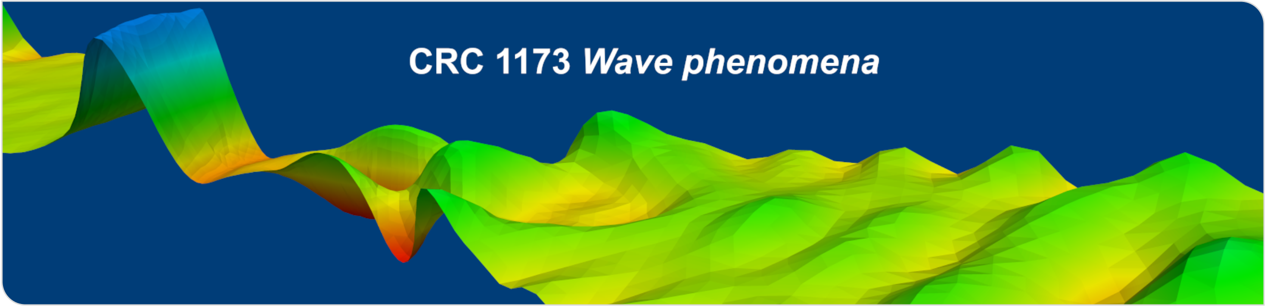
Multiscale method for the nonlinear Helmholtz equation

Barbara Verfürth, Karlsruhe Institute of Technology (KIT) | Solvers for frequency-domain wave problems and applications, Glasgow, 23rd June, 2022

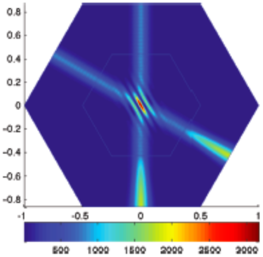


Motivation

- nonlinear constitutive laws required for, e.g., high intensities
- simplest case: Kerr-type media

$$\varepsilon(E) = \varepsilon_0 + \varepsilon_2 |E|^2$$

- new effects such as “crossing of beams”, optical bistability, ...



From [Wu, Zou, SINUM 56, 2018]

Model problem

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- Ω bounded, star-shaped domain, sufficiently regular
- $f \in L^2(\Omega)$, $g \in H^{1/2}(\partial\Omega)$
- wavenumber k possibly large
- $\varepsilon \in \mathbb{R}_{>0}$
- $D \subset\subset \Omega$ non-empty, Lipschitz

Nonlinear Helmholtz equation

Find $u \in H^1(\Omega)$ such that

$$\mathcal{B}(u; v) := (\nabla u, \nabla v) - k^2((1 + \varepsilon\chi_D|u|^2)u, v) + ik(u, v)_{\partial\Omega} = (f, v)_\Omega + (g, v)_{\partial\Omega}$$

for all $v \in H^1(\Omega)$.

Existing approaches

- constant coefficients + pseudospectral method [Yuan, Lu, 2017]
- layered media + finite volumes [Baruch, Fibich, Tsynkov, 2009]
- layered media + steady state of Schrödinger equation [Xu, Bao, 2010]
- constant coefficients + finite element method ($p = 1$) [Wu, Zou 2018]
- linear heterogeneous Helmholtz + finite element method [Graham, Sauter 2020], [Lafontaine, Spence, Wunsch], ...
- linear heterogeneous Helmholtz + localized orthogonal decomposition [Gallistl, Peterseim 2015], [Peterseim 2017],...

Today's talk

1. Introduction

2. Constant-coefficient case: Finite element method

3. Heterogeneous coefficients: Multiscale method

Constant-coefficient case: Finite element method

H. Wu, J. Zou. Finite element method and its analysis for a nonlinear Helmholtz equation with high wave numbers, *SIAM J. Numer. Anal.* 56(3): 1338–1359, 2018.

B. Verfürth. Higher-order finite element methods for the nonlinear Helmholtz equation. *in preparation*, 2022.

(Fixed-point) iteration schemes

1. frozen nonlinearity approach: find $u^{(l)}$ via

$$\mathcal{B}_{\mathrm{lin}}(u^{(l-1)}; u^{(l)}, v) = (f, v) + (g, v)_\Gamma \qquad \text{for all} \quad v \in H^1(\Omega),$$

$$\text{where} \quad \mathcal{B}_{\mathrm{lin}}(\Phi; v, w) := (\nabla v, \nabla w) - (k^2(1 + \chi_D \varepsilon |\Phi|^2)v, w) + i(kv, w)_\Gamma.$$

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2. approach of [Yua, Lu 2017]: find $u^{(l)}$ via

$$\mathcal{A}(u^{(l-1)}; u^{(l)}, v) = -k^2 \varepsilon (|u^{(l-1)}|^2 u^{(l-1)}, v)_D + (f, v) + (g, v)_\Gamma \qquad \text{for all} \quad v \in H^1(\Omega),$$

$$\text{where} \quad \mathcal{A}(\Phi; v, w) := (\nabla v, \nabla w) - (k^2(1 + 2\chi_D \varepsilon |\Phi|^2) v, w) + i(kv, w)_\Gamma.$$

Convergence of iteration schemes

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Theorem

If $k^{d-2}\varepsilon(\|f\|^2 + \|g\|^2)$ is sufficiently small, $u^{(l)} \to u$ in $H^1(\Omega)$.

- linearized Helmholtz equation as auxiliary problem: find $w \in H^1(\Omega)$ such that

$$\mathcal{B}_{\mathrm{lin}}(\Phi; w, v) := (\nabla w, \nabla v) - k^2((1 + \chi_D \varepsilon |\Phi|^2)w, v) + ik(w, v)_\Gamma = (f, v) + (g, v)_\Gamma .$$

- if $k\varepsilon\|\Phi\|^2_{L^\infty(D)}$ sufficiently small, show a priori estimates for $\|w\|_{1,k}$, $\|w\|_2$, $\|w\|_{L^\infty(D)}$

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- if $k\varepsilon\|\Phi\|^2_{L^\infty(D)}$ sufficiently small, show a priori estimates for $\|w\|_{1,k}$, $\|w\|_2$, $\|w\|_{L^\infty(D)}$
- deduce a priori estimates for $u^{(l)}$
- show contraction property for $u^{(l+1)} - u^{(l)}$
- convergence, existence and uniqueness via Banach fixed-point theorem

Finite element method

- standard finite element space $V_{h,p}$ with continuous, piece-wise polynomials of degree p
- combination with iteration schemes: obtain sequences $\{u_{h,p}^{(l)}\}$ to approximate FE solution $u_{h,p}$ of nonlinear problem

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Theorem

If $k(kh)^{2p}$ and $|\ln h|^{2r} k^{d-2}\varepsilon(\|f\|^2 + \|g\|^2)$ are sufficiently small, $u_{h,p}^{(l)} \to u_{h,p}$ and

$$\|u - u_{h,p}\|_{1,k} \lesssim (kh + k(kh)^{2p})(\|f\| + \|g\|).$$

Error analysis

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Theorem

If $k(kh)^{2p}$ and $|\ln h|^{2r} k^{d-2} \varepsilon(\|f\| + \|g\|)$ are sufficiently small, $u_{h,p}^{(l)} \to u_{h,p}$ and

$$\|u - u_{h,p}\|_{1,k} \lesssim (kh + k(kh)^{2p})(\|f\| + \|g\|).$$

Ideas of proof:

- convergence: transfer from the continuous case (show discrete stability estimates)
- prove an error estimate for $u^{(l)} - u_{h,p}^{(l)}$ and let $l \to \infty$
- write $u^{(l)} - u_{h,p}^{(l)} = u - \tilde{u}_h^{(l)} + \tilde{u}_h^{(l)} - u_{h,p}^{(l)}$, where $\tilde{u}_h^{(l)}$ is the discrete solution of the linearized problem using $u^{(l-1)}$

Numerical experiment: h-convergence

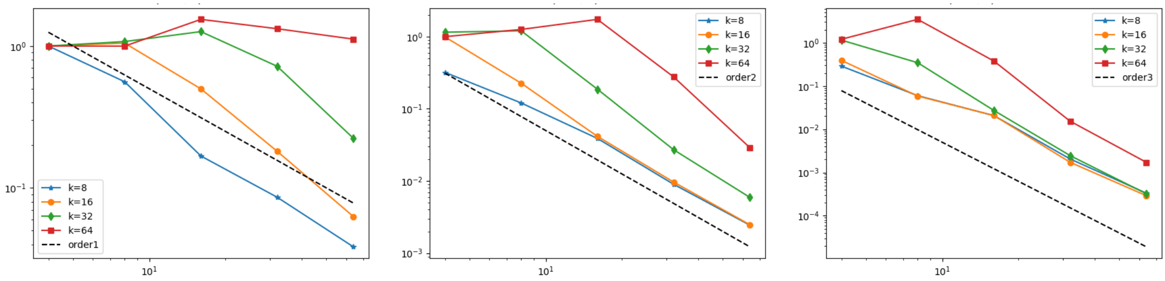
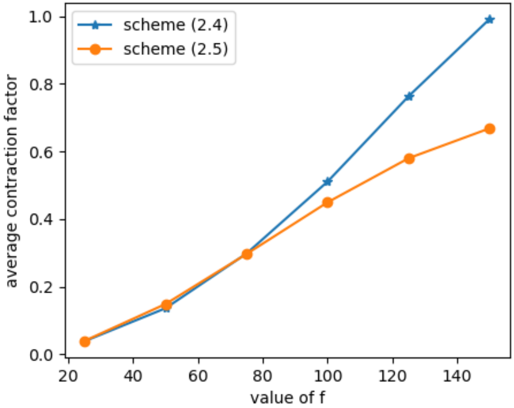
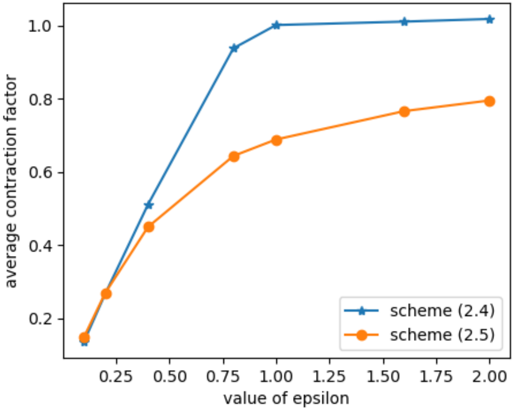


Figure: From left to right: $p = 1, 2, 3$

Numerical experiment: iteration convergence





Heterogeneous coefficients: Multiscale method

joint work with R. Maier (now Jena University)

R. Maier and B. Verfürth. Multiscale scattering in nonlinear Kerr-type media. *Math. Comp.* (online first), 2022, `DOI 10.1090/mcom/3722`.

Model problem

Kerr-type Helmholtz problem

Find $u \in H^1(\Omega)$ such that

$$\mathcal{B}(u; v) := (a\nabla u, \nabla v) - k^2((n + \varepsilon\chi_D|u|^2)u, v) + ik(u, v)_{\partial\Omega} = (f, v)_\Omega$$

for all $v \in H^1(\Omega)$.

- a, n, ε multiscale coefficients, k wave number (possibly large)

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- a, n, ε multiscale coefficients, k wave number (possibly large)
- semilinear term becomes dominant for large k
- LOD for linear Helmholtz: [Peterseim 2017], [Gallistl, Peterseim 2015]
  correction operators solve local Helmholtz problems

Localized Orthogonal Decomposition in a nutshell I

Elliptic multiscale problem: Find $u \in V \subseteq H_0^1$ such that

$$\mathcal{B}(u, v) := (A\nabla u, \nabla v) = (f, v) \qquad \forall v \in V,$$

where $A \in L^\infty$ entails multiscale features and $f \in L^2$.

Orthogonal Decomposition

$$V = V_H^{ms} \oplus W$$

- $W = \ker I_H$ for $I_H : V \to V_H$ stable interpolation operator
- $V_H^{ms} = \{v \in V | \mathcal{B}(v, w) = 0 \text{ for all } w \in W\}$
- $V_H^{ms} = (\mathrm{id} - \mathcal{Q}) V_H$, where $\mathcal{Q} : V \to W$ is the $\mathcal{B}$-orthogonal projection

Localized Orthogonal Decomposition in a nutshell II

- $U_m(T)$ m-layer patch around T, $W(U_m(T)) := \{v \in W | v = 0 \text{ outside } U_m(T)\}$
- localized correction operator $\mathcal{Q}_m = \sum_{T\in\mathcal{T}_H} \mathcal{Q}_{m,T}$, where $\mathcal{Q}_{m,T} : V \to W(U_m(T))$ defined via

$$(A\nabla(\mathcal{Q}_{m,T}v), \nabla w)_{U_m(T)} = (A\nabla v, \nabla w)_T \qquad \forall w \in W(U_m(T))$$

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Galerkin LOD

Find $u_{H,m} \in (\mathrm{id} - \mathcal{Q}_m) V_H$ such that

$$\mathcal{B}(u_{H,m}, v) = (f, v) \qquad \forall v \in (\mathrm{id} - \mathcal{Q}_m) V_H.$$

Iterative multiscale method

(Fixed-point) iteration + multiscale space in each step:

Sequence of multiscale solutions $u^j_{H,m} \in (\mathrm{id} - \mathcal{Q}^{j-1}_m) V_H$ such that

$$\mathcal{B}_{\mathrm{lin}}(u^{j-1}_{H,m}; u^j_{H,m}, v_H) = (f, v_H)_\Omega \qquad \forall v_H \in (\mathrm{id} - \mathcal{Q}^{*,j-1}_m) V_H$$

with $\mathcal{Q}^{j-1}_m = \mathcal{Q}_m[u^{j-1}_{H,m}]$

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with $\mathcal{Q}_m^{j-1} = \mathcal{Q}_m[u_{H,m}^{j-1}]$
Here, $\mathcal{Q}_m[\Phi] = \sum_{T\in\mathcal{T}_H} \mathcal{Q}_{T,m}[\Phi]$ and

$$\mathcal{B}_{\mathrm{lin},\mathrm{N}^m(T)}(\Phi; \mathcal{Q}_{T,m}[\Phi]v_H, w) = \mathcal{B}_{\mathrm{lin},T}(\Phi; v_H, w)$$

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Problem: recomputation of $\mathcal{Q}_m$ is expensive

Solution: only update correctors where nonlinearity changes significantly

Adaptive iterative multiscale method

(Fixed-point) iteration + multiscale space in each step + adaptive updates:

Sequence of multiscale solutions $\tilde{u}^j_{H,m} \in (\mathrm{id} - \tilde{\mathcal{Q}}^{j-1}_m) V_H$ such that

$$\mathcal{B}_{\mathrm{lin}}(\tilde{u}^{j-1}_{H,m}; \tilde{u}^j_{H,m}, v_H) = (f, v_H)_\Omega \qquad \forall v_H \in (\mathrm{id} - \tilde{\mathcal{Q}}^{*,j-1}_m) V_H,$$

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with $\tilde{\mathcal{Q}}^{j-1}_m = \sum_{T\in\mathcal{T}_H} \tilde{\mathcal{Q}}^{j-1}_{T,m}$ and

$$\tilde{\mathcal{Q}}^{j-1}_{T,m} := \begin{cases} \mathcal{Q}_{T,m}[\tilde{u}^{j-1}_{H,m}] & \text{(update)} \\ \tilde{\mathcal{Q}}^{j-2}_{T,m} & \text{(no update)} \end{cases}$$

based on error indicator and tolerance `tol`

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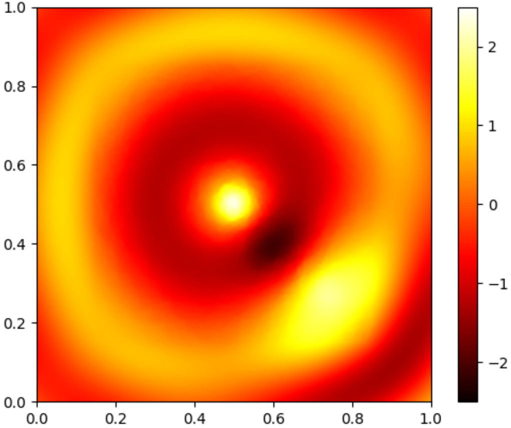
Theorem ([Maier, V. 2022])

If $kH \lesssim 1$, $m \gtrsim \log k$, $\mathtt{tol} \lesssim k^{-1}$ and $k^d n\varepsilon \|f\|^2 \lesssim \vartheta < 1$, it holds

$$\|\tilde{u}^j_{H,m} - u\|_{1,k} \lesssim \frac{1}{1-\vartheta}(\vartheta^j + H + \beta^m + C_{\mathrm{stab}}(k)\mathtt{tol})\|f\|.$$

Numerical experiment

- f (approximate) point source at $(0.5, 0.5)$; nonlinearity in $[0.55, 0.75] \times [0.25, 0.45]$
- variations on scale $\delta = 2^{-7}$, wave number $k = 17$



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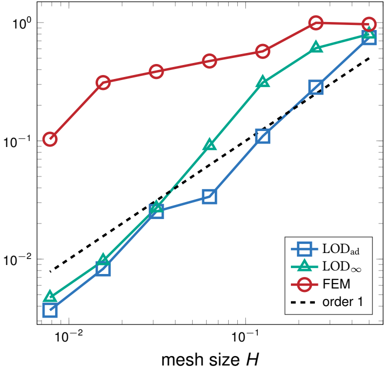

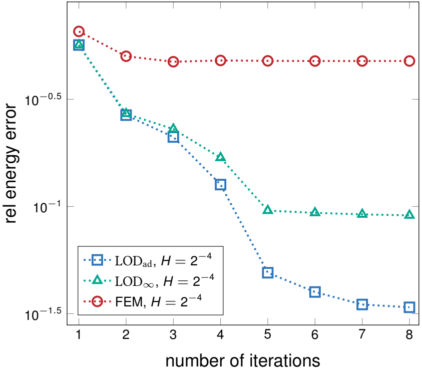

Summary and Outlook

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- finite element methods: error analysis for higher order methods
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- *hp*-error analysis
- (adaptive) iterative multiscale methods for quasilinear problems

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Thank you for your attention!

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