term is not meaningful because $d_x(x,\cdot)y$ does not belong to the Cameron–Martin space of $N_{\mathbb{Q}_T}$.

So, we shall provide a different expression of $I_2(n,x,y)$, using a suitable integration by parts formula as in

DP-Lunardi-Tubaro, Trans. AMS, 2018

.

Lemma

The following identity holds

$$I_2(n,x,y) = \int_{\{M_n(h+d(x,\cdot))\le r\}}$$

$$\times \varphi(h(T)) D_h \exp\left\{-\tfrac{1}{2}\, F(x) + G^n(x,h)\right\} \cdot (d_x(x,\cdot)y)\; N_{\mathbb{Q}_T}(dh)$$

$$+ \lim_{\epsilon\to 0} \frac{1}{2\epsilon} \int_{\{r-\epsilon\le M_n(h+d(x,\cdot))\le r+\epsilon\}}$$

$$\times \varphi(h(T)) M_n'(h+d(x,\cdot)) \cdot (d_x(x,\cdot)y) N_{\mathbb{Q}_T}(dh). \tag{10}$$

In identity (10) the bad term disappeared.

The price to pay, however, is that we have to compute the limit $\epsilon \to 0$ in identity (10) and then let $n \to \infty$.

This will require, as we shall see, that

$$\frac{dN_{\mathbb{Q}_T} \circ (M(h + d(x, \cdot)))^{-1}}{d\ell}(s) = D_r^+ \Lambda_x(s) =: \rho(x, s),$$

$$\frac{dN_{\mathbb{Q}_T} \circ (M_n(h + d(x, \cdot)))^{-1}}{d\ell}(s) = D_r^+ \Lambda_{x,n}(s) =: \rho_n(x, s)$$

where ℓ is the Lebesgue measure, $\rho(x, s)$ and $\rho_n(x, s)$ are locally integrable and $\rho_n \to \rho$.

This will require the Ehrhard inequality.

The Erhahrd Inequality

Let X be a separable Hilbert space, μ a gaussian measure in X, $A \subset X,\ B \subset X$ convex. Set

$$\Phi(r) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{r} e^{-\frac{1}{2}x^2}\, dx, \quad r \in \mathbb{R}.$$

Then the following concavity inequality holds

$$\Phi^{-1}[\mu((1-t)A+t\,B)] \geq (1-t)\Phi^{-1}[\mu(A)]+t\,\Phi^{-1}[\mu(B)], \quad \forall\, t \in [0,1].$$

As a consequence if $g : X \to \mathbb{R}$ is convex and

$$F(r) := \mu(g \leq r) = \int_{g\leq r} d\mu,$$

it follows that $\Phi^{-1} \circ F$ is concave, so that F is absolutely continuous and posseses right and left derivatives. See e.g.

Bogachev, Gaussian measures Th. 4.2.2

.

Using the Ehrhard inequality

Define

$$\Lambda_x(s) := N_{\mathbb{Q}_T}(M(h + d(x,\cdot)) \le s), \quad s \ge 0.$$

$$\Lambda_{n,x}(s) := N_{\mathbb{Q}_T}(M_n(h + d(x,\cdot)) \le s), \quad s \ge 0.$$

Since g is convex, mappings $M(\cdot + d(x,\cdot))$ and $M_n(\cdot + d(x,\cdot))$ are convex as well. So, by the Ehrhard inequality we see that the real functions

$$[0,+\infty) \to \mathbb{R},\ s \to S_x(s) = \Phi^{-1}(\Lambda_x(s)),$$

$$[0,+\infty) \to \mathbb{R},\ s \to S_{n,x}(s) = \Phi^{-1}(\Lambda_{n,x}(s)),$$

are concave. As a consequence, they are differentiable at any $s > 0$ up to a discrete set where there exist the left and the right derivative; we shall denote by $D_r^+\Lambda_x(s)$ and $D_r^+\Lambda_{n,x}(s)$ the right derivatives at any discontinuity point.

It follows that the measures

$$N_{\mathbb{Q}_T} \circ (M(h + d(x,\cdot)))^{-1}$$

and

$$N_{\mathbb{Q}_T} \circ (M_n(h + d(x,\cdot)))^{-1}$$

are absolutely continuous with respect to the Lebesgue measure ℓ in $\mathbb{R}$ and it results

$$\frac{dN_{\mathbb{Q}_T} \circ (M(h + d(x,\cdot)))^{-1}}{d\ell}(s) = D_r^+\Lambda_x(s) =: \rho(x,s)$$

ℓ in $\mathbb{R}$ and

$$\frac{dN_{\mathbb{Q}_T} \circ (M_n(h + d(x,\cdot)))^{-1}}{d\ell}(s) = D_r^+\Lambda_{x,n}(s) =: \rho_n(x,s)$$

Now the following lemma can be proved using the fact that $\Lambda_x(s)$ and $\Lambda_{n,x}(s)$ are increasing on s and decreasing on n and the selection principle of Helly.

Lemma

There exists $a > 0$ such that

$$\lim_{n\to\infty} \rho_n(x,s) = \rho(x,s) \quad \forall\, x \in \overline{\mathscr{O}_r}, \quad s \in [r-a, r+a].$$

Now, we can prove the existence of the limit

$$M_{2,2}(n,x,y) = \lim_{\epsilon\to 0}\frac{1}{2\epsilon}\int_{\{r-\epsilon\le M_n(h+d(x,\cdot))\le r+\epsilon\}}$$

$$\times\varphi(h(T))M_n'(h+d(x,\cdot))\cdot(d_x(x,\cdot)y)N_{\mathbb{Q}_T}(dh),$$

that we write as

$$M_{2,2}(n,x,y) = \lim_{\epsilon\to 0}\frac{1}{2\epsilon}\int_X \mathbb{E}_{N_{\mathbb{Q}_T}}\big[\mathbb{1}_{\{r-\epsilon\le M_n(h+d(x,\cdot))\le r+\epsilon\}}$$

$$\times\varphi(h(T))M_n'(h+d(x,\cdot))\cdot(d_x(x,\cdot)y)\big|M_n(h+d(x,\cdot))\big]N_{\mathbb{Q}_T}(dh),$$

that is

$$M_{2,2}(n,x,y) = \lim_{\epsilon\to 0}\frac{1}{2\epsilon}\int_{r-\epsilon}^{r+\epsilon}\mathbb{E}_{N_{\mathbb{Q}_T}}\big[\varphi(h(T))M_n'(h+d(x,\cdot))\cdot(d_x(x,\cdot)y$$

$$\big|M_n(h+d(x,\cdot))=s\big]\rho_{n,x}(s)\,ds.$$

It results

$$
\begin{aligned}
M_{2,2}(n,x,y) &= \mathbb{E}_{N_{\mathbb{Q}_T}}[\varphi(h(T))\,(M_n'(h+d(x,\cdot))\cdot(d_x(x,\cdot)y) \\
&\times \mathbb{E}[|M_n(h+d(x,\cdot))=r]\,\rho_{n,x}(r).
\end{aligned}
$$

Finally, letting $n\to\infty$ we obtain the main result.

Theorem, DP-Tubaro, European. J. Math. 22

For all $\varphi \in B_b(\overline{\mathscr{O}_r})$ there exists the gradient of $R_T^{\mathscr{O}_r}\varphi$ in all direction $y \in \mathbb{R}^d$ and it results

$$D_x R_T^{\mathscr{O}_r}\varphi(x)\cdot y = \int_{\{M(h+d(x,\cdot))\le r\}} \varphi(h(T))\exp\left\{-\tfrac{1}{2}F(x)+G(x,h)\right\}$$

$$\times\left(-\tfrac{1}{2}F_x(x)y+G_x(x,h)y\right)\,N_{\mathbb{Q}_T}(dh)$$

$$+\mathbb{E}\Big[\varphi(h(T))\,(M'(h+d(x,\cdot))\cdot(d_x(x,\cdot)y)M(h+d(x,\cdot))=r\Big]\,\rho_x(r).$$

References

- H. Airault and P. Malliavin. *Intégration géométrique sur l'espace de Wiener*, Bull. Sci. Math. **112**, 3–52, 1988.
- V.I. Bogachev, *Gaussian Measures*, A.M.S. 1998.
- P. Cattiaux, *Calcul stochastique et opèrateurs dègènèrès du second ordre*, Bull. Sciences. Math.(I) et (II), 1990-91.
- G. Da Prato, A. Lunardi and L. Tubaro, Trans. Amer. Math. Soc. **370**, 5795–5842, 2018.
- G. Da Prato and L. Tubaro, European. J. Math.
- E. B. Dynkin, *Markov processes I, II*, Springer–Verlag, 1965.
- R. R. Phelps, *Gaussian null sets and differentiability of Lipschitz map on Banach spaces*, Pac. J. Math., **77**, 523-531, 1978.