BORDISM CATEGORIES AND ORIENTATIONS OF MODULI SPACES.

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In many situations in Differential or Algebraic Geometry, one forms moduli spaces M of geometric objects, such that M is a manifold, or something close to a manifold (a derived manifold, Kuranishi space, ...). Then we can ask whether M is orientable, and if so, whether there is a natural choice of orientation.

This is important in the definition of enumerative invariants: we arrange that the moduli space M is a compact oriented manifold (or derived manifold), so it has a fundamental class in homology, and the invariants are the integrals of natural cohomology classes over this fundamental class.

For example, if X is a compact oriented Riemannian 4-manifold, we can form moduli spaces M of instanton connections on some principal G-bundle P over X, and the Donaldson invariants of X are integrals over M.

In the paper arXiv:2503.20456, Markus Upmeier and I develop a theory of ""bordism categories", which are a new tool for studying orientability and canonical orientations of moduli spaces. It uses a lot of Algebraic Topology, and computation of bordism groups of classifying spaces. We apply it to study orientability and canonical orientations of moduli spaces of G_2 instantons and associative 3-folds on G_2 manifolds, and of Spin(7) instantons and Cayley 4-folds on Spin(7) manifolds, and of coherent sheaves on Calabi-Yau 4-folds. These have applications to enumerative invariants, in particular, to Donaldson-Thomas type invariants of Calabi-Yau 4-folds.

All this is joint work with Markus Upmeier.