# Geodesically equivalent metrics and Nijenhuis geometry

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# Nijenhuis geometry and Nijenhuis operators

## Definition

A field of endomorphisms $L = (L^i_j)$ is called a *Nijenhuis operator*, if

$$\mathcal{N}_L(\xi,\eta) \stackrel{\text{def}}{=} L^2[\xi,\eta] - L[L\xi,\eta] - L[\xi,L\eta] + [L\xi,L\eta] = 0$$

for all vector fields $\xi$, $\eta$.

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Equivalently, in simple terms, *Nijenhuis structure* is defined by an $n \times n$ matrix $L(x) = \begin{pmatrix} L^1_1(x) & \dots & L^1_n(x) \\ \vdots & \ddots & \vdots \\ L^n_1(x) & \dots & L^n_n(x) \end{pmatrix}$, $L^i_j(x) = L^i_j(x_1, \dots, x_n)$, such that

$$(\mathcal{N}_L)^i_{jk} \stackrel{\text{def}}{=} \sum_\alpha \left( L^\alpha_j \frac{\partial L^i_k}{\partial x^\alpha} - L^\alpha_k \frac{\partial L^i_j}{\partial x^\alpha} - L^i_\alpha \frac{\partial L^\alpha_k}{\partial x^j} + L^i_\alpha \frac{\partial L^\alpha_j}{\partial x^k} \right) = 0,$$

$i, j, k = 1 \dots, n$.

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- $L(x) = J^{-1}SJ$, where $S = \begin{pmatrix} \sigma_1(x) & 1 & & \\ \sigma_2(x) & 0 & \ddots & \\ \vdots & \vdots & \ddots & 1 \\ \sigma_n(x) & 0 & \dots & 0 \end{pmatrix}$ and $J = \left( \dfrac{\partial \sigma_i}{\partial x_j} \right)$.

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- $L = A + b\, x^\top + x\, b^\top + K\, x\, x^\top, \quad b = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}, \; x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}.$

# Fundamental property of the characteristic polynomial

Theorem

*The characteristic polynomial*

$$\sigma(\lambda) = \det\big(\lambda \operatorname{Id} - L(x)\big) = \lambda^n - \sigma_1(x)\lambda^{n-1} - \sigma_2(x)\lambda^{n-2} - \ldots - \sigma_n(x)$$

*of a Nijenhuis operator $L$ satisfies the following identity*

$$(L - \lambda \operatorname{Id})^* \, \mathrm{d}\sigma(\lambda) = \sigma(\lambda) \, \mathrm{d}\operatorname{tr} L. \tag{1}$$

*Conversely, if* (1) *holds for a certain operator $L$ and the coefficients $\sigma_i$ of the characteristic polynomial are functionally independent, then $L$ is a Nijenhuis.*

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An operator $B = (B^i_j)$ is called a *symmetry* of $A$, if

- $AB = BA$
- $\langle A, B \rangle(\xi, \xi) = A[B\xi, \xi] + B[\xi, A\xi] - [A\xi, B\xi] = 0.$

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## Characteristic property of Nijenhuis operators.

Every conservation law $f = f_0$ of a Nijenhuis operator $L$ generates a hierarchy of conservation laws $\mathrm{d}f_k = (L^*)^k \, \mathrm{d}f$, $k = 0, 1, \dots$
Conversely, if an operator $A$ admits such an hierarchy for $k = 0, 1, \dots, n$ and $\mathrm{d}f_k$, $k = 0, \dots, n-1$, are linearly independent, then $A$ is Nijenhuis.

# Example

For a diagonal Nijenhuis operator

$$L = \begin{pmatrix} \lambda_1(x_1) & & \\ & \ddots & \\ & & \lambda_n(x) \end{pmatrix}$$

the conservation laws and symmetries (which will be automatically strong) are or the form

$$h = h_1(u_1) + h_2(u_2) + \cdots + h_n(u_n),$$

and

$$M = \begin{pmatrix} m_1(u_1) & & \\ & \ddots & \\ & & m_n(u_n) \end{pmatrix}$$

where $h_i$ and $m_i$ are arbitrary smooth functions.

# More interesting example

Let $L = L(u)$ be a Nijenhuis operator and $\sigma(\lambda) = \det\big(\lambda\,\mathrm{Id} - L(x)\big)$ is the characteristic polynomial of $L$. Consider the family of operators

$$A_\lambda = \sigma(\lambda)(L - \lambda\,\mathrm{Id})^{-1}. \tag{2}$$

All these operators are symmetries of each other. The functions $\dfrac{1}{\sigma(\mu)}$, $\mu \in \mathbb{R}$ are common conservation laws for these operators.

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One more useful description of common conservation laws for $A_\lambda$.

Let $L$ be $\mathrm{gl}$-regular Nijenhuis operator, and $M$ be an arbitrary symmetry of $L$. Then $M$ can be uniquely written as a linear combination

$$M = g_1 L^{n-1} + g_2 L^{n-2} + \dots + g_n \mathrm{Id},$$

with smooth coefficients $g_i$. The function $g_1$ is a common conservation law for all $A_\lambda$. Moreover, the following relation holds

$$A_\lambda^* \, \mathrm{d}g_1 = \mathrm{d}\big(\lambda^{n-1} g_1 + \lambda^{n-2} g_2 + \dots + g_n\big). \tag{3}$$

# Proof

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We use the fundamental property of the characteristic polynomial $\sigma(\mu)$:

$(L - \mu\,\mathrm{Id})^* \,\mathrm{d}\sigma(\mu) = \sigma(\mu)\,\mathrm{d}\mathrm{tr}\, L$ or in the new notation $A_\mu^*\,\mathrm{d}\sigma(\mu) = \mathrm{d}\mathrm{tr}\, L$

and the matrix identity $A_\lambda A_\mu = \frac{1}{\lambda - \mu}\big(\sigma(\mu) A_\lambda - \sigma(\lambda) A_\mu\big)$.

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Then

$$A_\lambda^*\,\mathrm{d}\left(\tfrac{1}{\sigma(\mu)}\right) = -\tfrac{1}{\sigma^2(\mu)}A_\lambda^*\,\mathrm{d}\sigma(\mu) = -\tfrac{1}{\sigma^2(\mu)}A_\lambda^* A_\mu^*\,\mathrm{dtr}\,L =$$

$$-\tfrac{1}{\lambda-\mu}\tfrac{1}{\sigma^2(\mu)}\big(\sigma(\mu)A_\lambda^*\,\mathrm{dtr}\,L - \sigma(\lambda)A_\mu^*\,\mathrm{dtr}\,L\big) =$$

$$-\tfrac{1}{\lambda-\mu}\tfrac{\sigma(\mu)\,\mathrm{d}\sigma(\lambda)-\sigma(\lambda)\,\mathrm{d}\sigma(\mu)}{\sigma^2(\mu)} = -\tfrac{1}{\lambda-\mu}\,\mathrm{d}\left(\tfrac{\sigma(\lambda)}{\sigma(\mu)}\right).$$

## Definition (Wikipedia)

In geometry, a *geodesic* is a curve representing in some sense the shortest path (arc) between two points in a surface, or more generally in a Riemannian manifold. It is a generalization of the notion of a "straight line".

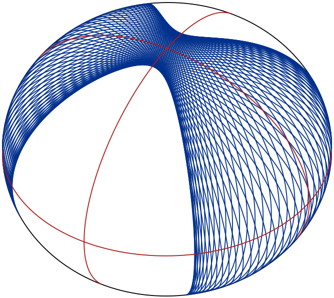

Рис.: A geodesic on a triaxial ellipsoid

By Cffk - Own work, CC BY-SA 3.0,
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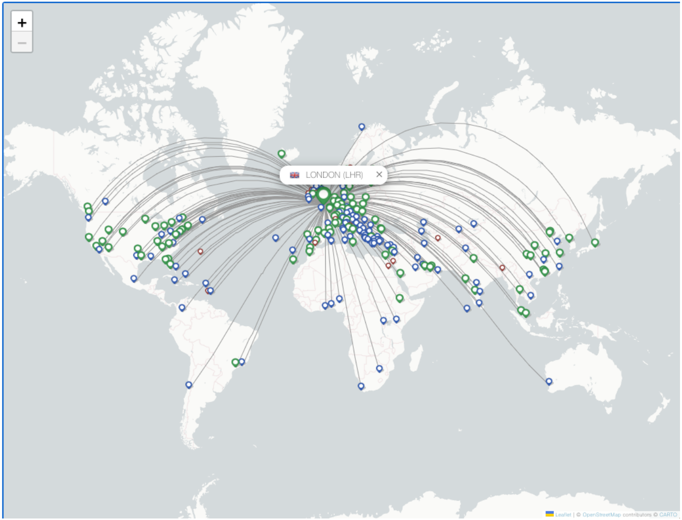
# FlightsFrom.com website




Flights from London Heathrow

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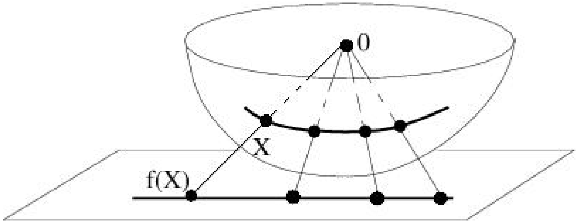

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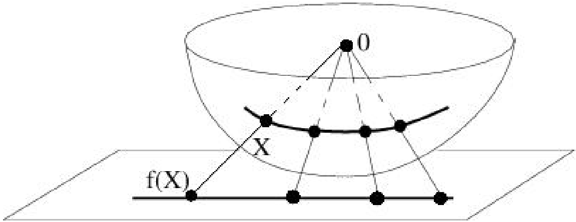


**Beltrami problem.** Describe/classify pairs of geodesically equivalent metrics.

# Classical result (everything is done?)

## Theorem (Levi-Civita (1896))

*Let $g$ and $\bar{g}$ be geodesically equivalent Riemannian metrics then (at a generic point)*

$$g = \sum_{i=1}^{n} \left( \pm \prod_{\alpha \neq i} (\lambda_i(x_i) - \lambda_\alpha(x_\alpha)) \right) \mathrm{d}x_i^2,$$

$$\bar{g} = \sum_{i=1}^{n} \left( \pm \frac{1}{\lambda_i(x_i) \prod_\alpha \lambda_\alpha(x_\alpha)} \prod_{\alpha \neq i} (\lambda_i(x_i) - \lambda_\alpha(x_\alpha)) \right) \mathrm{d}x_i^2.$$

*for some smooth functions $\lambda_i(x_i)$.*

# From $g$ and $\bar g$ to Nijenhuis operators: Sinjukov equation

Observation. It is more convenient to 'replace' $\bar g$ with the operator $L$ defined by

$$L = \left|\frac{\det \bar g}{\det g}\right|^{\frac{1}{n+1}} g\,\bar g^{-1}.$$

Notice that this matrix relation is equivalent to

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*Metrics $g$ and $\bar{g}$ are geodesically equivalent if and only if $L$ satisfies the PDE system*

$$\nabla_k L_{ij} = \frac{1}{2}\left( g_{jk} \frac{\partial \operatorname{tr} L}{\partial x_i} + g_{ik} \frac{\partial \operatorname{tr} L}{\partial x_j} \right), \quad L_{ij} = \sum_{\alpha} g_{i\alpha} L^{\alpha}_{j}. \tag{4}$$

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**Conclusion.** The study of geodesically equivalent metrics 'reduces' to the study of *geodesically compatible* pairs $(L, g)$ (i.e., such that $L$ is a $g$-selfadjoint and satisfies (4)).

# The key facts

Theorem (Sinjukov, Matveev, Topalov, Gover, Tabachnikov)

(a) *$L$ is a Nijenhuis operator.*

(b) *Operators* $A_\lambda = \det(L - \lambda \mathrm{Id}) \cdot (L - \lambda \mathrm{Id})^{-1}$ *are Killing* $(1,1)$*-tensors of the metric* $g$.

(c) *Functions* $F_\lambda : T^*\mathrm{M} \to \mathbb{R}$, $F_\lambda = g^{-1}(A_\lambda^* p, p)$, *are Poisson commuting first integrals of the geodesic flow of* $g$ *on* $T^*\mathrm{M}$.

(d) *If* $L$ *is* $\mathrm{gl}$*-regular, then among these integrals* $F_\lambda$, $\lambda \in \mathbb{R}$, *we can choose* $n = \dim \mathrm{M}$ *functionally independent integrals and, therefore, the geodesic flow of* $g$ *is Liouville integrable.*

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Let $F_A = g^{-1}(Ap, p)$ and $F_B = g^{-1}(Bp, p)$ be quadratic integrals of the geodesic flow of a metric $g$, and $AB = BA$.
Then $F_A$ and $F_B$ Poisson commute if and only if the evolutionary flows $u_{t_1} = Au_x$ and $u_{t_2} = Bu_x$ commute, i.e., $A$ and $B$ are symmetries of each other.

# Adding an integrable potential

Geodesically compatible $g$ and $L$ produce an integrable geodesic flow. Can we add a potential $V$ such that the system with the Hamiltonian

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## Theorem

*Let $M$ be a symmetry of a gl-regular operator $L$ written in the form*

$$M = g_1 L^{n-1} + g_2 L^{n-2} + \cdots + g_n \mathrm{Id}.$$

*Consider the natural Hamiltonian system on $\mathrm{M}$ with metric $g$ and potential $V = g_1(q)$, i.e., the system on $T^*\mathrm{M}$ with the Hamiltonian $H = K + V = \frac{1}{2} g^{-1}(p,p) + g_1(q)$. This system is Liouville integrable, its commuting integrals are functions of the form $\widetilde{F}_\lambda = F_\lambda + V_\lambda$, where*

$$V_\lambda(x) = g_1 \lambda^{n-1} + g_2 \lambda^{n-2} + \cdots + g_n.$$

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can be easily solved

$$L_{kj}(x) = A_{kj} + b_k x_j + b_j x_k + K x_k x_j$$

In matrix form:

$$L = A + b\,x^\top + x\,b^\top + K\,x\,x^\top, \quad b = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}, \; x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}. \qquad (5)$$

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In matrix form:

$$L = A + b\,x^\top + x\,b^\top + K\,x\,x^\top, \quad b = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}, \; x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}. \qquad (5)$$

## Theorem

*Let $\bar g$ be a metric on $\mathbb{R}^n$ whose geodesics are straight lines. Then*

$$\bar g = \frac{1}{\det L} L^{-1}, \quad \textit{where } L \textit{ is given by } (5).$$

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Consider $g = \sum_{m=1}^{n} \left( \sigma_{n-m} \sum_{i+j=m+1} \mathrm{d}u_i \, \mathrm{d}u_j \right)$ or in matrix form

$$g_{\mathrm{comp2}} = \begin{pmatrix} \sigma_{n-1} & \sigma_{n-2} & \dots & \sigma_1 & 1 \\ \sigma_{n-2} & & \sigma_1 & 1 & 0 \\ \vdots & \iddots & \iddots & \iddots & \vdots \\ \sigma_1 & 1 & 0 & & \\ 1 & 0 & 0 & \dots & 0 \end{pmatrix}$$

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## Theorem

*$L_{\mathrm{comp2}}$ and $g_{\mathrm{comp2}}$ are geodesically compatible.*

# Magic formula (continued)

This formula can be easily transformed to any other coordinate system $x_1, \dots, x_n$:

$$g = C^\top g_{\text{comp2}}\, C \tag{6}$$

where $C = \left(\frac{\partial u_i}{\partial x_j}\right)$ is the transition matrix, which is very special in this case

$$C = \begin{pmatrix} \mathrm{d}f \\ \mathrm{d}f \cdot L \\ \vdots \\ \mathrm{d}f \cdot L^{n-1} \end{pmatrix}, \quad \text{where } f \text{ is a conservation law of } L.$$

## Example (Levi-Civita formula)

Let $L = \operatorname{diag}\Big(\lambda_1(x_1), \lambda_2(x_2), \dots, \lambda_n(x_n)\Big)$. Take $f = x_1 + x_2 + \dots + x_n$. Then computing of (6) is just an algebraic exercise, leading to

$$g = \sum_i \sigma'(\lambda_i)\, \mathrm{d}x_i^2$$

where $\sigma'(\cdot)$ is the derivative of the characteristic polynomial of $L$ w.r.t. $\lambda$.

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$$g = \sum_i \sigma'(\lambda_i)\, \mathrm{d}x_i^2 = \sum_{i=1}^n \prod_{\alpha \neq i} \big(\lambda_i(x_i) - \lambda_\alpha(x_\alpha)\big)\, \mathrm{d}x_i^2,$$

where $\sigma'(\cdot)$ is the derivative of the characteristic polynomial of $L$ w.r.t. $\lambda$.

# Singularities in the context of geodesically equivalent metrics

*Singular points* are those at which the algebraic type of $L$ changes, e.g., the eigenvalues of $L$ collide.

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If $L$ is a $\mathrm{gl}$-regular operator, then its eigenvalues can still collide without violating the $\mathrm{gl}$-regularity condition. In the Nijenhuis geometry, scenarios of such collisions can be very different. However, regardless of any particular scenario, Magic Formula implies

**Theorem**

*Let $L$ be a $\mathrm{gl}$-regular real analytic Nijenhuis operator. Then (locally) there exists a pseudo-Riemannian metric $g$ geodesically compatible with $L$. Moreover, such a metric $g$ can be defined explicitly in terms of the second companion form of $L$.*

# How to describe all geodesically compatible partners for $L$?

Let $L$ be an admissible Nijenhuis operator (in the context of geodesic equivalence), i.e. there is at least one (pseudo)-Riemannian metric $g$ geodesically compatible with $L$.

**Open Problem 2.** Describe all geodesically compatible partners for $L$.

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Open Problem 2. Describe all geodesically compatible partners for $L$.

### Theorem

*Let $L$ and $g$ be geodesically compatible. Assume that $M$ is $g$-symmetric and is a strong symmetry of $L$, then $L$ and $gM := (g_{is} M^s_j)$ are geodesically compatible.*
*Moreover, if $L$ is $\mathrm{gl}$-regular, then every metric $\tilde{g}$ geodesically compatible with $L$ is of the form $\tilde{g} = gM$, where $M$ is a (strong) symmetry of $L$.*

Conclusion. Beltrami problem reduces essentially to the description of symmetries for Nijenhuis operators.

# Important example

## Theorem (Matveev, 2006)

*Consider a Riemannian metric $g$ and a Nijenhuis operator $L$ geodesically compatible with $g$. Assume that the eigenvalues of $L$ at a generic point are different, and $L(\mathrm{p}) = \lambda \cdot \mathrm{Id}$ at a singular point $\mathrm{p} \in \mathrm{M}$. Then up to a suitable coordinate transformation only the following three cases are possible:*

- $\dim \mathrm{M} = 2, \quad L = \lambda \cdot \mathrm{Id} \pm \begin{pmatrix} x^2 & xy \\ xy & y^2 \end{pmatrix};$
- $\dim \mathrm{M} = 2, \quad L = \lambda \cdot \mathrm{Id} + \begin{pmatrix} 2x & y \\ y & 0 \end{pmatrix};$
- $\dim \mathrm{M} = 3, \quad L = \lambda \cdot \mathrm{Id} + \begin{pmatrix} 2x & y & z \\ y & 0 & 0 \\ z & 0 & 0 \end{pmatrix};$

*As a geodesically compatible metric $g$ one can take the standerd Euclidean metric $g$, i.e. $\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2$ in dimension 2 and $\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2$ in dimension 3. Any other metric geodesically compatible with $L$ will be of the form $\tilde{g} = gM$, where $M$ is a certain symmetry of the operator $L$.*

# Classification in dimension 2: two examples from a long list

(joint project with D. Akpan)

- $$g = \frac{\mathrm{Im}\Big(W(ye^{-\mathrm{i}x})\Big)}{y}\,\mathrm{d}x\,\mathrm{d}y, \quad L = \begin{pmatrix} -\operatorname{Re} W & y^{-1}\operatorname{Im} W \\ -y\operatorname{Im} W & -\operatorname{Re} W \end{pmatrix}$$

  where $W(z) = H(z) + \mathrm{i}\,z\,h(z)$, $H$ and $h$ are arbitrary real analytic functions in a neighbourhood of $z = 0$ and $h(0) \neq 0$.

- $$g = \frac{X\left(y\left(1+\frac{2-s}{2}xy^{\frac{s-2}{2}}\right)^{\frac{2}{2-s}}\right) - Y\left(y\left(1-\frac{2-s}{2}xy^{\frac{s-2}{2}}\right)^{\frac{2}{2-s}}\right)}{2y^{s/2}}\,\mathrm{d}x\,\mathrm{d}y$$

  and

  $$L = \begin{pmatrix} X+Y & y^{-s/2}(X-Y) \\ y^{s/2}(X-Y) & X+Y \end{pmatrix},$$

  where $X = v^{s/2}h(v) + H(v)$, $Y = -v^{s/2}h(v) + H(v)$, $H$ and $h$ are arbitrary real analytic functions in a neighbourhood of $v = 0$ and $h(0) \neq 0$, $s > 2$.

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Happy Birthday, Sasha!