Invariant Gibbs measures for the three-dimensional cubic nonlinear wave equation

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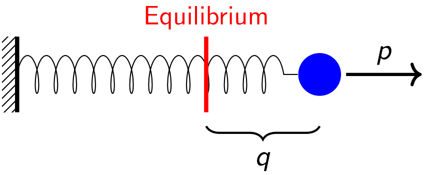
June 20th, 2022

Hamiltonian equations

Physical system:



Potential energy:

$$V(q) = \frac{q^2}{2} + \lambda \frac{q^4}{4} \qquad \text{where} \quad \lambda \geqslant 0.$$

Hamiltonian:

$$H(q,p) = \frac{p^2}{2} + V(q).$$

Hamiltonian ODE:

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} q \\ p \end{pmatrix} = \begin{pmatrix} \partial_p H \\ -\partial_q H \end{pmatrix} \quad \Longleftrightarrow \quad \frac{\mathrm{d}^2}{\mathrm{d}t^2} q = -q - \lambda q^3.$$

Gibbs measure

Definition (Gibbs measure). We define

$$\mathrm{d}\mu(q,p) = \mathcal{Z}^{-1} \exp\big(-H(q,p)\big)\,\mathrm{d}q\mathrm{d}p.$$

Example: If $V(q) = q^2/2 + \lambda \cdot q^4/4$, then

$$\mathrm{d}\mu(q,p) = \mathcal{Z}^{-1} \exp\Big(-\lambda q^4/4\Big) \exp\Big(-q^2/2 - p^2/2\Big)\mathrm{d}q\mathrm{d}p.$$

$\lambda = 0$: Gaussian measure.
$\lambda > 0$: Absolutely continuous w.r.t. the Gaussian measure.

Teaser: This can fail in the PDE-setting.

Invariance

$$\mathrm{d}\mu(q,p) = \mathcal{Z}^{-1} \exp\big(-H(q,p)\big)\,\mathrm{d}q\mathrm{d}p.$$

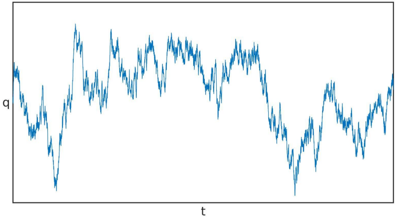
Theorem (Invariance). The Gibbs measure μ is invariant under the Hamiltonian flow.

In other words: If $\mathrm{Law}(q(0), p(0)) = \mu$, then $\mathrm{Law}(q(t), p(t)) = \mu$ for all $t \in \mathbb{R}$.

Proof: Conservation of energy &
Liouville's theorem for divergence-free vector fields. □

(Overdamped) Langevin equation

$$\mathrm{d}q = -V'(q)\mathrm{d}t + \sqrt{2}\,\mathrm{d}B$$



Theorem (Invariance). The measure

$$\mathrm{d}\nu(q) = \mathcal{Z}^{-1}\exp\Big(-V(q)\Big)\mathrm{d}q$$

is invariant under the Langevin dynamics.

Proof: Itô's formula. □

Note: μ and ν are the "same" measure, since

$$\mathrm{d}\mu(q,p) = \mathrm{d}\nu(q)\,\mathrm{d}\mathrm{Gaussian}(p).$$

Summary

Equipped with the potential energy V, we can define:

(1) A Gibbs measure.

(2) A Langevin equation.

(3) A Hamiltonian equation.

From ODEs to PDEs

Question: Can we prove the invariance of the Gibbs measure for *partial differential equations*?

Three parts:

(1) Construction of the Gibbs measure?

(2) *Local* well-posedness of the PDE?

(3) *Global* well-posedness and invariance?

From ODEs to PDEs

Previously:
- $q \in \mathbb{R}$.
- $V(q) = \frac{q^2}{2} + \frac{q^4}{4}$.

Now:
- $\phi \colon \mathbb{T}_x^d \to \mathbb{K}$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.
- $V(\phi) = \int_{\mathbb{T}_x^d} \mathrm{d}x \left(\frac{|\phi|^2}{2} + \frac{|\nabla \phi|^2}{2} + \frac{|\phi|^4}{4} \right)$.

Φ_d^4-models: With this potential energy, we can *formally* associate:

(1) A Gibbs measure.

(2) A cubic stochastic heat equation. ($\leftrightarrow$ Langevin)

(3) A cubic wave equation. ($\leftrightarrow$ real-valued Hamiltonian)

(4) A cubic Schrödinger equation. ($\leftrightarrow$ complex-valued Hamiltonian)

Literature on $\mathbb{T}^d$

Dimension	Measure	Heat	Wave	Schröd.
$d = 1$				
$d = 2$				
$d = 3$				
$d = 4$				
$d \geqslant 5$				

Timeline:

1960 2022

Literature on $\mathbb{T}^d$

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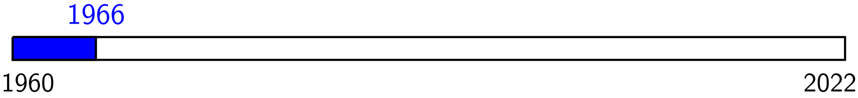


Literature on $\mathbb{T}^d$

Dimension	Measure	Heat	Wave	Schröd.
$d = 1$				
$d = 2$	[Nel66]			
$d = 3$	[GJ73]			
$d = 4$				
$d \geqslant 5$				

Timeline:

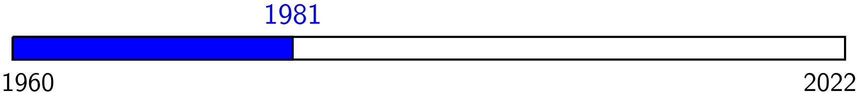
1973

1960 2022

Literature on $\mathbb{T}^d$

Dimension	Measure	Heat	Wave	Schröd.
$d = 1$				
$d = 2$	[Nel66]			
$d = 3$	[GJ73]			
$d = 4$				
$d \geqslant 5$	[Aiz81, Fro82]			

Timeline:



Literature on $\mathbb{T}^d$

Dimension	Measure	Heat	Wave	Schröd.
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$d = 3$	[GJ73]			
$d = 4$				
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Timeline:

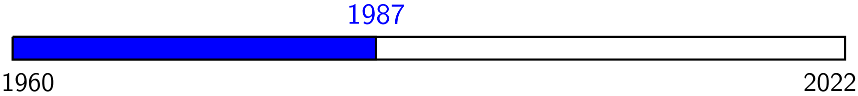
1981

1960 2022

Literature on $\mathbb{T}^d$

Dimension	Measure	Heat	Wave	Schröd.
$d = 1$		[Iwa87]		
$d = 2$	[Nel66]			
$d = 3$	[GJ73]			
$d = 4$				
$d \geqslant 5$	[Aiz81, Fro82]			

Timeline:



Literature on $\mathbb{T}^d$

Dimension	Measure	Heat	Wave	Schröd.
$d = 1$		[Iwa87]	[Zhi94]	
$d = 2$	[Nel66]			
$d = 3$	[GJ73]			
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Timeline:

1994

1960 2022

Literature on $\mathbb{T}^d$

Dimension	Measure	Heat	Wave	Schröd.
$d = 1$		[Iwa87]	[Zhi94]	[Bou94]
$d = 2$	[Nel66]			
$d = 3$	[GJ73]			
$d = 4$				
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Timeline:

1994

1960 2022

Literature on $\mathbb{T}^d$

Dimension	Measure	Heat	Wave	Schröd.
$d = 1$		[Iwa87]	[Zhi94]	[Bou94]
$d = 2$	[Nel66]			[Bou96]
$d = 3$	[GJ73]			
$d = 4$				
$d \geqslant 5$	[Aiz81, Fro82]			

Timeline:

1996

1960 2022

Literature on $\mathbb{T}^d$

Dimension	Measure	Heat	Wave	Schröd.
$d = 1$		[Iwa87]	[Zhi94]	[Bou94]
$d = 2$	[Nel66]		[Bou99]	[Bou96]
$d = 3$	[GJ73]			
$d = 4$				
$d \geqslant 5$	[Aiz81, Fro82]			

Timeline:

1999

1960 2022

Literature on $\mathbb{T}^d$

Dimension	Measure	Heat	Wave	Schröd.
$d = 1$		[Iwa87]	[Zhi94]	[Bou94]
$d = 2$	[Nel66]	[DPD03]	[Bou99]	[Bou96]
$d = 3$	[GJ73]			
$d = 4$				
$d \geqslant 5$	[Aiz81, Fro82]			

Timeline:

2003

1960 2022

Literature on $\mathbb{T}^d$

Dimension	Measure	Heat	Wave	Schröd.
$d = 1$		[Iwa87]	[Zhi94]	[Bou94]
$d = 2$	[Nel66]	[DPD03]	[Bou99]	[Bou96]
$d = 3$	[GJ73]	[Hai13]		
$d = 4$				
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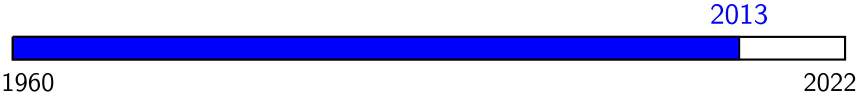
Timeline:



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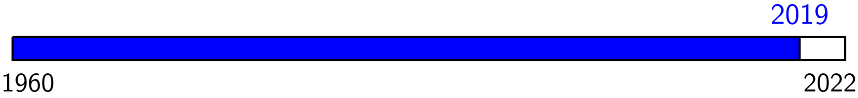
Timeline:



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Dimension	Measure	Heat	Wave	Schröd.
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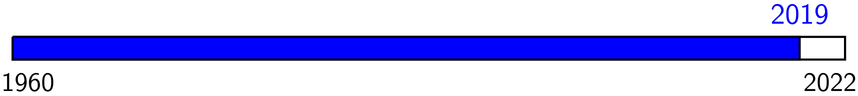
Timeline:



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$d = 3$	[GJ73]	[Hai13]	**[BDNY22]**	
$d = 4$	[ADC19]			
$d \geqslant 5$	[Aiz81, Fro82]			

Timeline:

Literature on $\mathbb{T}^d$

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$d = 2$	[Nel66]	[DPD03]	[Bou99]	[Bou96]
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$d = 4$	[ADC19]			
$d \geqslant 5$	[Aiz81, Fro82]			

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1960 2022

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$d = 3$			**[BDNY22]**	Open
$d = 4$				
$d \geqslant 5$				

Related works:

- Gubinelli-Koch-Oh '18 on quadratic NLW in $d = 3$.
- Deng-Nahmod-Yue '19, '21 on power-type or Hartree NLS.
- B. '20 on Hartree-NLW in $d = 3$.
- Oh-Okamoto-Tolomeo '21 on quadratic NLW in $d = 3$.

Main result

The (renormalized) three-dimensional cubic nonlinear wave equation is

$$\begin{cases} (\partial_t^2 + 1 - \Delta)\phi_{\mathrm{wave}} = -\phi_{\mathrm{wave}}^3 + \infty \cdot \phi_{\mathrm{wave}} & (t,x) \in \mathbb{R} \times \mathbb{T}^3 \\ \phi_{\mathrm{wave}}[0] = (\phi_0, \phi_1). \end{cases}$$

Theorem (B.-Deng-Nahmod-Yue '22).
The cubic nonlinear wave equation is probabilistically well-posed with respect to the Gibbs measure. Furthermore, the Gibbs measure is invariant under the dynamics.

The Gibbs measure

Formal definition: Formally, Gibbs $= \Phi_3^4 \otimes$ Gaussian, where

$$\text{“}\mathrm{d}\Phi_3^4(\phi) = \mathcal{Z}^{-1} \exp\left(-\int_{\mathbb{T}^3} \mathrm{d}x \left(\frac{|\phi|^2}{2} + \frac{|\nabla \phi|^2}{2} + \frac{|\phi|^4}{4} \right) \right) \prod_{x \in \mathbb{T}^3} \mathrm{d}\phi(x)\text{”}.$$

Rigorous treatment in [GJ73] and [BG20, GH21, MW20]:
The Φ_3^4-measure can be defined rigorously (via a finite-dimensional approximation).

Properties of samples from Φ_3^4-measure:

- Spatial regularity $< 1 - \text{dim.}/2 = -1/2$.
- Fourier coefficients are probabilistically dependent (singularity w.r.t. Gaussian free field).

Caloric representation

We construct *caloric* initial data $\color{blue}{\bigcirc}$, $\color{green}{\diamond}$, and $\color{red}{\pentagon}$, which satisfy

$$\text{Law}\left(\bigcirc - \diamond + \pentagon\right) = \Phi^4_3$$

and have the following properties:

Data	Probabilistic Structure	Regularity $<$
$\bigcirc$	Gaussian	$-1/2$
$\diamond$	cubic Gaussian chaos	$1/2$
$\pentagon$	not available	1

Remark.

- We use the cubic stochastic heat equation, which has been treated in [Hai13] and [CC18, GIP15].
- This is motivated by Tao's caloric gauge [Tao04].

The caloric initial value problem

$$\begin{cases} (\partial_t^2 + 1 - \Delta)\phi_{\text{wave}} = -\phi_{\text{wave}}^3 + \infty \cdot \phi_{\text{wave}} & (t,x) \in \mathbb{R} \times \mathbb{T}^3 \\ \phi_{\text{wave}}[0] = {\color{blue}\bigcirc} - {\color{green}\Diamond} + {\color{red}\pentagon} \end{cases}$$

Why is this difficult?

Spatial regularity of ${\color{blue}\bigcirc}$ is only $s = -1/2-$.

Random structure

Para-controlled Ansatz:

$\phi_{\text{wave}} =$  $-$  $-$  $+ 3$  $+ X^{(1)} + X^{(2)} + Y$.

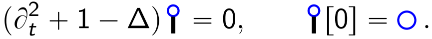
(explicit objects: the first four terms; para-controlled: $X^{(1)} + X^{(2)}$; remainder: Y)

Overview:

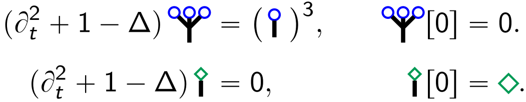
Regularity	$-1/2-$	$0-$	$1/2-$				$1/2+$
Component					$X^{(1)}$	$X^{(2)}$	Y

Explicit stochastic objects

Linear object:



Cubic objects:



Quintic object:

Para-controlled components and remainder

Para-controlled components:

$$(\partial_t^2 + 1 - \Delta)X^{(1)} \simeq (\text{High}\times\text{low}\times\text{low})\Big(\overset{\circ}{\mathrm{I}},\, *,\, *\Big).$$

$$(\partial_t^2 + 1 - \Delta)X^{(2)} \simeq (\text{High}\times\text{high}\times\text{low})\Big(\overset{\circ}{\mathrm{I}},\, \overset{\circ}{\mathrm{I}},\, *\Big).$$

"Find all terms with regularity $< 1/2$ and exhibit a useful structure."

Remainder:

$$(\partial_t^2 + 1 - \Delta)Y = \Big\{\text{All interactions}\Big\} - \Big\{\text{Removed interactions}\Big\}$$

$$\#\simeq 87 \qquad\qquad \#\simeq 6$$

"Main task"

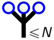
Techniques

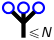
The article is ~ 200 pages long and combines ingredients from several different fields:

- Gaussian hypercontractivity (Probability theory)
- Lattice point estimate (Number theory/Geometry)
- Molecular graph analysis (Combinatorics)
- Multi-linear dispersive estimates (PDE)
- Multiple stochastic integrals (Probability theory)
- Para-product estimates (Harmonic analysis)
- Para-controlled calculus (SPDE)
- Random tensor estimates (Probability theory)

Personal highlight: 1533-cancellation.

Let $N \geqslant 1$ be a frequency-truncation parameter.

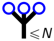
Lemma (33-divergence). The square of , i.e.,

$$\Big($$  $$(t,x)\Big)^2,$$

diverges as $N \to \infty$ in the sense of space-time distributions.

But, we are lucky:

Proposition (1533-cancellation). The linear combination

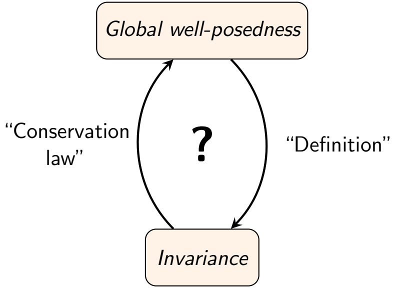
$$6 \cdot$$  $$(t,x)$$  $$(t,x) + \Big($$  $$(t,x)\Big)^2$$

has a well-defined limit as $N \to \infty$.

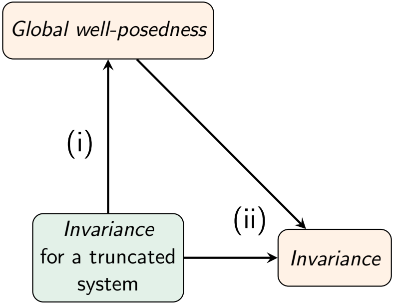
Global well-posedness and invariance

Goal (Invariance): If $\mathrm{Law}(\phi[0]) = \mu$, then $\mathrm{Law}(\phi[t]) = \mu$ for all $t \in \mathbb{R}$.

Danger: Circular argument?



Solution: Bourgain!



Remark:

- The *singularity* of the Gibbs measure introduces *severe* difficulties.
- We improve on earlier works [Bri20] and [OOT21].

Thank you!