Algebraic lattices for cryptography

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Fundations and applications of lattice-based cryptography workshop

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What about security:

- most of the time no better attacks than for unstructured lattices
- but for some problems, we have specific attacks using the algebraic structure (cf second talk)

Outline of the talk

1. A bit of number theory
2. Algebraic lattices
3. Algorithmic problems for cryptography
4. Some more number theory

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Ring of integers: $\mathcal{O}_K \subset K$, for this talk $\mathcal{O}_K = \mathbb{Z}[X]/P(X)$
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Embeddings

$(K = \mathbb{Q}[X]/P(X), \quad \alpha_1, \cdots, \alpha_d$ complex roots of $P(X))$

Coefficient embedding: $\Sigma : \begin{array}{rcl} K & \to & \mathbb{R}^d \\ \sum_{i=0}^{d-1} y_i X^i & \mapsto & (y_0, \cdots, y_{d-1}) \end{array}$

Canonical embedding: $\sigma : \begin{array}{rcl} K & \to & \mathbb{C}^d \\ y(X) & \mapsto & (y(\alpha_1), \cdots, y(\alpha_d)) \end{array}$

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Which embedding should we choose?

- coefficient embedding is used for constructions (efficient implementation)
- canonical embedding is used in cryptanalysis / reductions (nice mathematical properties)
- for fields used in crypto, both geometries are $\approx$ the same

Ideals

Ideal: $I \subseteq \mathcal{O}_K$ is an ideal if

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- $a \cdot x \in I$ for all $a \in \mathcal{O}_K$ and $x \in I$

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- $\mathcal{N}(I_1) = 2$ and $\mathcal{N}(J_1) = 6$
- $\mathcal{N}(I_2) = 2$

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- g is a generator of $\langle g\rangle$
- $\{$ generators of $\langle g\rangle$ $\} = \{gu \,|\, u \in O_K^\times\}$
- $\mathcal{N}(\langle g\rangle) = |\mathcal{N}(g)|$, where $\mathcal{N}(g) = \prod_i g(\alpha_i)$ (α_i complex roots of $P(X)$)

Outline of the talk

Ideal lattices

$\mathcal{O}_K$ is a lattice:

- $\mathcal{O}_K = 1 \cdot \mathbb{Z} + X \cdot \mathbb{Z} + \cdots + X^{d-1} \cdot \mathbb{Z}$
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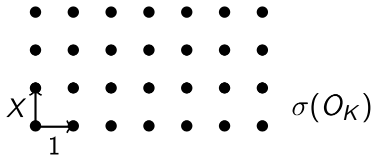
(this is also true for non principal ideals)

Ideal lattices (2)



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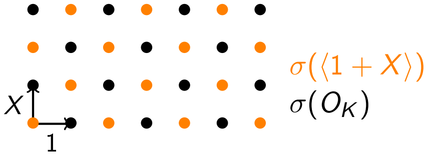

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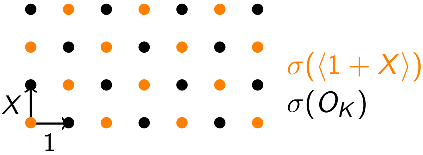

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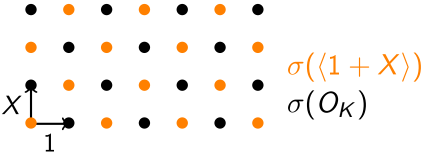

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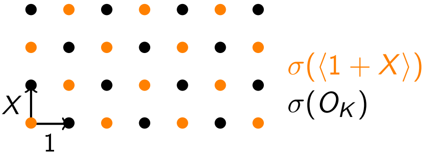

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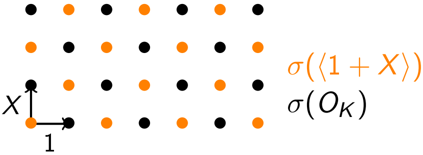

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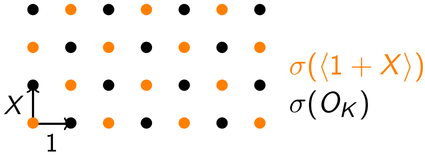

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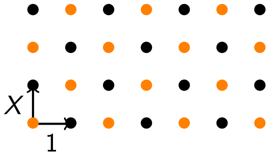

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Discriminant: $\Delta_K := \sqrt{\mathrm{vol}(\sigma(\mathcal{O}_K))}$

Volume of an ideal: $\mathrm{vol}(\sigma(I)) = \mathcal{N}(I) \cdot \sqrt{\Delta_K}$

Module lattices

(Free) module:

$$M = \{B \cdot x \,|\, x \in \mathcal{O}_K^k\} \text{ for some matrix } B \in \mathcal{O}_K^{k \times k} \text{ with } \det_K(B) \neq 0$$

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- $\mathrm{vol}(M) = |\mathcal{N}(\det_K(B))| \cdot \Delta_K^{k/2}$

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In $K = \mathbb{Q}[X]/(X^d + 1)$:

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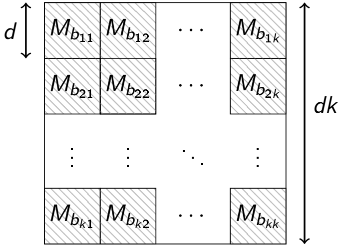
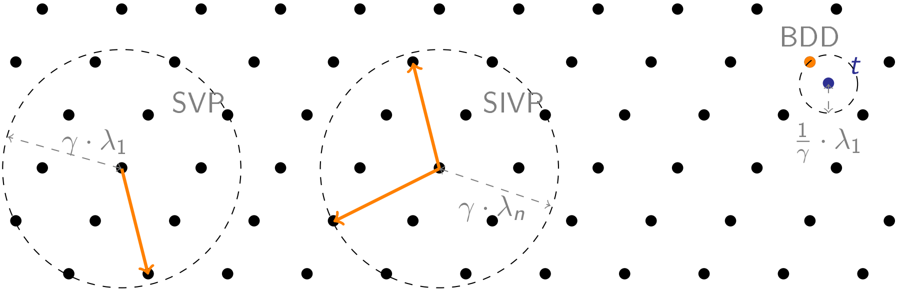
basis of a
principal ideal lattice



basis of a free module lattice
of rank k

Algorithmic problems

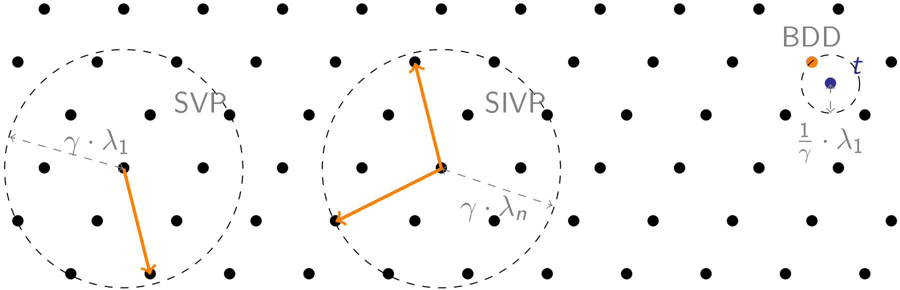


γ-SVP
shortest vector problem

γ-SIVP
shortest independent vector problem

γ-BDD
bounded distance decoding problem

Algorithmic problems



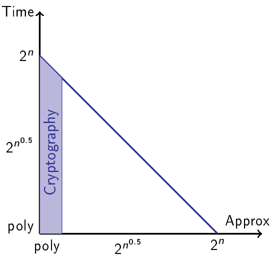
γ-SVP	γ-SIVP	γ-BDD
shortest vector problem	shortest independent vector problem	bounded distance decoding problem

Notations:

- id-X = problem X restricted to ideal lattices
- mod-X_k = problem X restricted to module lattices of rank k

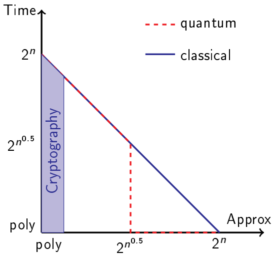
Hardness of SVP

Asymptotics:



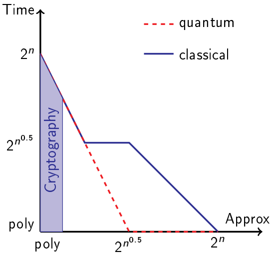
SVP and mod-SVP$_k$

($k \geq 2$)



id-SVP [CDW17]

(in cyclotomic fields)



id-SVP [PHS19,BR20]

(with $2^{O(n)}$ pre-processing)

[CDW17] Cramer, Ducas, Wesolowski. Short stickelberger class relations and application to ideal-SVP. Eurocrypt.

[PHS19] Pellet-Mary, Hanrot, Stehlé. Approx-SVP in ideal lattices with pre-processing. Eurocrypt.

[BR20] Bernard, Roux-Langlois. Twisted-PHS: using the product formula to solve approx-SVP in ideal lattices. AC.

Hardness of SVP

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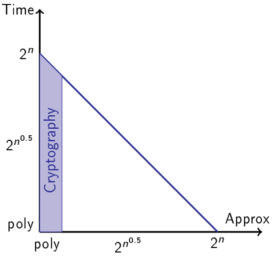

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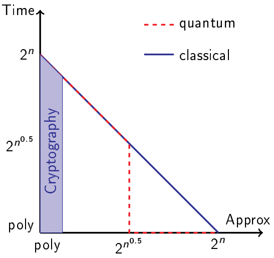

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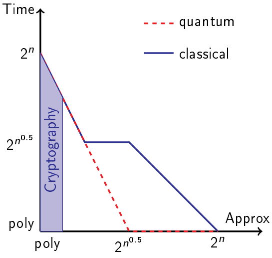

id-SVP [PHS19,BR20]
(with $2^{O(n)}$ pre-processing)

Practice: Darmstadt challenge[1]

$\rightsquigarrow$ max dim for SVP: 180
$\rightsquigarrow$ max dim for id-SVP: 150

[1] https://www.latticechallenge.org/

Outline of the talk

Ring and Module-LWE

(search) mod-LWE$_k$

Parameters: $k, m, q \in \mathbb{Z}_{>0}$ and $\alpha \in \mathbb{R}_{>0}$

Objective: given $(A, b) \in \mathcal{O}_K^{m \times k} \times \mathcal{O}_K^m$, with

- A uniform in $\mathcal{O}_K^{m \times k}$
- s uniform in $\mathcal{O}_K^k$ and $e \in \mathcal{O}_K^m$ such that $\sigma(e) \leftarrow D_{\sigma(\mathcal{O}_K), \alpha \cdot q}$
  ($D_{L,\sigma}$ discrete Gaussian distribution over L with parameter σ)
- $b = As + e$

output s

(can also be defined using Σ instead of σ)

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$$\text{RLWE} = \text{mod-LWE}_1$$

Decision mod-LWE

dec-mod-LWE$_k$

Parameters: $k, m, q \in \mathbb{Z}_{>0}$ and $\alpha \in \mathbb{R}_{>0}$

Objective: distinguish between (A, b) and (A, u), where

- A and b are as on the previous slide
- u is uniform in $\mathcal{O}_K^m$

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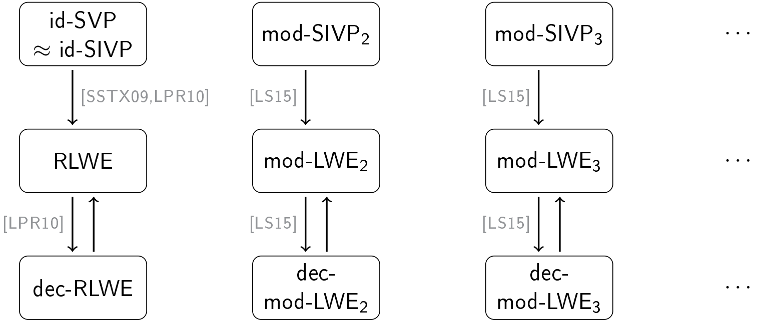
- u is uniform in $\mathcal{O}_K^m$

mod-LWE$_k$ reduces to dec-mod-LWE$_k$ [LS15]

[LS15] Langlois, Stehlé. Worst-case to average-case reductions for module lattices. DCC.

Reductions



⚠ Arrows may not all compose (different parameters) ⚠

(References are for the first reductions. Better, more recent reductions may exist.)

[SSTX09] Stehlé, Steinfeld, Tanaka, Xagawa. Efficient public key encryption based on ideal lattices. Asiacrypt.

[LPR10] Lyubashevsky, Peikert, Regev. On ideal lattices and learning with errors over rings. Eurocrypt.

[LS15] Langlois, Stehlé. Worst-case to average-case reductions for module lattices. DCC.

From mod-LWE$_k$ to mod-SIVP$_{k+1}$

Reminder mod-LWE$_k$: $(A, b = A \cdot s + e \bmod q)$

with $s \in \mathcal{O}_K^k$, $e \in \mathcal{O}_K^m$ and $\|\sigma(e)\| \approx \alpha \cdot q$

mod-LWE$_k$ is a BDD in the rank-m module lattice

$$\Lambda = \sigma\Big(\big\{x \in \mathcal{O}_K^m \,|\, \exists z \in \mathcal{O}_K^k,\, x = A \cdot z \bmod q\big\}\Big)$$

- BDD only if m is large enough

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  - we need roughly $m = k \cdot \frac{\log(q)}{\log(1/\alpha)}$
  - for $k = 1$, $m = 2$ is possible if $\alpha \cdot q \lesssim \sqrt{q}$

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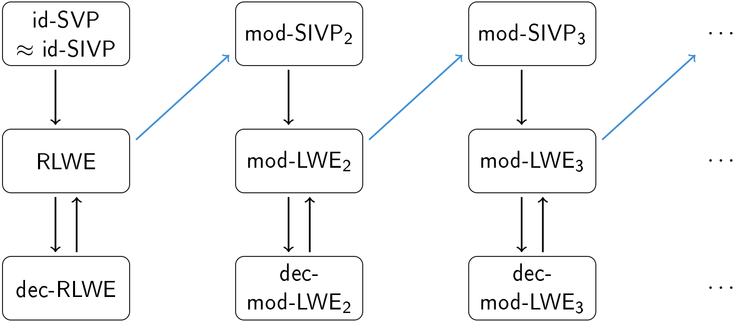
RLWE is at best a special case of mod-BDD$_2$

Reductions



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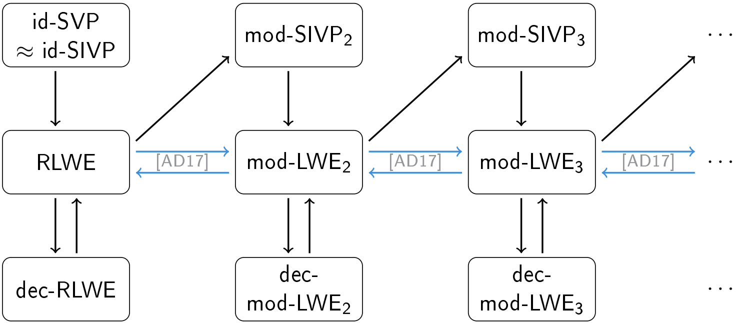

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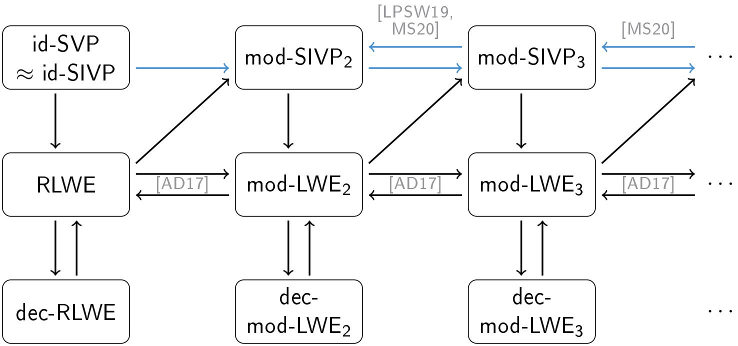
[AD17] Albrecht, Deo. Large modulus ring-LWE $\geq$ module-LWE. Asiacrypt.

Reductions



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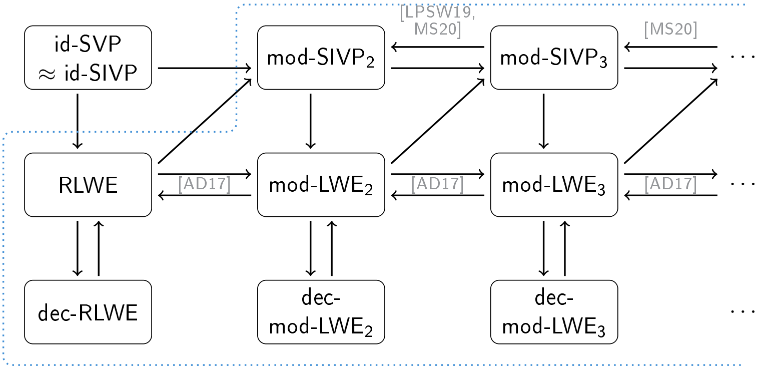

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NTRU (a.k.a, partial Fourier recovery problem [HPS98])

(search) NTRU

Parameters: $q \geq B > 1$ and ψ distribution over $\mathcal{O}_K$ outputting elements $\leq B$

Objective: given $h \in \mathcal{O}_K/(q\mathcal{O}_K)$, with

- $f, g \leftarrow \psi$ conditioned on g invertible modulo q
- $h = f \cdot g^{-1} \bmod q$

output (f, g)

(can also be defined using Σ instead of σ)

[HPS98] Hoffstein, Pipher, and Silverman. NTRU: a ring based public key cryptosystem. ANTS.

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dec-NTRU

Parameters: q, B and ψ

Objective: distinguish between h as above and u uniform in $\mathcal{O}_K/(q\mathcal{O}_K)$

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Two regimes of NTRU

If $B \geq \sqrt{q} \cdot \mathrm{poly}(d)$

If $B \leq \sqrt{q}/\mathrm{poly}(d)$

Two regimes of NTRU

If $B \geq \sqrt{q} \cdot \mathrm{poly}(d)$

- h is statistically close to uniform mod q [SS11,WW18]
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[WW18] Wang and Wang. Provably secure NTRUEncrypt over any cyclotomic field. SAC.

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For the rest of the talk, we consider $B \ll \sqrt{q}$

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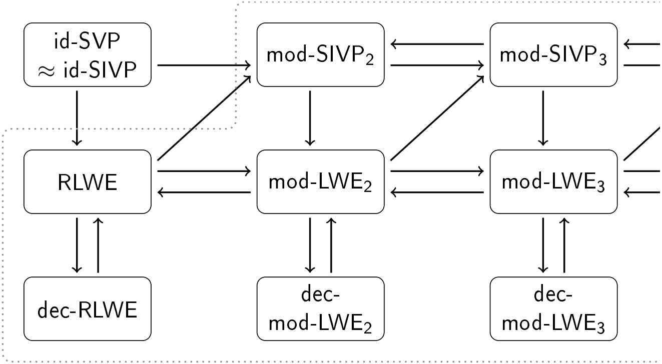

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[Pei16] Peikert. A decade of lattice cryptography. Foundations and Trends in TCS.

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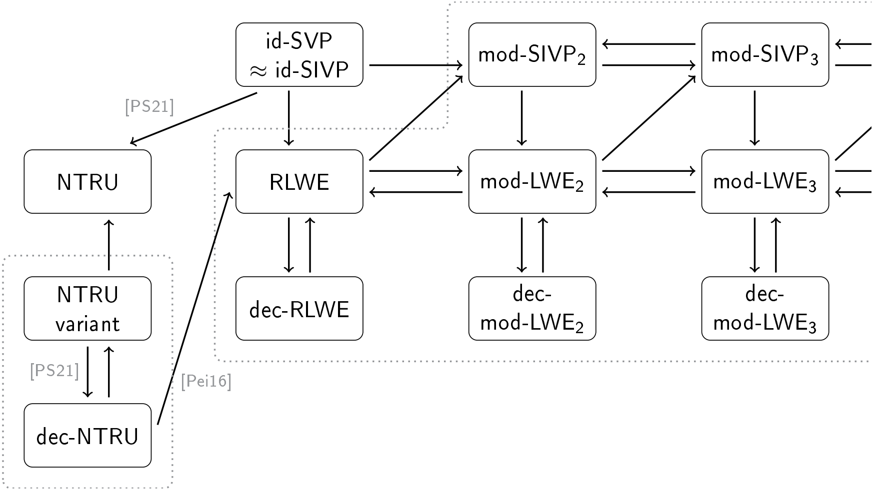

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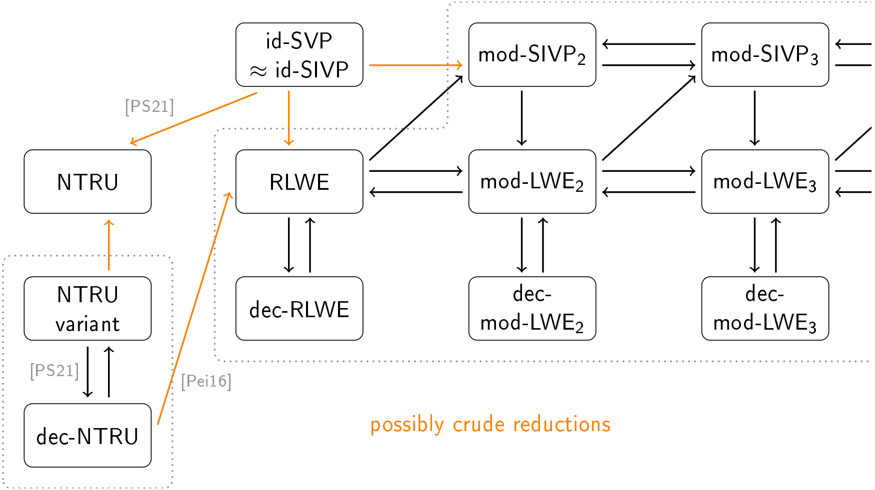

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Breaking id-SVP do break:

- some early FHE schemes
- the PV-Knap problem (see next slides)

PV-Knap (a.k.a, partial Fourier recovery problem)

Notations:

- $K = \mathbb{Q}[X]/\Phi_N(X)$ with Φ_N cyclotomic polynomial
  - $\Phi_N(\alpha) = 0$ if and only if α is a primitive N-th root of unity

---

[HPS+14] Hoffstein, Pipher, Schanck, Silverman, and Whyte. Practical signatures from the partial Fourier recovery problem. ACNS.

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Partial Vandermonde Knapsack (PV-Knap) [HPS+14]

Parameters: q, S_t and $B > 1$

Objective: recover f from $(f(\omega) \bmod q)_{\omega \in S_t}$, where

- $f = f(X) \in \mathcal{O}_K$ is sampled randomly such that $\|\sigma(f)\| \leq B$

(The original article worked in $\mathbb{Q}[X]/(X^N - 1)$ and with Σ)

[HPS+14] Hoffstein, Pipher, Schanck, Silverman, and Whyte. Practical signatures from the partial Fourier recovery problem. ACNS.

PV-Knap is an (ideal) lattice problem

PV-Knap

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A few observations:

- easy to recover a large $\tilde{f}$ such that $\tilde{f}(\omega) = f(\omega) \bmod q$, $\forall \omega \in S_t$
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- Recovering small f from large $\tilde{f}$ is a BDD in

$$\Lambda = \sigma\Big(\{g \in \mathcal{O}_K \mid g(\omega) = 0 \bmod q,\ \forall \omega \in S_t\}\Big)$$

(if parameters are well chosen)

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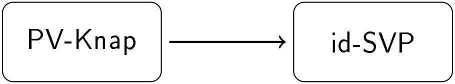
  (if parameters are well chosen)
- Λ is an ideal lattice [BSS22]

[BSS22] Boudgoust, Sakzad, and Steinfeld. Vandermonde meets Regev: Public Key Encryption Schemes Based on Partial Vandermonde Problems. DCC.

Hardness of PV-Knap



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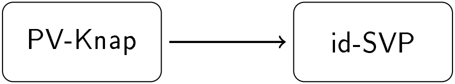
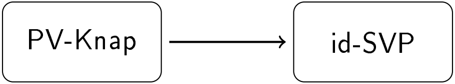


Warning:

- The reduction produces specific ideals
  (they divide $\langle q \rangle$)
  - PV-Knap might be easier than id-SVP

Hardness of PV-Knap



Warning:

- The reduction produces specific ideals
  (they divide $\langle q \rangle$)
  - PV-Knap might be easier than id-SVP
- if S_t is badly chosen, id-SVP can be solved in poly time [BGP22]
  - attacks on PV-Knap for bad choices of S_t

[BGP22] Boudgoust, Gachon, and Pellet-Mary. Some Easy Instances of Ideal-SVP and Implications on the Partial Vandermonde Knapsack Problem. Crypto.

Outline of the talk

The Log function

$$\text{Log} : K \to \mathbb{R}^d$$
$$y \mapsto (\log|y(\alpha_1)|, \cdots, \log|y(\alpha_d)|)$$

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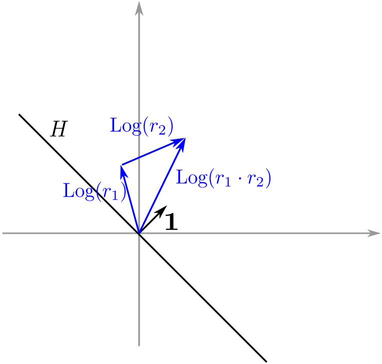
Let $\mathbf{1} = (1, \cdots, 1)$ and $H = \mathbf{1}^\perp$.

Properties ($r \in O_K$)

$\text{Log}\, r = h + a \cdot \mathbf{1}$, with $h \in H$

- $\text{Log}(r_1 \cdot r_2) = \text{Log}(r_1) + \text{Log}(r_2)$



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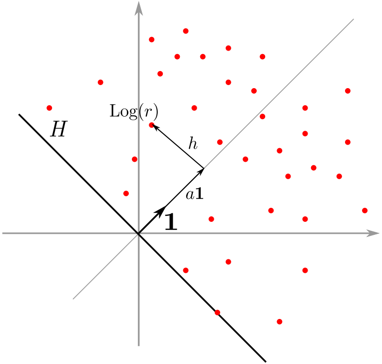
- $a \geq 0$



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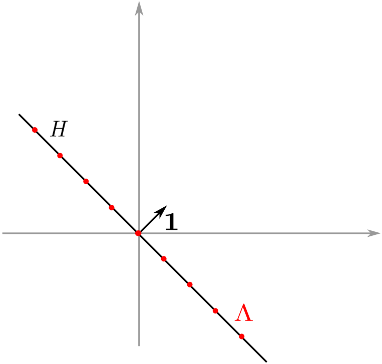
Let $\mathbf{1} = (1, \cdots, 1)$ and $H = \mathbf{1}^\perp$.

Properties ($r \in O_K$)

$\mathrm{Log}\, r = h + a \cdot \mathbf{1}$, with $h \in H$

- $\mathrm{Log}(r_1 \cdot r_2) = \mathrm{Log}(r_1) + \mathrm{Log}(r_2)$
- $a \geq 0$
- $a = 0$ iff r is a unit



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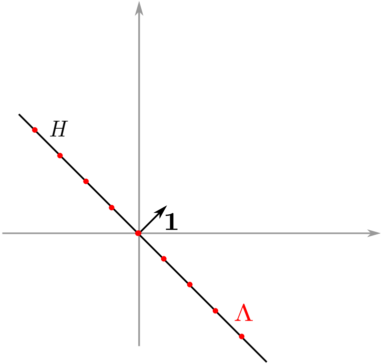

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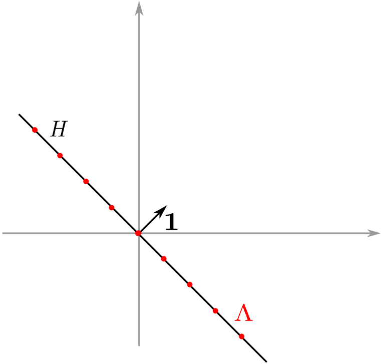

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Example:

$\vdots$
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$\quad | \; 2$
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Subfields: If L subfield of K, there exist $S_L \subseteq \{1, \cdots, d\}$ s.t.

- $|S_L| = [K : L] - 1$
- for all $f \in K$,

$$\mathcal{N}_{K/L}(f) := f \cdot \prod_{i \in S_L} \sigma_i(f) \in L$$

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Thank you