Weak universality for a class of nonlinear wave equations

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The microscopic model

• For $N \geq 1$, consider a microscopic model of the wave dynamics

$$\partial_t^2 u + |D_x|^{2\alpha} u + N^{-\theta} \Pi_N V'(u) = 0, \ (t,x) \in \mathbb{R} \times \mathbb{T}_N^2, \alpha \leq 1,$$

where $\theta > 0$, $|D_x|^{2\alpha} = (-\Delta)^\alpha$ and

$$\mathbb{T}_N^2 \equiv (\mathbb{R}/2\pi N)^2, \ \ V(u) = \sum_{j=0}^{m} a_j u^{2j}, \quad m \geq 2, \, a_m > 0.$$

• Π_N is a Dirichlet projector defined as

$$\Pi_N \Big(\sum_{k \in \mathbb{Z}^2} \widehat{f}(k/N) \ e^{i \frac{k \cdot x}{N}} \Big) = \sum_{|k| \leq N} \widehat{f}(k/N) \ e^{i \frac{k \cdot x}{N}} \ .$$

• We will consider smooth Gaussian initial data of amplitude ~ 1, spatially localized in a box of size $\sim N^2$ and frequency localized in a box of size $\sim 1/N^2$ (behaving essentially as $f(x/N)$, where f is a Schwartz function on $\mathbb{R}^2$).

• **The question we study :** Understand how the weak nonlinear interaction $N^{-\theta} \Pi_N V'(u)$ modifies the free evolution for $N \gg 1$.

The initial data

• Consider the equation

$$\partial_t^2 u + |D_x|^{2\alpha} u + N^{-\theta} \Pi_N V'(u) = 0, \ (t,x) \in \mathbb{R} \times \mathbb{T}_N^2$$

with gaussian initial data

$$u(0,x) = \phi_N(x) \ , \quad (\partial_t u)(0,x) = \psi_N(x),$$

where

$$\phi_N(x) = \frac{1}{(2\pi)^2} N^{\alpha-1} \sum_{|k|\leq N} \frac{g_k(\omega)}{\langle k\rangle^\alpha} e^{i\frac{k\cdot x}{N}} \ ,$$

with $\langle k\rangle^\alpha := (1+|k|^{2\alpha})^{\frac{1}{2}}$ and

$$\psi_N(x) = \frac{1}{(2\pi)^2} N^{-1} \sum_{|k|\leq N} h_k(\omega) \ e^{i\frac{k\cdot x}{N}} \ .$$

• Here g_k and h_k are standard complex Gaussians such that $g_k = \overline{g_{-k}}$, $h_k = \overline{h_{-k}}$ and otherwise independent.
• The initial position $\phi_N(x)$ and the initial speed $\psi_N(x)$ are gaussians with variances ~ 1, independent of x.

Assumptions on the potential

- Note that ϕ_N has a stationary Gaussian distribution. More precisely

$$\phi_N(x) \sim \mathcal{N}(0, \sigma_N^2) \ , \quad \forall\, x \in \mathbb{T}_N^2,$$

where for $\alpha < 1$

$$\sigma_N^2 = \frac{1}{4\pi^2 N^{2(1-\alpha)}} \sum_{|k| \le N} \frac{1}{\langle k \rangle^{2\alpha}} = \underbrace{\frac{1}{4\pi^2} \int_{|\xi|<1} \frac{1}{|\xi|^{2\alpha}} \mathrm{d}\xi}_{\sigma^2} + \mathcal{O}(N^{-2(1-\alpha)}) \ .$$

- Let $\mu = \mathcal{N}(0, \sigma^2)$, and

$$\langle V \rangle(z) := \int_{\mathbb{R}} V(z+y) \mu(\mathrm{d}y)$$

be the average of V under μ. Our main assumption on the polynomial V is the criticality and the positivity of its averaged version $\langle V \rangle$.

Assumptions on the potential (sequel)

More precisely, we suppose that V is an even polynomial, given by

$$V(z) = \sum_{j=0}^{2m} a_j z^{2j}, \quad m \geq 2$$

and we assume that the averaged polynomial

$$\langle V \rangle(z) := \int_{\mathbb{R}} V(z+y)\mu(\mathrm{d}y) = \sum_{j=0}^{m} \overline{a}_j z^{2j}$$

satisfies

1. $\langle V \rangle''(0) = 0$.

2. $\langle V \rangle(z) - \langle V \rangle(0) > 0$ for all $z \neq 0$.

With these hypothesis, later when we rescale to the unit torus, we obtain directly the renormalized version of the equation.

Assumptions on the potential (sequel)

- We have that

$$\langle V\rangle(z) = \sum_{j=0}^{m} \overline{a}_j z^{2j}$$

where

$$\overline{a}_j = \frac{1}{(2j)!}\mathbb{E}\Big[V^{(2j)}\Big(\mathcal{N}(0,\sigma^2)\Big)\Big] \ ,$$

and we can compute

$$\overline{a}_j = \frac{1}{(2j)!}\sum_{k=j}^{m} \frac{(2k)!}{(2k-2j)!!} \cdot a_k \cdot \sigma^{2(k-j)} \ .$$

- Then the the first assumption is $\overline{a}_1 = 0$ and the second one is

$$\sum_{j=2}^{m} \overline{a}_j \, z^{2(j-2)} > 0, \quad \forall\, z \in \mathbb{R}.$$

- If we fix $a_2 > 0$, ... ,$a_m > 0$, we can find $a_1 < 0$ such that our assumptions on V are satisfied. For example:

$$V(z) = z^6 - 45\sigma^2 z^2.$$

The macroscopic model

• Define the rescaled process u_N on $\mathbb{R} \times \mathbb{T}^2$ by

$$u_N(t,x) := N^{1-\alpha} u(N^\alpha t, Nx) \ .$$

• The spatial domain of u_N becomes the standard torus $\mathbb{T}^2$ and the equation for u_N then becomes

$$\partial_t^2 u_N + |D_x|^{2\alpha} u_N + N^{1+\alpha-\theta} \Pi_N V'(N^{\alpha-1} u_N) = 0$$

with initial datum

$$u_N(0,x) = N^{1-\alpha} \phi_N(Nx) = \frac{1}{(2\pi)^2} \sum_{|k| \le N} \frac{g_k(\omega)}{\langle k \rangle^\alpha} \, e^{ik\cdot x}$$

and

$$\partial_t u_N(0,x) = N \psi_N(Nx) = \frac{1}{(2\pi)^2} \sum_{|k| \le N} h_k(\omega) \, e^{ik\cdot x}.$$

The macroscopic model

• In order for the cubic power in the macroscopic dynamics

$$\partial_t^2 u_N + |D_x|^{2\alpha} u_N + N^{1+\alpha-\theta} \Pi_N V'(N^{\alpha-1} u_N) = 0$$

to have $\mathcal{O}(1)$ coefficient, one necessarily needs to set α and θ such that

$$1 + \alpha - \theta = 3(1 - \alpha) \quad \Longleftrightarrow \quad \theta = 4\alpha - 2 \, .$$

• Therefore we expect that under such a scaling at macroscopic level the dynamics is governed by a "cubic equation" (even of there is no cubic term in the polynomial V' !).

• The criticality condition on the averaged potential assures that the linear term has a limit.

Solving the cubic equation

- Consider

$$\partial_t^2 u_N + |D_x|^{2\alpha} u_N + \Pi_N (u_N)^3 = 0,$$

posed on $\mathbb{T}^2$ with gaussian initial data

$$(u_N(0,x), \partial_t u_N(0,x)) = \frac{1}{(2\pi)^2} \sum_{|k| \leq N} \Big(\frac{g_k(\omega)}{\langle k \rangle^\alpha} \; e^{ik\cdot x}, h_k(\omega) \; e^{ik\cdot x} \Big). \qquad (1)$$

Theorem 1
There is a divergent sequence $(c_N)_{N\geq 1}$ *such that the solutions of*

$$\partial_t^2 u_N + |D_x|^{2\alpha} u_N + \Pi_N \Big((u_N)^3 - c_N u_N \Big) = 0$$

with initial data (1) *converge almost surely in the sense of distribution on* $\mathbb{R} \times \mathbb{T}^2$*, as* $N \to \infty$.

The full model

- We now consider the full macroscopic problem

$$\partial_t^2 u + |D_x|^{2\alpha} u + N^{3(1-\alpha)} \Pi_N V'(N^{\alpha-1}u) = 0.$$

- We have that

$$N^{3(1-\alpha)} \Pi_N V'(N^{\alpha-1}u) = \Pi_N \Big(N^{4(1-\alpha)} V(N^{\alpha-1}u) \Big)' = \Pi_N (V_N'(u)),$$

where

$$V_N(u) := N^{4(1-\alpha)} V(N^{\alpha-1}u).$$

Therefore, we have

$$V_N'(u) = \sum_{j=1}^{m} (2j) \overline{a}_{j,N} N^{-(2j-4)(1-\alpha)} H_{2j-1}(u_N; \tilde{\sigma}_N^2),$$

where $\overline{a}_{j,N} \to \overline{a}_j$ and $H_\ell(x;\sigma)$ denotes the Hermite polynomial of degree ℓ and

$$\tilde{\sigma}_N^2 := \frac{1}{4\pi^2} \sum_{k \in \mathbb{Z}^2, |k| \leq N} \frac{1}{\langle k \rangle^{2\alpha}} \sim N^{2(1-\alpha)} \quad \text{if } 0 < \alpha < 1 \, .$$

Moreover, the limit of $N^{2(1-\alpha)} \overline{a}_{1,N}$ exists.

Main result

Theorem 2 (Weak Universality for the cubic wave dynamics) *Let $1 > \alpha > \frac{8}{9}$ and $\sigma < \alpha - 1$ and suppose that V satisfies our assumptions. Let u_N be the solution of*

$$\partial_t^2 u_N + |D_x|^{2\alpha} u_N + \Pi_N V_N'(u_N) = 0,$$

with initial data

$$(u_N(0,x), \partial_t u_N(0,x)) = \frac{1}{(2\pi)^2} \sum_{|k| \leq N} \Big(\frac{g_k(\omega)}{\langle k \rangle^\alpha} \, e^{ik\cdot x}, h_k(\omega) \, e^{ik\cdot x} \Big). \qquad (2)$$

There is $\lambda > 0$ ($\lambda = 4\overline{a}_2$) and a divergent sequence $(c_N)_{N\geq 1}$ such that the solutions of

$$\partial_t^2 v_N + |D_x|^{2\alpha} v_N + \Pi_N(\lambda (v_N)^3 - c_N v_N) = 0$$

with initial data (2) converge almost surely in the sense of distribution on $\mathbb{R} \times \mathbb{T}^2$, as $N \to \infty$ and satisfy

$$\lim_{N\to\infty} \|u_N - v_N\|_{\mathcal{C}([-T,T],H^\sigma(\mathbb{T}^2))} = 0, \, \forall \, T > 0.$$

Comments

• This type of weak universality was first studied by Hairer-Quastel in deriving the KPZ equation from a large class of microscopic growth models. It has later been extended in various directions in the setting of parabolic singular stochastic PDEs (Hairer-Xu, Furlan-Gubinelli...).

• A key feature in this type of this problem is that every term in the expansion of the nonlinearity has the same size and hence the constant λ of this limiting equation depends on the whole nonlinearity rather than the naive guess of the corresponding power only. Few results for dispersive models fitting in this situation.

• For V of high degree, the data is of supercritical regularity, even with respect to the threshold of probabilistic well-posedness proposed by Deng-Nahmod-Yue.

• Our techniques can be used to extend the weak universality result of Gubunelli-Koch-Oh for the 2D stochastic nonlinear wave equation to the stochastic nonlinear fractional wave equation with space-time white noise, formally written as

$$\partial_t^2 u + |D_x|^{2\alpha} u + \partial_t u + \lambda u^{\diamond 3} = \xi, \quad (t, x) \in \mathbb{R}^+ \times \mathbb{T}^2$$

when $1 > \alpha > \frac{8}{9}$. Gubunelli-Koch-Oh treat the case $\alpha = 1$.

• The weak universality result of Gubunelli-Koch-Oh is a consequence of the almost sure global well-posedness for the two-dimensional nonlinear wave equation ($\alpha = 1$) with *any order nonlinearity*, while for the fractional wave equation with $\alpha < 1$, the situation is radically different.

The Gibbs measure

- Let μ be the gaussian measure induced by the map

$$\omega \longmapsto \frac{1}{(2\pi)^2} \sum_{k\in\mathbb{Z}^2} \frac{g_k(\omega)}{\langle k\rangle^\alpha} e^{ik\cdot x} .$$

- Let ν_N be the probability measure given by

$$\nu_N(\mathrm{d}\phi) = \frac{1}{\mathcal{Z}_N} e^{-\int_{\mathbb{T}^2}\left(V_N(\Pi_N\phi)-1/2((\Pi_N\phi)^2-\tilde{\sigma}_N^2)\right)\mathrm{d}x} \mu(\mathrm{d}\phi) .$$

The measure ν_N is well defined as long as $a_m > 0$.

- If $\lambda := 4\overline{a}_2 > 0$, then for any $c \in \mathbb{R}$ the measure

$$\nu(c)(\mathrm{d}\phi) = \frac{1}{\mathcal{Z}} e^{-\lambda\int_{\mathbb{T}^2}\phi^{\diamond 4}\mathrm{d}x + c\int_{\mathbb{T}^2}\phi^{\diamond 2}\mathrm{d}x} \mu(\mathrm{d}\phi)$$

is also well-defined, where $\phi^{\diamond k}$ denotes the k-th Wick power of ϕ with respect to the Gaussian structure induced by μ.

The Gibbs measure (sequel)

Theorem 3

Let $\alpha \in (\frac{3}{4}, 1)$. Suppose that V satisfies our assumptions. Then

$$\sup_N |\log \mathcal{Z}_N| < +\infty$$

and there exists $c \in \mathbb{R}$ such that ν_N converges to $\nu(c)$ in total variation. In particular, $\nu_N(A)$ converges to $\nu(c)(A)$ for every Borel set A.

• The restriction $\alpha > \frac{3}{4}$ is natural in the sense that in this range, one can define the ϕ^4 measure by an absolutely continuous density with respect to the Gaussian measure μ. The fourth Wick power $\phi^{\diamond 4}$ fails to exist under μ when $\alpha = \frac{3}{4}$, in which case one expects to end up with a measure (after further renormalizations) that is mutually singular with respect to μ.

On the optimality of our assumptions

Proposition 4
If there exists $\theta \in \mathbb{R}$ such that

$$\sum_{j=1}^{m} \overline{a}_j \theta^{2(j-2)} < 0$$

then there exists $c > 0$ such that $\log \mathcal{Z}_N > cN^{4(1-\alpha)}$ for all N. Consequently, the Radon-Nikodym density $\frac{d\nu_N}{d\mu_N}$ cannot converge in L^1 with respect to μ.

• The proof for both positive and negative results for the measure convergence uses Barashkov-Gubinelli's approach based on the Boué-Depuis variational formula.

• When $\overline{a}_m > 0$, it is very likely that the sequence of measures

$$\nu_N(d\phi) = \frac{1}{\mathcal{Z}_N} e^{-\int_{\mathbb{T}^2} V_N(\Pi_N \phi) - 1/2(\Pi_N \phi)^2 - \widetilde{\sigma_N}^2} \mu(d\phi)$$

do not converge.

Mains ingredients for the convergence of the dynamics

• Compare to the parabolic equations, the wave group does not have nice mapping properties on L^∞ type spaces. Hence, the heuristic reasoning that negative powers of N balance out high powers of singular objects needs more involved justification.

• Globalisation argument: In the parabolic setting, the global-in-time convergence follows from the global well-posedness of the limiting equation and stability. However in the current dispersive setting, even though the limiting equation is globally wellposed, the stability properties are not good enough here, and we need to make an essential use of invariant measure to get global convergence.

• Our argument contains two main ingredients :

1. Bourgain-Bulut type argument: A priori bounds resulting from the invariance of the Gibbs measures associated both to the cubic equation and to the full model.

2. Dispersive effects giving $L_t^2 L_x^\infty$ local bounds.

Basic ideas behind the proof

• We have for any $\delta > 0$,

$$\|\Pi_N \phi\|_{L^\infty_x} \leq C_\delta N^{1-\alpha+\delta}$$

in a set of residual probability $\lesssim e^{-cN^2}$. Thanks to the invariant measure considerations, we can propagate this information to the full solution u_N.

• This information is not enough to treat terms like

$$u_N^3 \left(N^{-(1-\alpha)} u_N\right)^{2k+1} = N^{-(1-\alpha)} u_N^4 \left(N^{-(1-\alpha)} u_N\right)^{2k}.$$

We need to combine a $L^2_t L^\infty_x$ type control coming from Strichartz estimates. This leads to the local convergence.

• The global in time convergence crucially relies on the a priori bounds on the global cubic dynamics. These bounds are again relying on invariant measure considerations but this time for the limit dynamics.

Perspectives and open problems

- Other dispersive models: NLS, Φ_3^4 NLW, quasi-linear equations,...
- More general initial data?

Thank you !