ON A LONGSTANDING OPEN PROBLEM IN THE THEORY OF MARKOV PROCESSES

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Abstract

We define a class of not necessarily linear C_0-semigroups $(P_t)_{t\geq 0}$ on $C_b(E)$ (more generally, on $C_\kappa(E) := \frac{1}{\kappa}C_b(E)$, for some growth bounding continuous function κ) equipped with the mixed topology $\tau_1^{\mathscr{M}}$ for a large class of topological state spaces E. In the linear case we prove that such $(P_t)_{t\geq 0}$ can be characterized as integral operators given by measure kernels satisfying certain properties. We prove that the strong and weak infinitesimal generators of such C_0-semigroups coincide. As a main result we prove that transition semigroups of Markov processes are C_0-semigroups on $(C_b(E), \tau_1^{\mathscr{M}})$, if they leave $C_b(E)$ invariant and they are jointly weakly continuous in space and time. In particular, they are infinitesimally generated by their generator $(L, D(L))$ and thus reconstructable through an Euler formula from their strong derivative at zero in $(C_b(E), \tau_1^{\mathscr{M}})$. This solves a long standing open problem on Markov processes. Our results apply to a large number of Markov processes given as the laws of solutions to SDEs and SPDEs, including the stochastic 2D Navier-Stokes equations and the stochastic fast and slow diffusion porous media equations. Furthermore, we introduce the notion of a Markov core operator $(L_0, D(L_0))$ for the above generators $(L, D(L))$ and prove that uniqueness of the Fokker-Planck-Kolmogorov equations corresponding to $(L_0, D(L_0))$ for all Dirac initial conditions implies that $(L_0, D(L_0))$ is a Markov core operator for $(L, D(L))$. As a consequence we can identify the Kolmogorov operator of a large number of SDEs on finite and infinite dimensional state spaces as Markov core operators for the infinitesimal generators of the C_0-semigroups on $\left(C_\kappa(E), \tau_\kappa^{\mathscr{M}}\right)$ given by their transition semigroups. If each P_t is merely convex, we prove that $(P_t)_{t\geq 0}$ gives rise to viscosity solutions to the Cauchy problem of its associated (non linear) infinitesimal generators. Furthermore, we prove that each P_t has a stochastic representation as a convex expectation in terms of a nonlinear Markov process.

Joint work with:
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References

[1] Ben Goldys, Max Nendel and Michael Röckner. Operator semigroups in the mixed topology and the infinitesimal description of Markov processes, 2022; arXiv:2204.07484.