SOLVERS FOR FREQUENCY-DOMAIN WAVE PROBLEMS AND APPLICATIONS
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Stable approximation of Helmholtz solutions by evanescent plane waves

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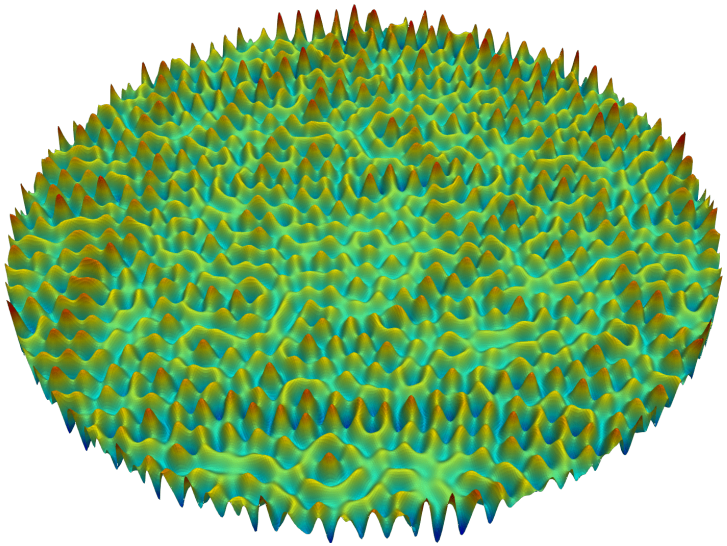
Helmholtz equation

Homogeneous Helmholtz equation:

$$-\Delta u - \kappa^2 u = 0$$

Wavenumber $\kappa = \omega/c > 0$,
$\lambda = \frac{2\pi}{\kappa}$ = wavelength.



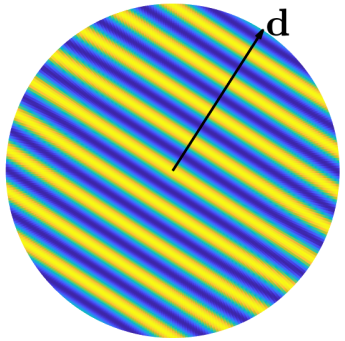
$u(\mathbf{x})$ represents the space dependence of time-harmonic solutions $U(\mathbf{x}, t) = \Re\{\mathrm{e}^{-\mathrm{i}\omega t} u(\mathbf{x})\}$ of the wave equation $\frac{1}{c^2}\frac{\partial^2 U}{\partial t^2} - \Delta U = 0$.

- "Easy" PDE for small κ: perturbation of Laplace eq.
- "Difficult" PDE for large κ: high-frequency problems

Propagative plane waves

A difficulty for $\kappa \gg 1$ is the approximation of Helmholtz solutions.

One can beat (piecewise) polynomial approximations using propagative plane waves (PPWs):



$$e^{i\kappa \mathbf{d}\cdot\mathbf{x}} \qquad \mathbf{d} \in \mathbb{R}^n \quad \mathbf{d}\cdot\mathbf{d} = 1$$

Some uses of PPWs:

- Trefftz methods:
  Galerkin schemes whose basis functions are local PDE solutions.
  E.g.: UWVF, TDG, PWDG, DEM, VTCR, WBM, LS, PUM ...
- reconstruction of sound fields from point measurements (microphones) in experimental acoustics.

PPWs are complex exponentials:
easy & cheap to manipulate, evaluate, differentiate, integrate...
$\rightarrow$ preferred against other Trefftz functions (e.g. circular waves)

Approximation and instability

Rich PPW approximation theory for Helmholtz solutions:

- Cessenat, Després 1998, Taylor-based, h
- Melenk 1995; Moiola, Hiptmair, Perugia 2011, Vekua theory, hp

κ-explicit, better rates vs DOFs than polynomials.

So why isn't everybody using plane waves?

The issue is "instability".
Increasing # of PPWs, at some point convergence stagnates.

Numerical phenomenon: due to computer arithmetic+cancellation.

PPW instability already observed in all PPW-based Trefftz methods.
Usually described and treated as ill-conditioning issue.

Adcock–Huybrechs theory

BEN ADCOCK, DAAN HUYBRECHS, SiRev 2019 & JFAA 2020,
"Frames and numerical approximation I & II"

Goal: Approximate some $v \in V$ with linear combination of $\{\phi_m\} \subset V$.

Result: If there exists $\sum_m a_m \phi_m$ with
- good approximation of v,
- small coefficients a_m,

then the approximation of v in computer arithmetic is stable,
if one uses oversampling and SVD regularization.

Stability does not depend on (LS, Galerkin,. . .) matrix conditioning.

Spoiler:
- − PPWs can not approximate u with small coefficients.
- + Evanescent PWs $\to$ small-coefficient approx. $\to$ stability.

Here we consider only the approximation in the unit disk $B_1 \subset \mathbb{R}^2$.
(But numerical results on convex polygons are very promising!)

Part I

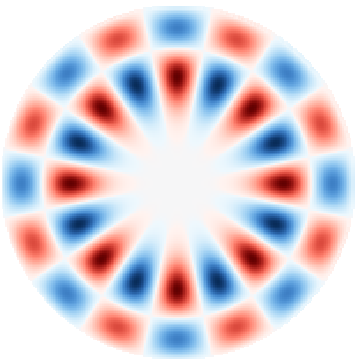
Circular and propagative plane waves

Circular waves — Fourier–Bessel functions

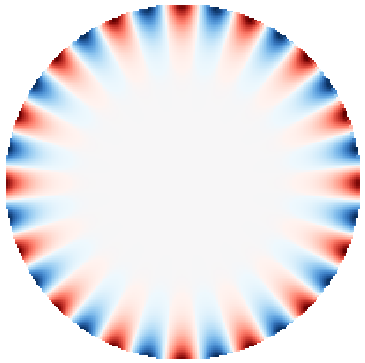
Separable solutions in polar coordinates:

$$b_p(r,\theta) := \beta_p J_p(\kappa r)\mathrm{e}^{\mathrm{i}p\theta} \qquad \forall p \in \mathbb{Z}, \quad (r,\theta) \in B_1$$

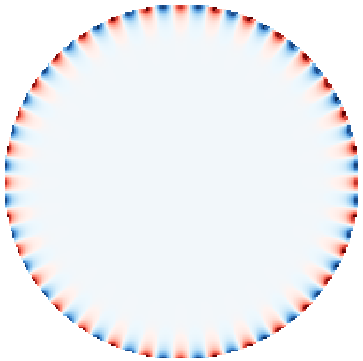
β_p = normalization, e.g. in $H^1(B_1)$ norm. $\qquad \beta_p \sim \kappa\left(\frac{2|p|}{\mathrm{e}\kappa}\right)^{|p|}$ as $p \to \infty$.



$p = 8 = \kappa/2$
Propagative mode



$p = 16 = \kappa$



$p = 32 = 2\kappa$
Evanescent mode

$\{b_p\}_{p\in\mathbb{Z}}$ is orthonormal basis of $\quad \mathcal{B} := \{u \in H^1(B_1) : -\Delta u - \kappa^2 u = 0\}$

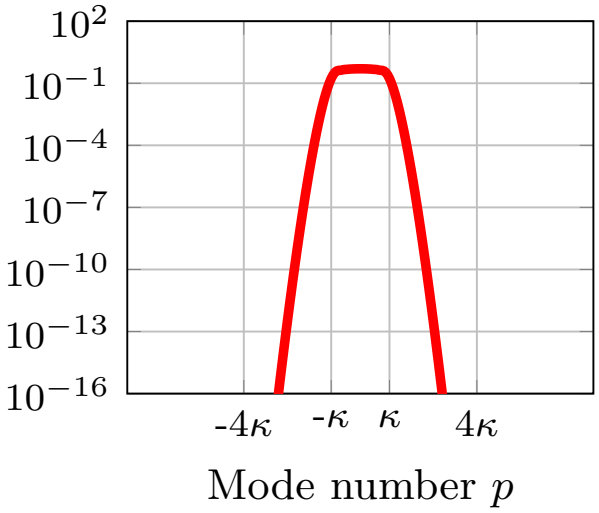
PPW instability

The Jacobi–Anger expansion relates PPWs and circular waves b_p:

$$\mathrm{PW}_\varphi(\mathbf{x}) := \mathrm{e}^{\mathrm{i}\kappa\mathbf{d}\cdot\mathbf{x}} = \sum_{p\in\mathbb{Z}} \mathrm{i}^p J_p(\kappa r)\mathrm{e}^{\mathrm{i}p(\theta-\varphi)}$$

$$= \sum_{p\in\mathbb{Z}} \left(\mathrm{i}^p \mathrm{e}^{-\mathrm{i}p\varphi}\beta_p{}^{-1}\right) b_p(r,\theta)$$

$$\begin{cases} \mathbf{d} = (\cos\varphi,\ \sin\varphi) \\ \mathbf{x} = (r\cos\theta,\ r\sin\theta) \end{cases}$$



Modulus of Fourier coefficient
$|\mathrm{i}^p\mathrm{e}^{-\mathrm{i}p\varphi}\beta_p^{-1}| = |\beta_p^{-1}| \sim |p|^{-|p|}$ indep. of φ.

Approximation of $u = \sum_p \widehat{u}_p b_p \in \mathcal{B}$
requires exponentially large coefficients.

$u \in H^s(B_1), s \geq 1 \iff |\widehat{u}_p| \sim o(|p|^{-s+\frac{1}{2}})$
but $|\beta_p^{-1}| \sim |p|^{-|p|}$ is much smaller!

$$\begin{matrix} \forall p \in \mathbb{Z} \\ \forall M \in \mathbb{N} \\ \forall \boldsymbol{\mu} \in \mathbb{C}^M \\ \forall \eta \in (0,1) \end{matrix} \quad \left\| b_p - \sum_{m=1}^{M} \mu_m \mathrm{PW}_{\frac{2\pi m}{M}} \right\|_{\mathcal{B}} \leq \eta \implies \|\boldsymbol{\mu}\|_{\ell^1(\mathbb{C}^M)} \geq (1-\eta)\ \underbrace{|\beta_p|}_{\sim|p|^{|p|}}$$

Part II

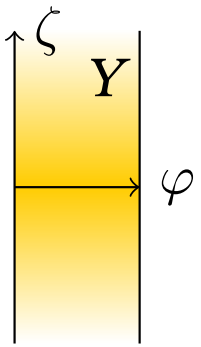
Evanescent plane waves

Evanescent plane waves

Idea from WBM (wave-based method) by Wim Desmet etc (Leuven).

Stability improves using PPWs & evanescent plane waves (EPW):

$$\mathrm{e}^{\mathrm{i}\kappa\mathbf{d}\cdot\mathbf{x}} \qquad \mathbf{d}\in\mathbb{C}^2 \quad \mathbf{d}\cdot\mathbf{d}=1$$

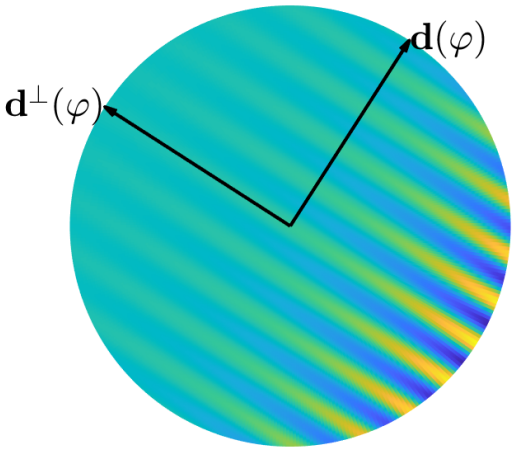


Complex $\mathbf{d}$! Again: exponential Helmholtz solutions.

Parametrised by φ = direction, ζ = "evanescence".
Parametric cylinder: $\mathbf{y} := (\varphi,\zeta) \in Y := [0,2\pi)\times\mathbb{R}$.

$$\mathbf{d}(\mathbf{y}) := \big(\cos(\varphi+\mathrm{i}\zeta),\,\sin(\varphi+\mathrm{i}\zeta)\big)\in\mathbb{C}^2$$

$$\begin{aligned}\mathrm{EW}_{\mathbf{y}}(\mathbf{x}) &:= \mathrm{e}^{\mathrm{i}\kappa\mathbf{d}(\mathbf{y})\cdot\mathbf{x}}\\ &= \mathrm{e}^{\mathrm{i}\kappa(\cosh\zeta)\mathbf{x}\cdot\mathbf{d}(\varphi)}\;\mathrm{e}^{-\kappa(\sinh\zeta)\mathbf{x}\cdot\mathbf{d}^{\perp}(\varphi)},\end{aligned}$$



oscillations along $\mathbf{d}(\varphi) := (\cos\varphi,\,\sin\varphi)$

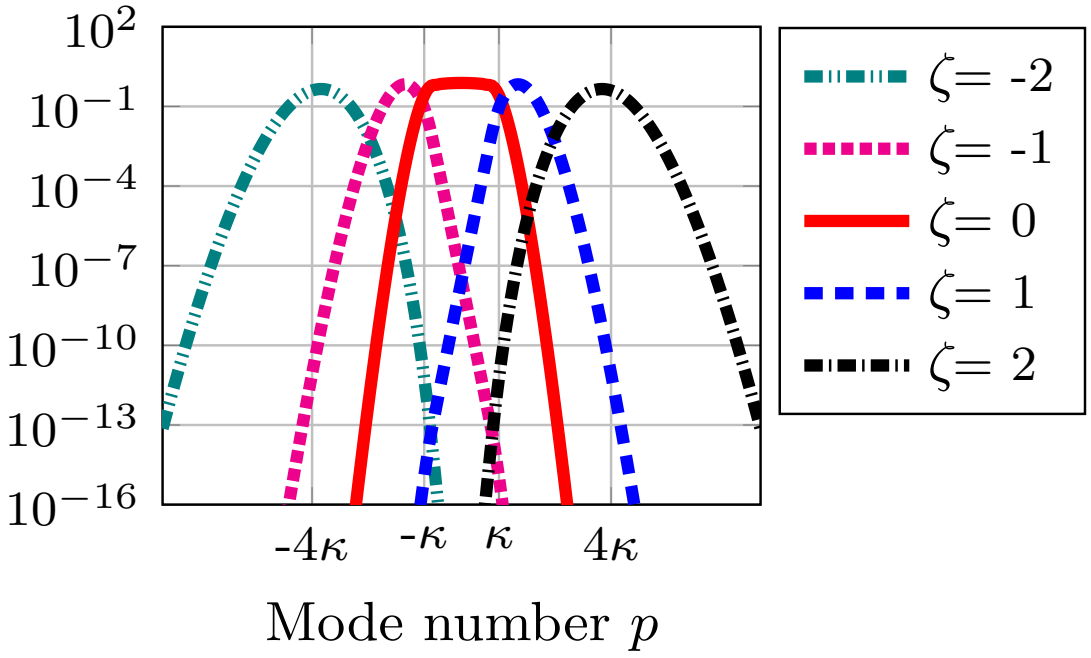
decay along $\mathbf{d}^{\perp}(\varphi) := (-\sin\varphi,\,\cos\varphi)$

EPW modal analysis

Jacobi–Anger expansion holds also for EPWs:

$$\mathrm{EW}_{\mathbf{y}}(\mathbf{x}) = \mathrm{e}^{\mathrm{i}\kappa\mathbf{d}(\mathbf{y})\cdot\mathbf{x}} = \sum_{p\in\mathbb{Z}} \left(\mathrm{i}^p \mathrm{e}^{-\mathrm{i}p\varphi} \mathrm{e}^{p\zeta} \beta_p^{-1}\right) b_p(\mathbf{x}).$$

Absolute values of Fourier coefficients $\left|\mathrm{i}^p \mathrm{e}^{-\mathrm{i}p\varphi} \mathrm{e}^{p\zeta} \beta_p^{-1}\right|, \quad \kappa = 16$:



Looks promising!

We can hope to approximate large-p Fourier modes with EPWs & small coefficients.

Herglotz representation with EPWs

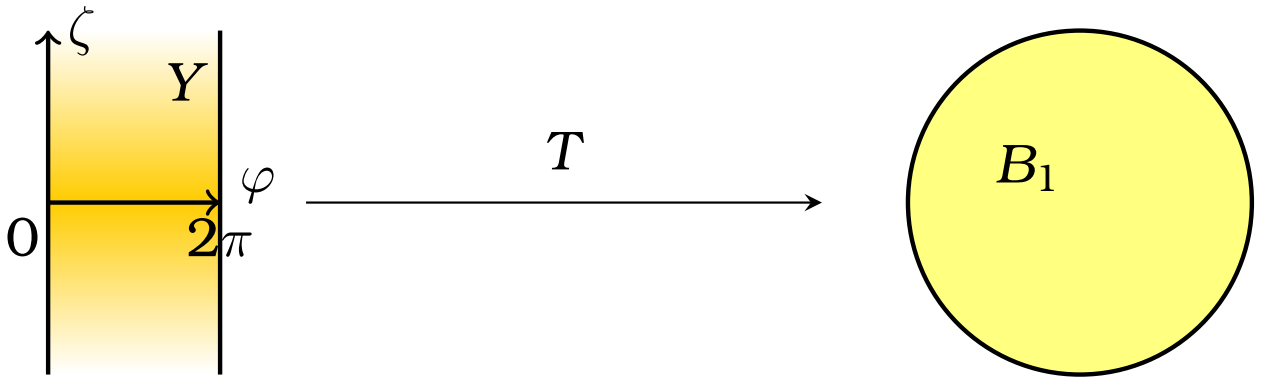
We want to represent $u \in \mathcal{B}$ as continuous superposition of EPWs:

$$u(\mathbf{x}) = (Tv)(\mathbf{x}) = \int_Y \mathrm{EW}_{\mathbf{y}}(\mathbf{x})\, v(\mathbf{y})\, w^2(\mathbf{y})\, \mathrm{d}\mathbf{y} \qquad \mathbf{x} \in B_1$$

with density $v \in L^2(Y; w^2)$ and weight $w^2 = \mathrm{e}^{-2\kappa \sinh|\zeta| + \frac{1}{2}|\zeta|}$

Parametric space

Physical space



Herglotz density

$v \in \mathcal{A} = \overline{\mathrm{span}\{a_p\}} \subset L^2(Y; w^2)$

Helmholtz solution

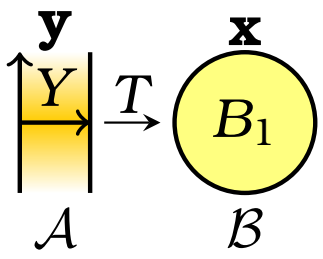
$u \in \mathcal{B} = \overline{\mathrm{span}\{b_p\}} \subset H^1(B_1)$

$a_p(\mathbf{y}) := \alpha_p\, \mathrm{e}^{p(\zeta + \mathrm{i}\varphi)}$ $\qquad$ $\alpha_p > 0$ normalization in $\|\cdot\|_{\mathcal{A}} = \|\cdot\|_{L^2(Y;w^2)}$, $p \in \mathbb{Z}$

Helmholtz solutions are superposition of EPWs

Define Herglotz transform: (synthesis operator)

$$(Tv)(\mathbf{x}) := \int_Y \mathrm{EW}_{\mathbf{y}}(\mathbf{x})\, v(\mathbf{y})\, w^2(\mathbf{y})\, \mathrm{d}\mathbf{y} \qquad \begin{array}{r} T : \mathcal{A} \to \mathcal{B} \\ v \mapsto u \end{array}$$



Jacobi–Anger $\Rightarrow$ T is diagonal in ONB's $\{a_p\}, \{b_p\}$:

$$\mathrm{EW}_{\mathbf{y}}(\mathbf{x}) = \sum_{p\in\mathbb{Z}} \tau_p \overline{a_p(\mathbf{y})} b_p(\mathbf{x}), \qquad \tau_p := \frac{\mathrm{i}^p}{\alpha_p \beta_p}, \qquad 0 < \tau_- \le |\tau_p| \le \tau_+ < \infty.$$

The operator $T : \mathcal{A} \to \mathcal{B}$ is bounded and invertible:

$$Ta_p = \tau_p b_p, \qquad \tau_- \|v\|_{\mathcal{A}} \le \|Tv\|_{\mathcal{B}} \le \tau_+ \|v\|_{\mathcal{A}} \qquad \forall v \in \mathcal{A}$$

Every Helmholtz solution is (continuous) linear combination of EPW with small coefficients: $\|v\|_{\mathcal{A}} \le \tau_-^{-1} \|u\|_{\mathcal{B}}$

How to sample $\mathcal{A} \subset L^2(Y; w^2)$? How to choose points $\{\mathbf{y}_m\}_m \in Y$?

Parameter sampling in Y

Idea from COHEN, MIGLIORATI, 2017.

Fix $P \in \mathbb{N}$, set $\mathcal{A}_P := \mathrm{span}\{a_p\}_{|p|\le P} \subset \mathcal{A}$. Define probability density

$$\rho(\mathbf{y}) := \frac{w^2}{2P+1} \sum_{|p|\le P} |a_p(\mathbf{y})|^2 \qquad \text{on } Y$$

$\rho^{-1} =$ "Christoffel function"

and generate $M \in \mathbb{N}$ nodes $\{\mathbf{y}_m\}_{m=1,\dots,M}$ distributed according to ρ.

We expect that any $u \in \mathrm{span}\{b_p\}_{|p|\le P}$ can be approximated by EPWs

$$\left\{ \mathbf{x} \mapsto \frac{1}{\sqrt{M \sum_{|p|\le P} |a_p(\mathbf{y}_m)|^2}} \mathrm{EW}_{\mathbf{y}_m}(\mathbf{x}) \right\}_{m=1,\dots,M} \subset \mathcal{B}$$

with small coefficients.

$\rightarrow$ Stable approx. in computer arithmetic using SVD & oversampling.

The M-dimensional EPW space depends on truncation parameter P: the space is tuned to approximate the Fourier modes b_p with $|p| \le P$.

Part III

Numerical results

Boundary sampling method

Given (PPW, EPW,...) approximation set $\operatorname{span}\{\phi_m\}_{m=1,\dots,M}$,
how do we approximate $u \in \mathcal{B}$ in practice?

We use boundary sampling on $\left\{\mathbf{x}_s = \binom{r=1}{\theta_s=\frac{2\pi s}{S}}\right\}_{s=1,\dots,S} \subset \partial B_1$:

$$A\xi = \mathbf{c} \qquad \text{with} \qquad \begin{array}{c} A_{s,m} := \phi_m(\mathbf{x}_s), \\ c_s := u(\mathbf{x}_s) \end{array} \quad \begin{array}{c} s=1,\dots,S \\ m=1,\dots,M \end{array} \quad \rightarrow \quad u_M = \sum_m \xi_m \phi_m \approx u.$$

Choose $\kappa^2 \neq$ Laplace–Dirichlet eigenvalue on B_1.

Could use instead: $\left\{\begin{array}{l} \text{sampling in the bulk of } B_1, \\ \text{impedance trace,} \\ \mathcal{B} \;/\; L^2(B_1) \;/\; L^2(\partial B_1) \text{ projection...} \end{array}\right.$

- Oversampling: $S > M$
- SVD regularization, threshold ϵ:

$\Big\}$ required by Adcock–Huybrechs

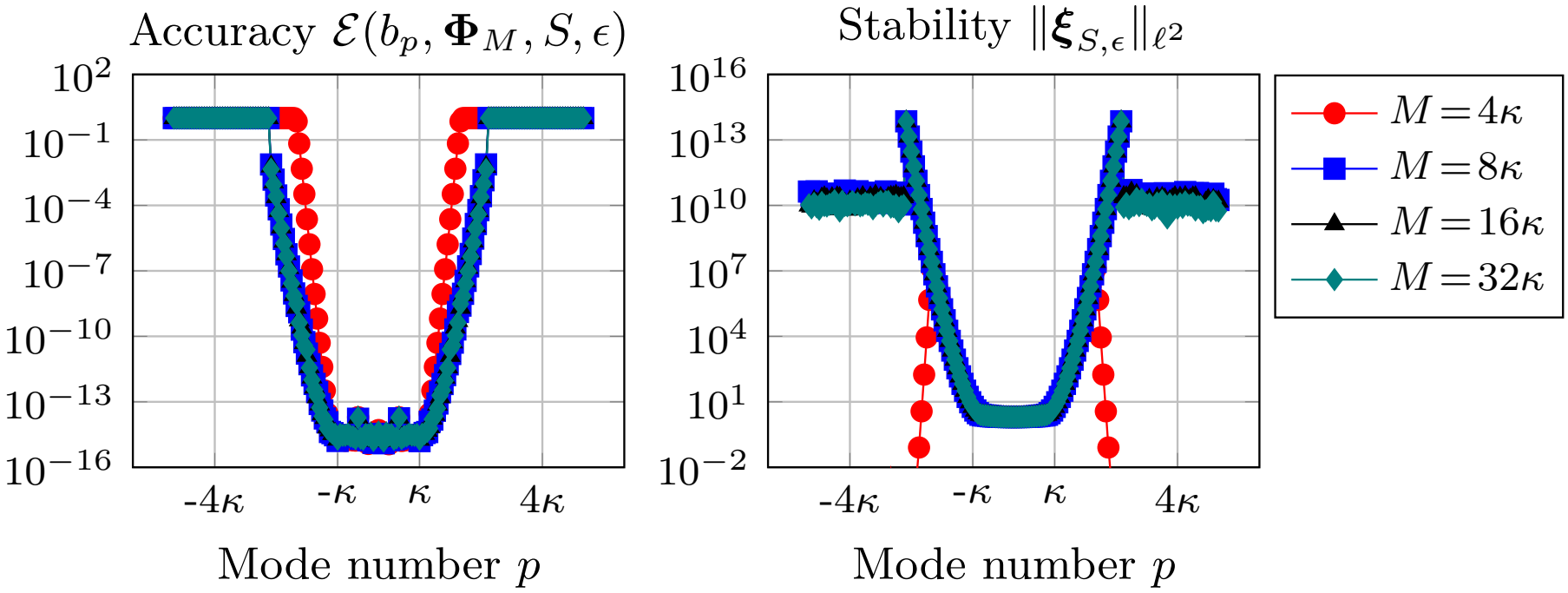
$$A = U\,\mathrm{diag}(\sigma_1,\dots,\sigma_M)\,V^*, \qquad \Sigma_\epsilon := \mathrm{diag}(\{\sigma_m > \epsilon \max_{m'} \sigma_{m'}\}),$$

$$\boxed{\xi_\epsilon = V\Sigma_\epsilon^\dagger U^* \mathbf{c}}$$

Approximation by PPWs

Approximation of circular waves $\{b_p\}_p$ by equispaced PPWs

$\kappa = 16, \qquad \epsilon = 10^{-14}, \qquad S = \max\{2M, 2|p|\},$ residual $\mathcal{E} = \frac{\|A\boldsymbol{\xi}_\epsilon - \mathbf{c}\|}{\|\mathbf{c}\|}$

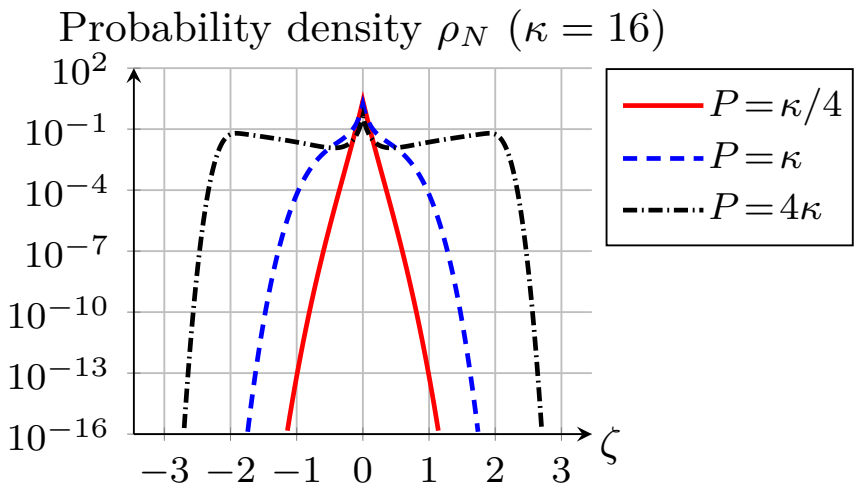


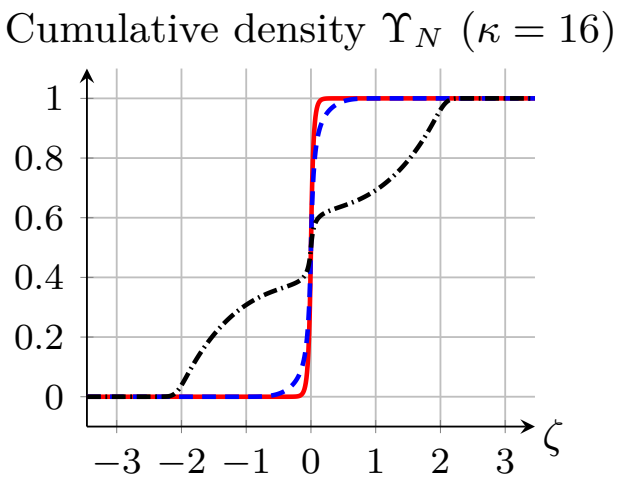
- Propagative modes $|p| \lesssim \kappa$: $\mathcal{O}(\epsilon)$ error $\forall M$, $\mathcal{O}(1)$ coeff.'s
- Evanescent modes $|p| \gtrsim 3\kappa$: $\mathcal{O}(1)$ error $\forall M$, large coeff.'s

Condition number is irrelevant!

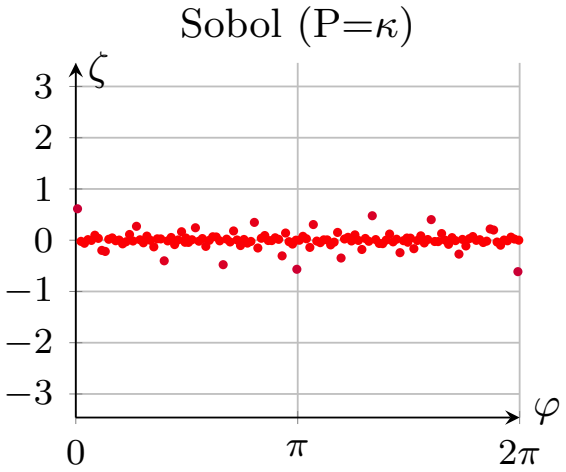
Probability measure ρ on Y and samples

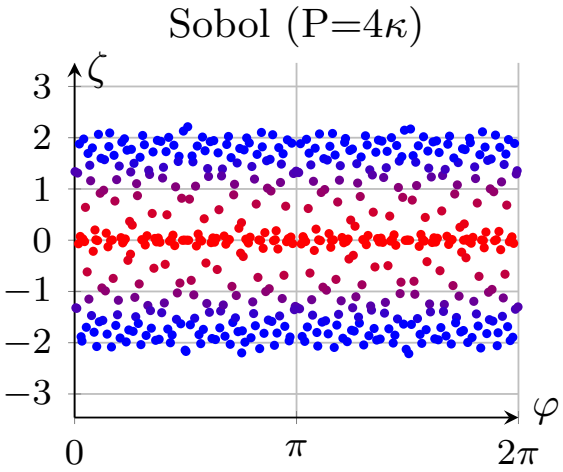
Probability density ρ & cumulative d.f. as functions of evanescence ζ:





They depend on P: target functions in $\mathrm{span}\{\boldsymbol{b}_p\}_{|p|\leq P}$.

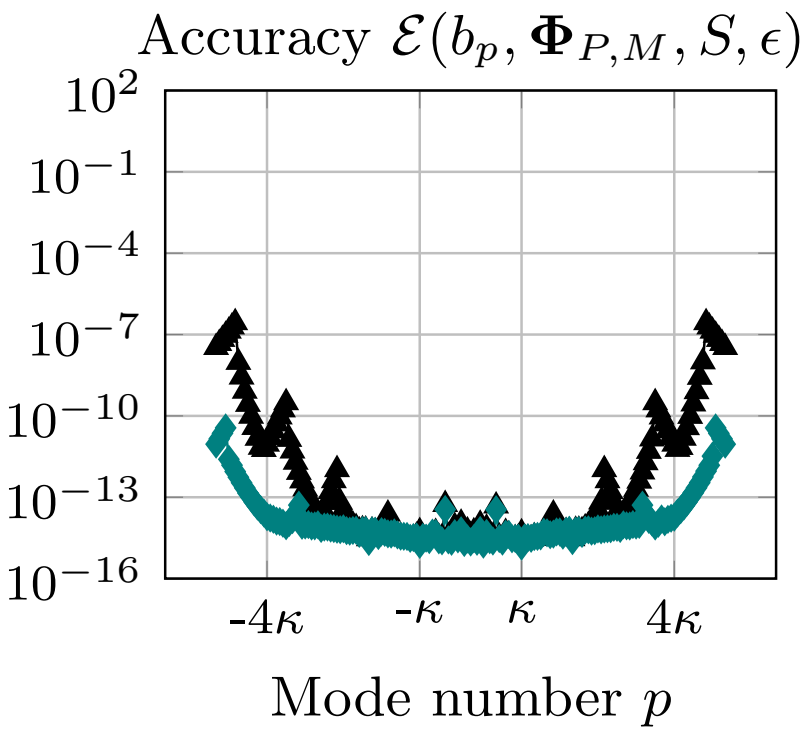


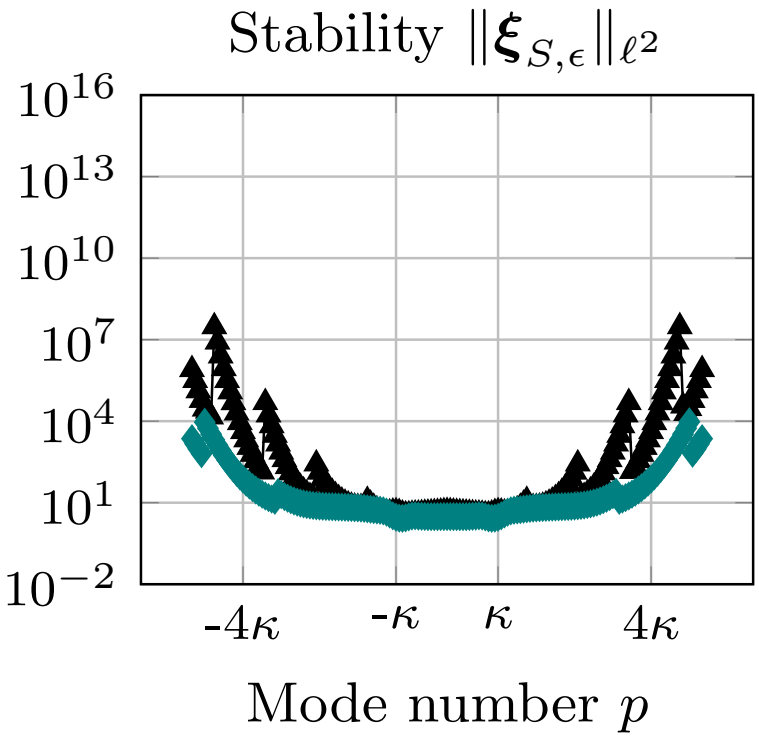


Samples computed on $(0,1)^2$ & uniform prob., mapped to Y by Υ^{-1}.

Approximation by EPWs

Approximation of $\{b_p\}$, $\boxed{P = 4\kappa}$, $\kappa = 16$, ▲ $M = 4P$, ♦ $M = 8P$



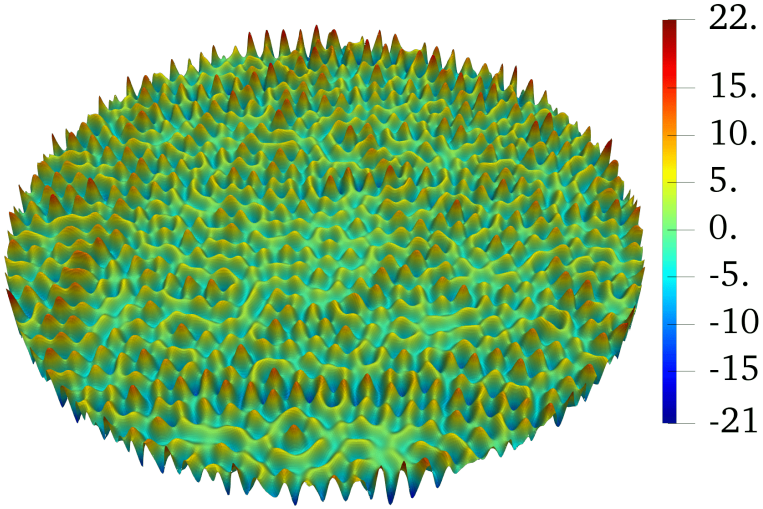


Discrete EPW space approximates all b_ps for $|p| \leq P$!

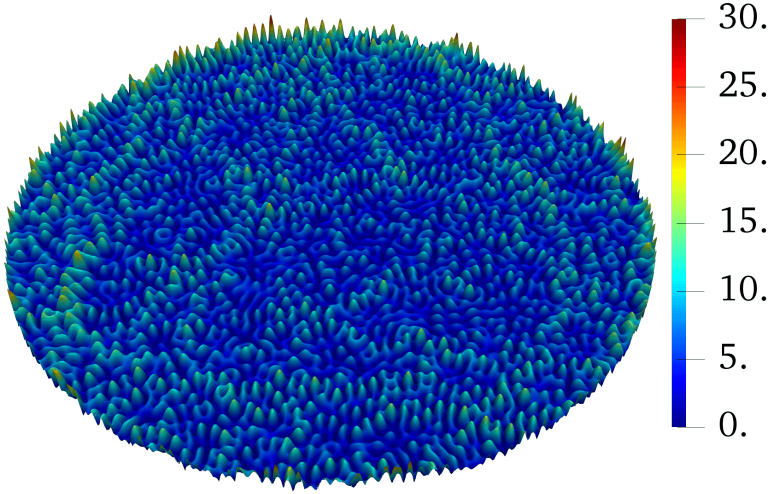
Solution and error plots

$u = \sum_{|p|\le P} \hat{u}_p b_p, \quad \hat{u}_p \sim (\max\{1, |p| - \kappa\})^{-1/2}, \quad \kappa = 100, \quad P = 2\kappa, \quad M = 802$

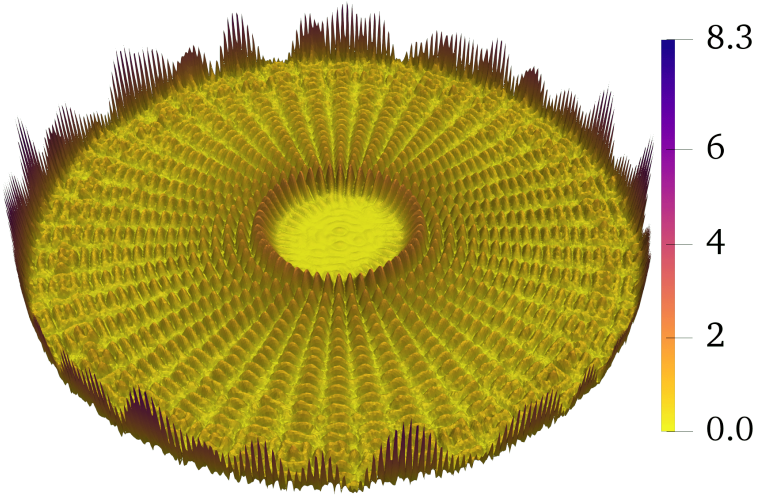
$\Re\{u\}$



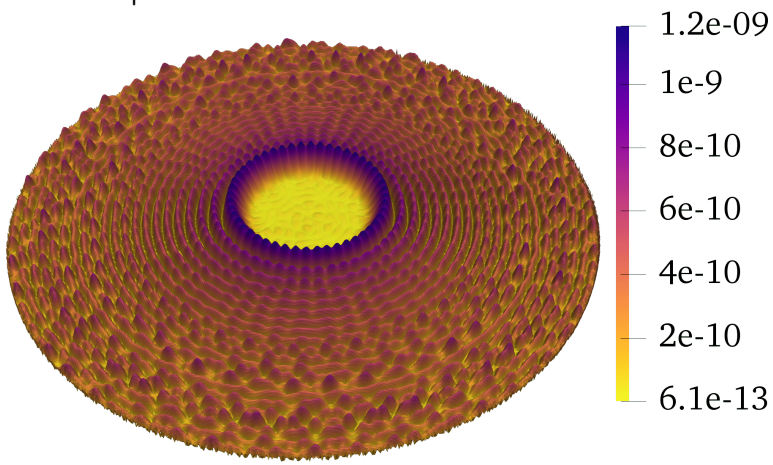
$|u|$



$|u - PPW|$



$|u - EPW|$

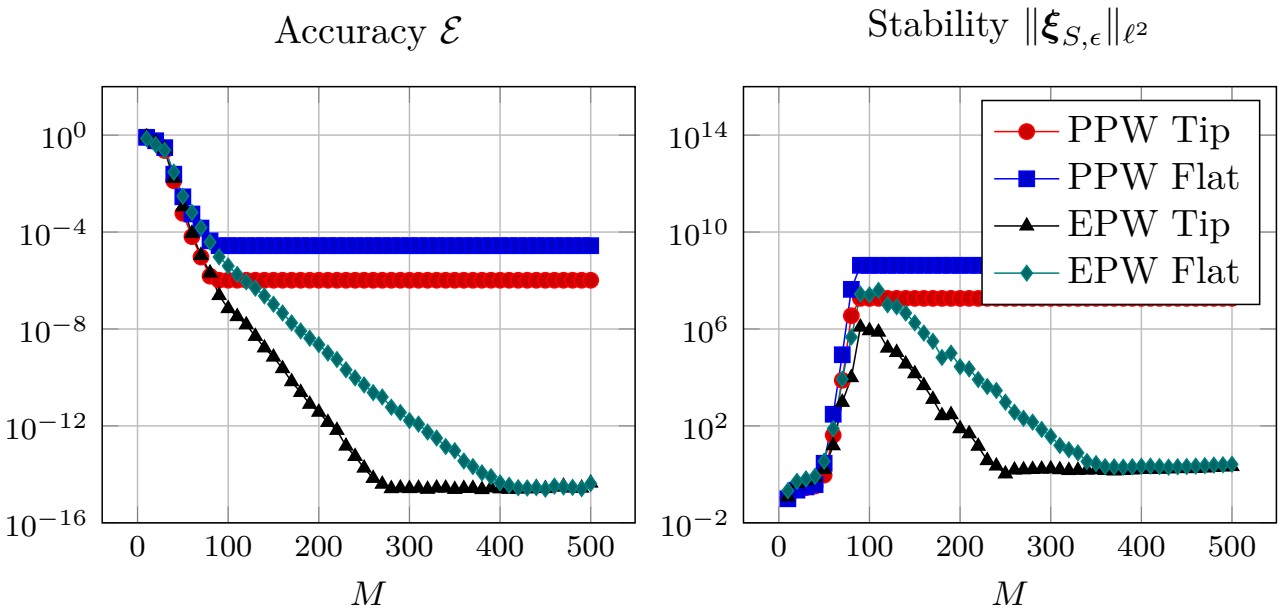


$\|u - PPW\|_{L^\infty} \gtrsim 7 \cdot 10^9 \, \|u - EPW\|_{L^\infty}$

DOFs/wavelength $= \lambda\sqrt{M/|B_1|} \approx 1$

Bonus: polygonal domain, same discrete space

$\kappa = 16, \qquad M = 200, \qquad u =$ fundamental solution at distance 0.25



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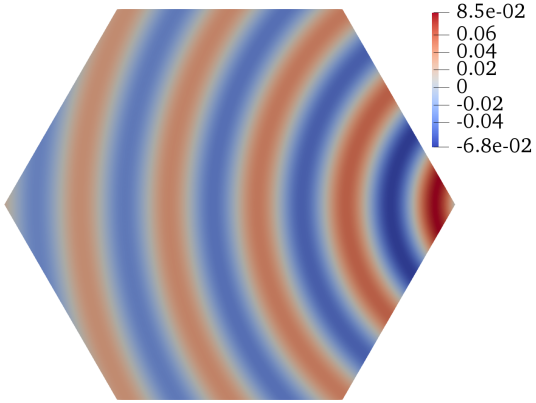
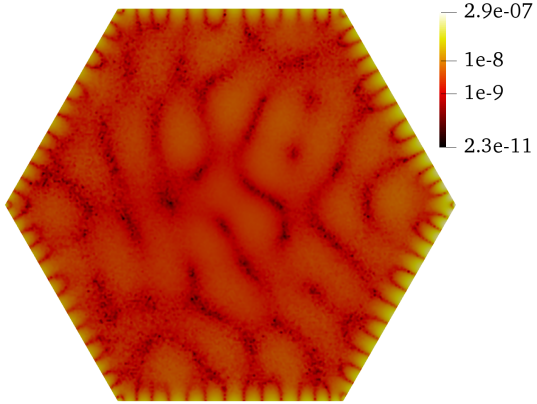
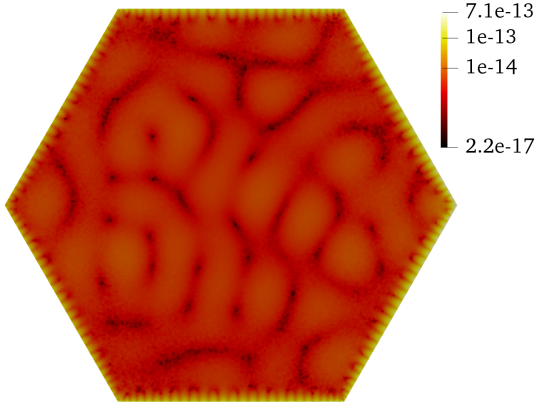


$|u - PPW|$



$|u - EPW|$



Summary

- Approximation of Helmholtz solutions by PPWs is unstable: accuracy only with large coefficients.
- Approximation by evanescent PWs seems to be stable.
- EPWs parameters chosen with sampling in Y.
- Key new result is stable Herglotz transform $u = Tv$.

Next steps:

- General geometries
- 3D
- Maxwell & elasticity
- Proof of EPW stability
- Use in Trefftz-DG & sampling
- ...

Thank you!

E. Parolin, D. Huybrechs, A. Moiola arXiv:2202.05658
Stable approximation of Helmholtz solutions by evanescent plane waves
Julia code on:
`https://github.com/EmileParolin/evanescent-plane-wave-approx`