Refined horoball counting for Kleinian group actions

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Joint work with Jonathan Fraser

Modelling hyperbolic space

We work in $\overline{\mathbb{R}}^{d+1}$, and model $(d+1)$-dimensional hyperbolic space with the ball

$$\mathbb{D}^{d+1} = \{z \in \mathbb{R}^{d+1} \mid |z| < 1\}$$

equipped with the hyperbolic metric $d_{\mathbb{H}}$ defined by

$$dt = \frac{2|dz|}{1-|z|^2}.$$

This is referred to as the Poincaré ball model. Denote the 'boundary at infinity' of $\mathbb{D}^{d+1}$ by

$$\mathbb{S}^d = \{z \in \mathbb{R}^{d+1} \mid |z| = 1\}.$$

Modelling hyperbolic space

We will also make use of the upper half-space model $\mathbb{H}^{d+1} = \mathbb{R}^d \times (0, \infty)$ with boundary $\mathbb{R}^d \times \{0\}$ and equipped with the analogous metric, noting that we can move between these models by applying a Möbius transformation (the Cayley transformation).

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Kleinian groups act 'properly discontinuously' on $\mathbb{D}^{d+1}$, but this may fail on parts of the boundary.

Limit Sets

Definition

Let $\Gamma \leq \text{Con}^{+}(d)$ *be a Kleinian group. Then the* **limit set** *of* Γ*, denoted as* $L(\Gamma)$*, is*

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Limit sets capture where the Kleinian group fails to be discontinuous on the boundary. It is easy to show that limit sets are closed, Γ-invariant, and (assuming they contain at least 3 points) perfect.

Limit Sets

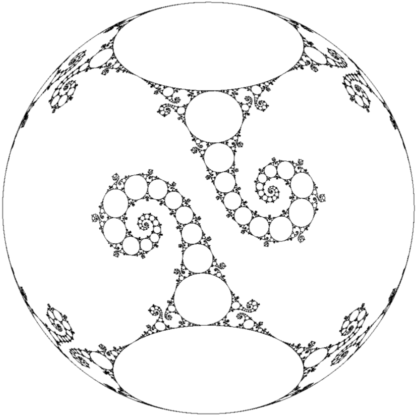

Figure: A Kleinian limit set

Parabolic elements and horoballs

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It is known that we can fix a standard set of horoballs (Euclidean balls in $\mathbb{D}^{d+1}$ which are tangent at some $p \in P$) $\{H_p\}_{p \in P}$ such that they are all pairwise disjoint, do not contain $\mathbf{0}$, and given any $p \in P$ and $g \in \Gamma$, we have $g(H_p) = H_{g(p)}$.

Geometric Finiteness and Poincaré Exponent

We restrict our attention to non-elementary geometrically finite Kleinian groups.

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δ turns out to be closely related to the dimension theory of $L(\Gamma)$, with $\dim_{\mathrm{H}} L(\Gamma) = \dim_{\mathrm{B}} L(\Gamma) = \delta$.

The Patterson-Sullivan Measure

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Limit sets of geometrically finite Kleinian groups are known to support ergodic conformal measures with maximal Hausdorff dimension.

These are often referred to as Patterson-Sullivan measures, and as much of the theory is the same for this family of measures, we will simply fix one and refer to *the* Patterson-Sullivan measure, which we will denote by μ_δ.

Counting horoballs

Question: is it possible to count horoballs of certain sizes? E.g. given $r > 0$, how many horoballs of radius $\approx r$ should we expect to see?

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Theorem (Stratmann-Velani '95)

There exists $\tau \in (0, 1)$ such that for all sufficiently large $k \in \mathbb{N}$, we have

$$\# \left\{ p \in P \mid \tau^{k+1} \leq |H_p| < \tau^k \right\} \approx \tau^{-k\delta}.$$

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So roughly speaking, given sufficiently small $r > 0$, we would expect to see $\approx r^{-\delta}$ horoballs of that size.

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- Horoballs with radius $\lesssim R^2$.
- Horoballs with radius $\geq R$.
- Intermediate horoballs which lie between the above two cases.

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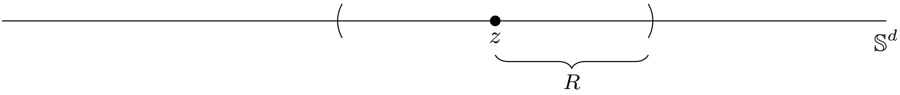
- Horoballs with radius $\lesssim R^2$.
- Horoballs with radius $\geq R$.
- Intermediate horoballs which lie between the above two cases.

The second case is trivial, clearly any ball $B(z, R)$ can only have at most 1 such horoball due to disjointness.

Small horoballs



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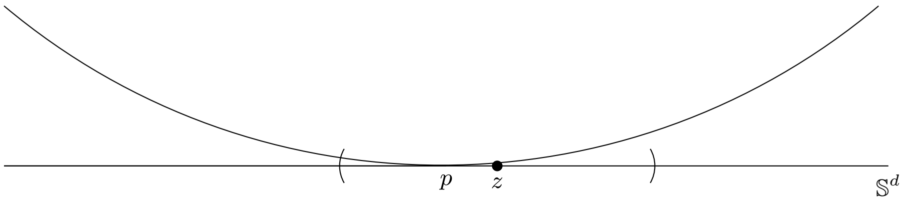

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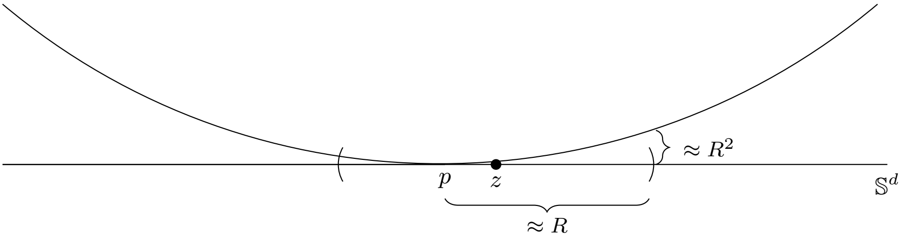

Upper bound

For the other cases, we always have the following upper bound.

Theorem (Fraser-S')

Let $\tau \in (0,1)$*,* $z \in L(\Gamma)$*, and* $R \in (0,1)$*. If* $k \in \mathbb{N}$ *is such that* $\tau^k \lesssim R$*, then*

$$\#\left\{p \in P \cap B(z,R) \mid \tau^{k+1} \leq |H_p| < \tau^k\right\} \lesssim_\tau \tau^{-k\delta}\mu_\delta(B(z,R)).$$

Moreover, if $k \in \mathbb{N}$ *is such that* $\tau^{k+1} > 2R$*, then*

$$\#\left\{p \in P \cap B(z,R) \mid \tau^{k+1} \leq |H_p| < \tau^k\right\} \leq 1.$$

Small horoballs

For the first case, this upper bound turns out to be sharp.

Theorem (Fraser-S')

For all sufficiently small $\tau \in (0,1)$ *there exists* $C \in (0,1)$ *such that for all* $z \in L(\Gamma)$*, all sufficiently small* $R > 0$ *and all* $k \in \mathbb{N}$ *such that* $\tau^k < CR^2$*, we have*

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Proof sketch: A result of Stratmann and Velani says that there exists a constant $\kappa > 0$ such that for sufficiently small $r > 0$,

$$L(\Gamma) \subseteq \bigcup_{\substack{p \in P \\ |H_p| \geq r}} \Pi\left(\kappa\sqrt{\frac{r}{|H_p|}} H_p\right)$$

with multiplicity $\lesssim 1$. For notational convenience we write $\lambda_p = \kappa\sqrt{\frac{r}{|H_p|}}$

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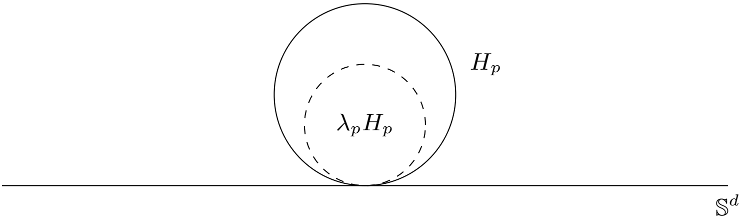

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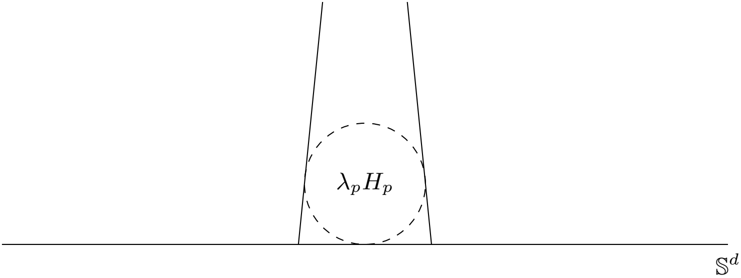

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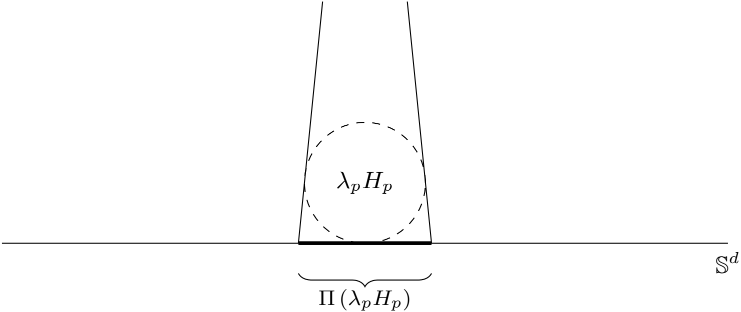

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In particular, this means that $B(z, R)$ can be covered efficiently by projecting horoballs associated to parabolic points which lie in $B(z, R)$. The condition on R also guarantees that large horoballs (radius $\geq R$) contribute negligibly to this cover.

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$$\mu_\delta(B(z,R)) \approx \sum_{\substack{p \in P \cap B(z,R) \\ r \leq |H_p| < R}} \mu_\delta\left(\Pi\left(\lambda_p H_p\right)\right).$$

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Following the Stratmann-Velani argument regarding horoballs, this allows us to show that for all sufficiently large $\alpha > 0$, we have

$$\sum_{\substack{p \in P \cap B(z,R) \\ r \leq |H_p| < \alpha r}} 1 \approx r^{-\delta} \mu_\delta(B(z,R))$$

and the result follows provided we can choose $\alpha = 1/\tau$, which is possible dependant on various fixed constants used throughout the proof.

Intermediate horoballs

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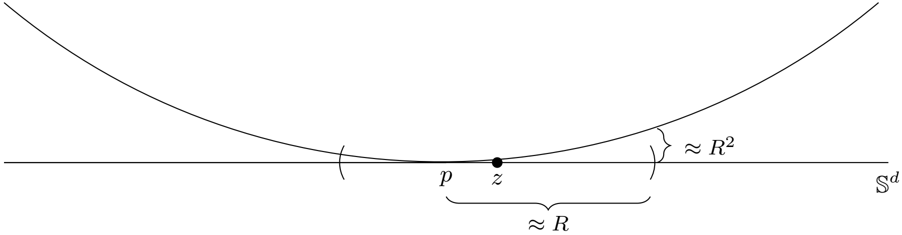

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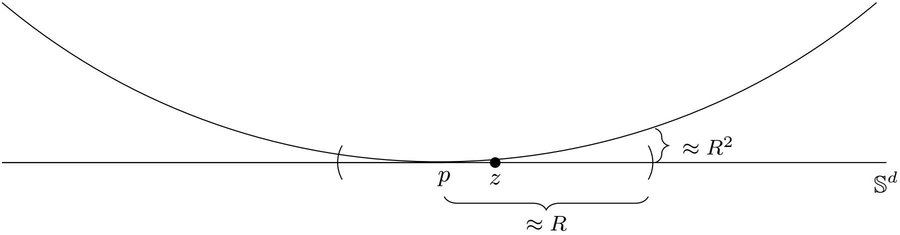

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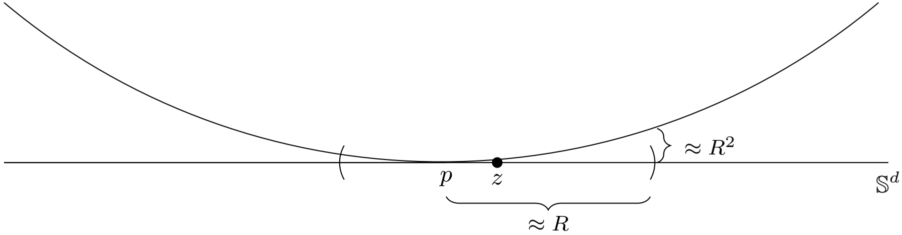

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Is this enough to ensure that intermediate-sized horoballs will appear?

Rank

To state the intermediate result, we require the notion of 'rank'. Given $p \in P$, we write $k(p)$ to denote the **rank** of p, i.e. the largest integer n such that there is a subgroup of Stab(p) isomorphic to $\mathbb{Z}^n$.

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$$k_{\min} = \min\{k(p) \mid p \in P\}$$
$$k_{\max} = \max\{k(p) \mid p \in P\}.$$

Intermediate horoballs

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For all sufficiently small $\tau \in (0,1)$*, all sufficiently small* $R > 0$ *and all* $k \in \mathbb{N}$ *such that* $R^2 < \tau^k < R$ *and* $z \in L(\Gamma)$ *for which there exists* $p_0 \in P$ *with*

$$\tau^{k/2} \lesssim |z - p_0| \lesssim \sqrt{R|H_{p_0}|}$$

we have

$$\# \left\{ p \in P \cap B(z,R) \mid \tau^{k+1} \le |H_p| < \tau^k \right\} \gtrsim_\tau \tau^{-k\delta} \mu_\delta(B(z,\tau^{k/2})) \left(\frac{R}{\tau^{k/2}} \right)^{k(p_0)}.$$

In particular, if $\delta = k_{\min} = k_{\max}$*, then*

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Intermediate horoballs

Proof sketch for $d = 1$: Switch to the upper half plane model $\mathbb{H}^2 = \{x + iy \mid y > 0\}$. We may assume without loss of generality that $0 \in P$ which will be fixed by some parabolic $g \in \Gamma$.

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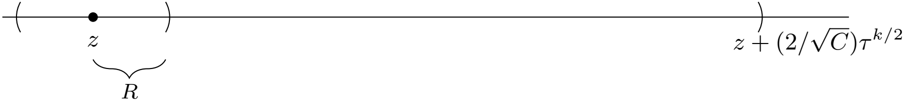

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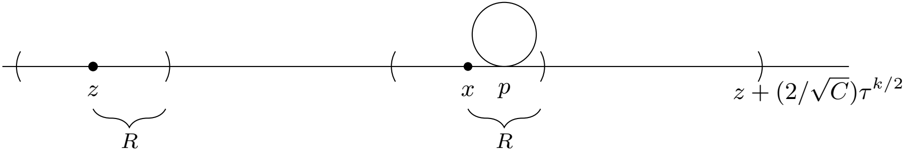

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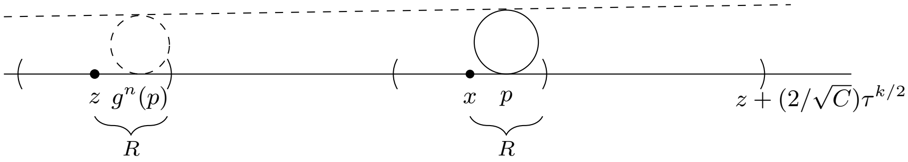

Intermediate horoballs

The case when $d \geq 2$ is trickier, but we can simplify a bit by mapping p_0 to ∞ by applying a circle inversion. In this case, p_0 will have $k(p_0)$ parabolic maps fixing it which will take the form

$$f_i(z) = A_i z + t_i$$

where A_i is a finite order rotation matrix.

Proximity limits

We can also ask the converse question, i.e. how close to a large horoball does z need to be to stop intermediate horoballs from appearing?

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Let $\lambda \in (1,2)$, $z \in L(\Gamma)$, *and* $R > 0$. *If there exists* $p_0 \in P$ *with*

$$|H_{p_0}|R^{\lambda} > R^{2\lambda} + (|z - p_0| + R)^2,$$

then

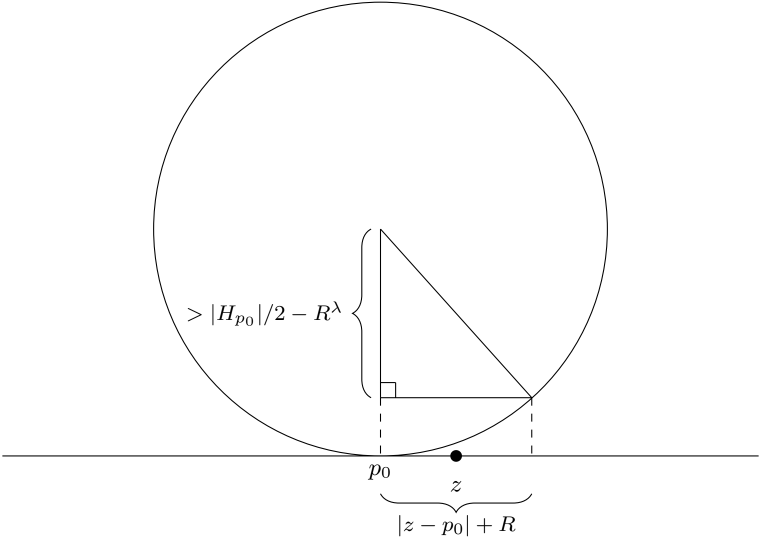
$$\#\left\{p \in P \cap B(z,R) \mid R^{\lambda} \leq |H_p|\right\} \leq 1.$$

In particular, this condition guarantees

$$R^{\lambda/2} \geq |z - p_0|$$

which forbids the assumption of the previous theorem with $\tau^k = R^{\lambda}$.

Proximity limits



Diophantine approximation applications

Consider the group $\Gamma = \mathsf{PSL}(2, \mathbb{Z})$ acting on the upper half plane $\mathbb{H}^2$.

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It is an easy exercise to show that $L(\Gamma) = \mathbb{R} \cup \{\infty\}$, $P = \mathbb{Q} \cup \{\infty\}$, and given coprime $p \in \mathbb{Z}$, $q \in \mathbb{N}$, we can choose an appropriate 'top representation' to ensure that $|H_{p/q}| = 1/q^2$.

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This can be viewed as a special case of Diophantine approximation on Kleinian groups, where we ask how well points in $L(\Gamma)$ can be approximated by points in P.

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This can be viewed as a special case of Diophantine approximation on Kleinian groups, where we ask how well points in $L(\Gamma)$ can be approximated by points in P. In this setting, our notion of 'cost' is the radius of the horoball of the parabolic point.

Diophantine approximation applications

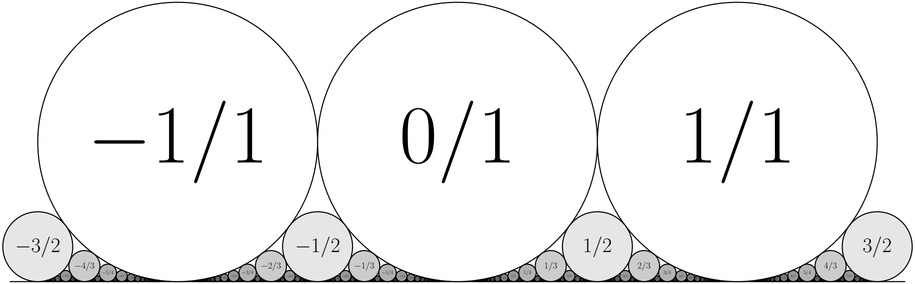

Figure: An illustration of the horoballs in the case where $|H_{p/q}| = 1/q^2$.

Diophantine approximation applications

Corollary

For all sufficiently small $\tau \in (0,1)$, there exists $C \in (0,1)$ such that for all sufficiently small $R > 0$, all $z \in \mathbb{R}$, and for all $k \in \mathbb{N}$ such that $\tau^k < CR^2$, we have

$$\sum_{\substack{q \in \mathbb{N}: \\ \tau^{-k} < q^2 \leq \tau^{-k-1}}} \# \{p \in \mathbb{Z} \mid \gcd(p,q) = 1, \ |p/q - z| \leq R\} \approx \tau^{-k} R.$$

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Let $\lambda \in (1,2)$, $z \in \mathbb{R}$, *and* $R > 0$. *If there exist coprime* $p_0 \in \mathbb{Z}$ *and* $q_0 \in \mathbb{N}$ *such that*

$$\frac{R^\lambda}{q_0^2} > R^{2\lambda} + (|z - p_0/q_0| + R)^2,$$

then

$$\#\left\{p \in \mathbb{Z},\ q \in \mathbb{N} \mid \gcd(p,q) = 1,\ |p/q - z| \le R,\ R^\lambda \le 1/q^2\right\} \le 1.$$

Dimension theory applications

One feature of the Patterson-Sullivan measure is that it is an example of a *conformal measure*. For $s > 0$, we say that a Borel probability measure μ is *s-conformal* for Γ if for any $g \in \Gamma$ and for any measurable $A \subset \mathbb{S}^d$,

$$\mu(g(A)) = \int_A |g'|^s \mathrm{d}\mu.$$

In the case of the Patterson-Sullivan measure, we have $s = \delta$.

Dimension theory applications

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Also, the measure of a point can be related to the radius of its associated horoball, and as many notions of dimensions of measures involve estimating measures of balls, calculating the dimensions of μ_s is related to counting horoballs of certain sizes.

Thank you for listening!

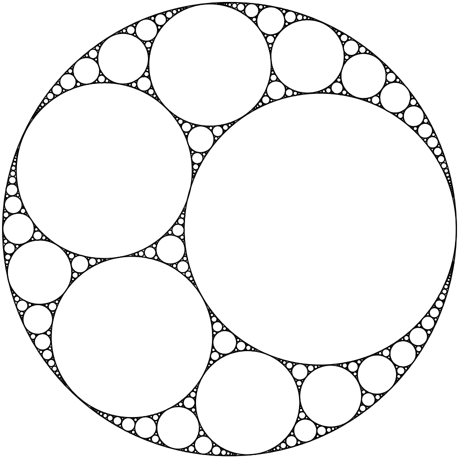

Figure: An Apollonian gasket viewed as the limit set of a Kleinian group acting on $\mathbb{H}^3$.