A *non-overlapping DDM* with *PML transmission conditions* and *checkerboard partitions* for Helmholtz problems

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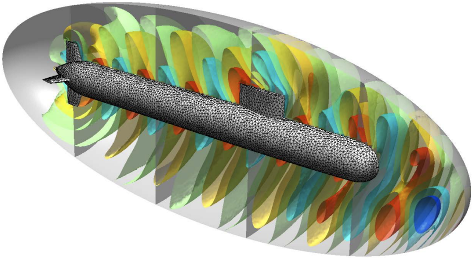
Exemple: Scattering problem

Figure from Boubendir-Antoine-Geuzaine 2012



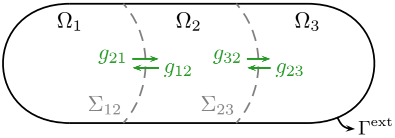
Helmholtz problem with exterior ABC

Find $u \in H^1(\Omega)$ such that
$$\left| \begin{array}{rl} -\Delta u - k^2 u = 0 & \text{on } \Omega \\ \partial_{\mathbf{n}} u + \mathcal{T}^{\text{ext}} u = 0 & \text{on } \Gamma^{\text{ext}} \\ \partial_{\mathbf{n}} u = g & \text{on } \Gamma^{\text{int}} \end{array} \right.$$

$\mathcal{T}^{\text{ext}}$ = DtN operator *(boundary)*

Finite-element approach $\Longrightarrow$ $\begin{cases} \text{Large sparse linear system} \\ \text{Slow convergence with standard iterative solvers} \end{cases}$

Non-overlapping domain decomposition — Layered partition



$\mathcal{T}_{IJ}$ = DtN operator *(interface)*

$$\text{with } g_{IJ} = \partial_{\mathbf{n}_I} u_J + \mathcal{T}_{IJ} u_J$$

Which transmission operator $\mathcal{T}_{IJ}$ at the interfaces?

- Ideal operator → DtN related to the complementary of each subdomain
- Good operators → Operators used for artificial/absorbing boundary conditions

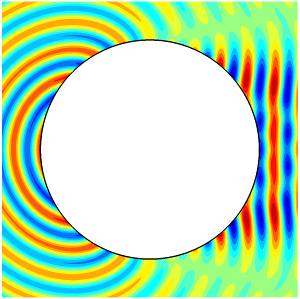
[e.g. Hagstrom-Tewarson-Jazcilevich 1988, Nataf-Rogier-de Sturler 1994]

Current approaches *(references for Helmholtz)*: *[e.g. Gander-Zhang 2019]*

- Impedance conditions *[Després 1991, Gander-Magoulès-Nataf 2002, Boubendir 2007 …]*
- 2nd-order conditions *[Gander et al 2002, Boubendir et al 2008, Després-Nicolopoulos-Thierry 2021 …]*
- High-order conditions / HABCs *[Boubendir-Antoine-Geuzaine 2012, Kim-Zhang 2015 …]*
- Perfectly matched layers / PMLs *[Stolk 2013, Vion-Geuzaine 2014 …]*
- Non-local approaches *[Lecouvez et al 2014, Claeys-Parolin 2020 …]*

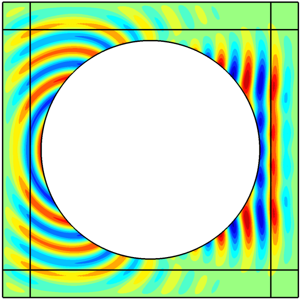
PMLs for DD preconditionning: *[Tosseli 1998, Schädle-Zschiedrich 2007]*

Original **scattering problem**



$$\begin{cases} -\Delta u - k^2 u = 0 & \text{in } \mathbb{R}^2 \backslash \Omega^{\text{disk}} \\ \text{Sommerfeld B.C.} & \text{at } \|\mathbf{x}\| \to \infty \\ \partial_{\mathbf{n}} u = g & \text{on } \Gamma^{\text{int}} \end{cases}$$

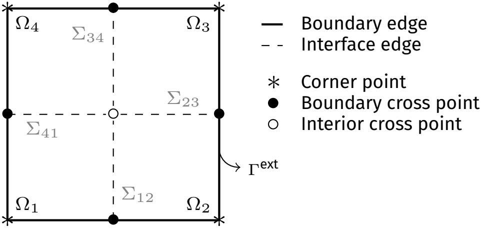
Finite element solution with **Perfectly Matched Layer (PML)**



$$\begin{cases} -\nabla \cdot (\mathbb{D}\nabla u) - E k^2 u = 0 & \text{in } \Omega_{\text{all}} \\ \mathbf{n} \cdot (\mathbb{D}\nabla u) = 0 & \text{on } \Gamma_{\text{all}}^{\text{ext}} \\ \partial_{\mathbf{n}} u = g & \text{on } \Gamma_{\text{all}}^{\text{int}} \end{cases}$$

Material scalar/tensor fields: E and $\mathbb{D}$

Non-overlapping domain decomposition — Checkerboard partition



Strategies for/with cross points *(specific for the Helmholtz equation)*:

- Impedance conditions *[Farhat et al 2020, Boubendir-Bendali-Fares 2008, ...]*
- 2nd-order conditions *[Després-Nicolopoulos-Thierry 2021, 2022, ...]*
- High-order conditions / HABCs *[Modave-Royer-Antoine-Geuzaine 2022]*
- Perfectly matched layers / PMLs ***[Royer-Geuzaine-Béchet-Modave 2022]*** *← This talk*
- Non-local approaches *[Zepeda-Núñez-Demanet 2016, Claeys et al 2021, Claeys-Parolin 2022 ...]*

PMLs for DD precond. for checkerboard partitions: *[Astaneh-Guddati 2016, Leng-Ju 2019]*

Challenges in this talk
- Writting the PML as a DtN transmission operator
- PML as a transmission operator for *checkerboard partions*
- Discretization aspects

DDM iterative process
The PML as a DtN operator
Finite element discretization
Numerical results

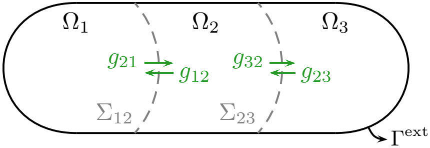
DDM iterative process
The PML as a DtN operator
Finite element discretization
Numerical results

[Després 1991, Collino-Ghanemi-Joly 2000]

Helmholtz problem

Find $u(\mathbf{x})$ with $\mathbf{x} \in \Omega$ s.t.

$$\begin{cases} -\Delta u - k^2 u = f & \text{in } \Omega \\ \partial_{\mathbf{n}} u - \imath k u = 0 & \text{on } \Gamma^{\text{ext}} \end{cases}$$



DDM iterative process *(substructuring method — non-overlapping partition)*

For each Ω_I, find $u_I^{\ell+1}(\mathbf{x})$ with $\mathbf{x} \in \Omega_I$ s.t.

$$\begin{cases} -\Delta u_I^{\ell+1} - k^2 u_I^{\ell+1} = f & \text{in } \Omega_I \\ \partial_{\mathbf{n}_I} u_I^{\ell+1} - \imath k u_I^{\ell+1} = 0 & \text{on } \partial\Omega_I \cap \Gamma^{\text{ext}} \\ \partial_{\mathbf{n}_I} u_I^{\ell+1} + \mathcal{T}_{IJ} u_I^{\ell+1} = g_{IJ}^{\ell} & \text{on each } \Sigma_{IJ} \end{cases}$$

$\longrightarrow$ **Exterior boundary condition**

$\longrightarrow$ **Transmission condition**

Data transfer at the interfaces

$\ell{+}{+}$

Transmission variables:

$$g_{IJ}^{\ell+1} := \partial_{\mathbf{n}_I} u_J^{\ell} + \mathcal{T}_{IJ} u_J^{\ell} = -g_{JI}^{\ell} + 2\mathcal{T}_{IJ} u_J^{\ell}$$

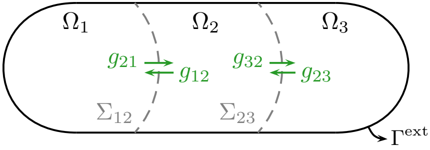
DtN operator: $\mathcal{T}_{IJ}$

[Després 1991]

Helmholtz problem

Find $u(\mathbf{x})$ with $\mathbf{x} \in \Omega$ s.t.

$$\begin{cases} -\Delta u - k^2 u = f & \text{in } \Omega \\ \partial_{\mathbf{n}} u - \imath k u = 0 & \text{on } \Gamma^{\text{ext}} \end{cases}$$



DDM iterative process with impedance operator

For each Ω_I, find $u_I^{\ell+1}(\mathbf{x})$ with $\mathbf{x} \in \Omega_I$ s.t.

$$\begin{cases} -\Delta u_I^{\ell+1} - k^2 u_I^{\ell+1} = f & \text{in } \Omega_I \\ \partial_{\mathbf{n}_I} u_I^{\ell+1} - \imath k u_I^{\ell+1} = 0 & \text{on } \partial\Omega_I \cap \Gamma^{\text{ext}} \\ \partial_{\mathbf{n}_I} u_I^{\ell+1} - \imath k u_I^{\ell+1} = g_{IJ}^{\ell} & \text{on each } \Sigma_{IJ} \end{cases}$$

→ **Exterior boundary condition**

→ **Transmission condition**

Data transfer at the interfaces

$\ell{+}{+}$

Transmission variables:

$$g_{IJ}^{\ell+1} := \partial_{\mathbf{n}_I} u_J^{\ell} - \imath k u_J^{\ell} = -g_{JI}^{\ell} - 2\imath k u_J^{\ell}$$

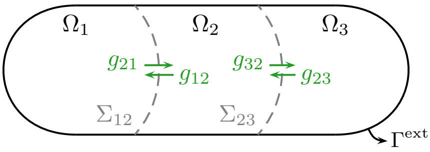
DTN operator: $\mathcal{T}_{IJ} = -\imath k$

[Boubendir-Antoine-Geuzaine, 2012]

Helmholtz problem

Find $u(\mathbf{x})$ with $\mathbf{x} \in \Omega$ s.t.

$$\begin{cases} -\Delta u - k^2 u = f & \text{in } \Omega \\ \partial_{\mathbf{n}} u - \imath k u = 0 & \text{on } \Gamma^{\text{ext}} \end{cases}$$



DDM iterative process with high-order operator based on Padé-type HABC

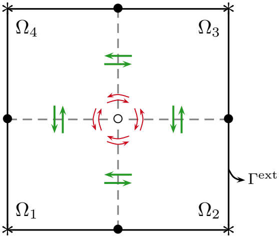
For each Ω_I, find $u_I^{\ell+1}(\mathbf{x})$ with $\mathbf{x} \in \Omega_I$ s.t.

$$\begin{cases} -\Delta u_I^{\ell+1} - k^2 u_I^{\ell+1} = f & \text{in } \Omega_I \\ \partial_{\mathbf{n}_I} u_I^{\ell+1} - \imath k u_I^{\ell+1} = 0 & \text{on } \partial\Omega_I \cap \Gamma^{\text{ext}} \\ \partial_{\mathbf{n}_I} u_I^{\ell+1} - \imath k \big(u_J^{\ell+1} + \frac{2}{M} \sum_{n=1}^{N} c_n (u_J^{\ell+1} + \varphi_n|_{JI}^{\ell+1}) \big) = g_{IJ}^{\ell} & \text{on each } \Sigma_{IJ} \\ \Delta_\Sigma \varphi_n|_{IJ}^{\ell+1} + k^2 (c_n + 1) \big(\varphi_n|_{IJ}^{\ell+1} + u_I^{\ell+1} \big) = 0 & \text{on each } \Sigma_{IJ}, \forall n \end{cases}$$

Data transfer at the interfaces

ℓ++

[Modave-Royer-Antoine-Geuzaine, 2020]



Helmholtz problem

Find $u(\mathbf{x})$ with $\mathbf{x} \in \Omega$ s.t.

$$\begin{cases} -\Delta u - k^2 u = f & \text{in } \Omega \\ \partial_{\mathbf{n}} u - \imath k u = 0 & \text{on } \Gamma^{\text{ext}} \end{cases}$$

DDM iterative process with high-order operator

For each Ω_I, find $u_I^{\ell+1}(\mathbf{x})$ with $\mathbf{x} \in \Omega_I$ s.t.

$$\begin{cases} -\Delta u_I^{\ell+1} - k^2 u_I^{\ell+1} = f & \text{in } \Omega_I \\ \partial_{\mathbf{n}_I} u_I^{\ell+1} - \imath k u_I^{\ell+1} = 0 & \text{on } \partial\Omega_I \cap \Gamma^{\text{ext}} \\ \partial_{\mathbf{n}_I} u_I^{\ell+1} - \imath k \left(u_J^{\ell+1} + \frac{2}{M} \sum_{n=1}^{N} c_n \left(u_J^{\ell+1} + \varphi_n|_{JI}^{\ell+1} \right) \right) = g_{IJ}^{\ell} & \text{on each } \Sigma_{IJ} \\ \Delta_\Sigma \varphi_n|_{IJ}^{\ell+1} + k^2 \left((\alpha^2 c_n + 1) \varphi_n|_{IJ}^{\ell+1} + \alpha^2 (c_n + 1) u_I^{\ell+1} \right) = 0 & \text{on each } \Sigma_{IJ}, \forall n \\ \partial_{\boldsymbol{\tau}_{IJ}} \varphi_n|_{IJ}^{\ell+1} - \imath k \varphi_n|_{IJ}^{\ell+1} = 0 & \text{at each } \partial\Sigma_{IJ} \cap \Gamma^{\text{ext}}, \forall n \\ \partial_{\boldsymbol{\tau}_{IJ}} \varphi_n|_{IJ}^{\ell+1} + \mathcal{T}_{\Sigma_{IJ'}} \varphi_n|_{IJ}^{\ell+1} = g_n|_{IJ'}^{\ell} & \text{at each } \partial\Sigma_{IJ} \cap \partial\Sigma_{IJ'}, \forall n \end{cases}$$

DDM iterative process
The PML as a DtN operator
Finite element discretization
Numerical results

Helmholtz subproblem with PML

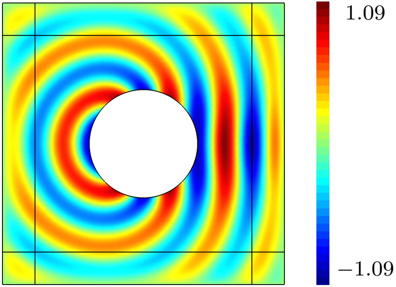
Find each $u_{\text{all}} \in H^1(\Omega_{\text{all}})$ such that

$$\begin{cases} -\nabla \cdot (\mathbb{D}\nabla u_{\text{all}}) - Ek^2 u_{\text{all}} = f & \text{in } \Omega_{\text{all}} \\ \mathbf{n} \cdot (\mathbb{D}\nabla u_{\text{all}}) = 0 & \text{on } \partial\Omega_{\text{all}} \end{cases}$$

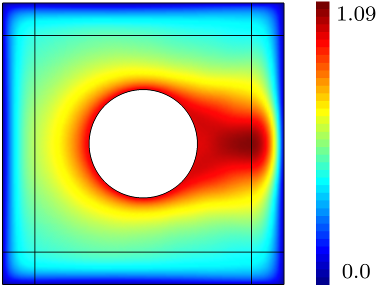
where E and $\mathbb{D}$ are material scalar/tensor fields.

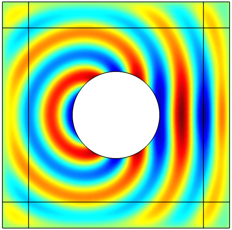
Exemple with a Dirichlet BC on the disk

Real part of the solution



Norm of the solution





Helmholtz subproblem with PML

Find each $u_{\text{all}} \in H^1(\Omega_{\text{all}})$ such that

$$\begin{cases} -\nabla \cdot (\mathbb{D}\,\nabla u_{\text{all}}) - Ek^2 u_{\text{all}} = f & \text{in } \Omega_{\text{all}} \\ \mathbf{n} \cdot (\mathbb{D}\,\nabla u_{\text{all}}) = 0 & \text{on } \partial\Omega_{\text{all}} \end{cases}$$

with material tensor/scalar fields $\mathbb{D}$ and E:

$$\mathbb{D}(x,y) = \operatorname{diag}\left(\gamma_y/\gamma_x, \gamma_x/\gamma_y\right)$$
$$E(x,y) = \gamma_x \gamma_y$$

with stretching functions $\gamma_x(x)$ and $\gamma_y(y)$ and absorption functions $\sigma_x(x)$ and $\sigma_y(y)$:

$$\gamma_x(x) = 1 + \sigma_x(x)/\imath k \qquad \sigma_x(x) \geq 0$$
$$\gamma_y(y) = 1 + \sigma_y(y)/\imath k \qquad \sigma_y(y) \geq 0$$

Inside the truncated domain Ω: $\sigma_x = 0\,;\, \sigma_y = 0 \implies \mathbb{D} = \mathbb{I}\,;\, E = 1$

Variational formulation

Find each $u_{\text{all}} \in H^1(\Omega_{\text{all}})$ such that

$$\int_{\Omega_{\text{all}}} \nabla u_{\text{all}}\, \mathbb{D}\, \overline{\nabla v_{\text{all}}}\, \mathrm{d}\Omega - \int_{\Omega_{\text{all}}} k^2\, E\, u_{\text{all}}\, \overline{v_{\text{all}}}\, \mathrm{d}\Omega = \int_{\Omega} f\, \overline{v_{\text{all}}}\, \mathrm{d}\Omega, \quad \forall v_{\text{all}} \in H^1(\Omega_{\text{all}})$$

The PML as a DtN operator — Decomposed problem [1/3]

Helmholtz subproblem with PML *(decomposed version)*

- For the domain Ω:

$$\begin{cases} -\Delta u - k^2 u = f & \text{in } \Omega \\ \mathbf{n}\cdot\nabla u = \lambda_i & \text{on each } \Gamma_i \end{cases}$$

with $\lambda_i := \mathbf{n}\cdot\nabla u_i$.

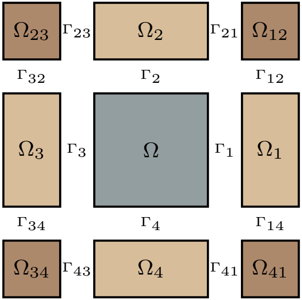
- For each edge PML Ω_i:

$$\begin{cases} -\nabla\cdot(\mathbb{D}\,\nabla u_i) - Ek^2 u_i = 0 & \text{in } \Omega_i \\ \mathbf{n}\cdot(\mathbb{D}\,\nabla u_i) = 0 & \text{on } \Gamma_i^{\text{ext}} \\ u_i = u & \text{on } \Gamma_i \\ \mathbf{n}\cdot(\mathbb{D}\,\nabla u_i) = \lambda_{ij} & \text{on each } \Gamma_{ij} \end{cases}$$

with $\lambda_{ij} := \mathbf{n}\cdot(\mathbb{D}\,\nabla u_{ij})$.

- For each corner PML Ω_{ij}:

$$\begin{cases} -\nabla\cdot(\mathbb{D}\,\nabla u_{ij}) - Ek^2 u_{ij} = 0 & \text{in } \Omega_{ij} \\ \mathbf{n}\cdot(\mathbb{D}\,\nabla u_{ij}) = 0 & \text{on } \Gamma_{ij}^{\text{ext}} \\ u_{ij} = u_i & \text{on } \Gamma_{ij} \\ u_{ij} = u_j & \text{on } \Gamma_{ji} \end{cases}$$



$$\Gamma_i := \Omega \cup \Omega_i$$
$$\Gamma_{ij} := \Omega_i \cup \Omega_{ij}$$
$$\Gamma_i^{\text{ext}} := \partial\Omega_i \cap \partial\Omega$$
$$\Gamma_{ij}^{\text{ext}} := \partial\Omega_{ij} \cap \partial\Omega$$

Helmholtz subproblem with PML *(decomposed version)*

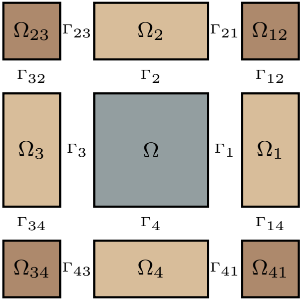
- For the domain Ω:

$$\begin{cases} -\Delta u - k^2 u = f & \text{in } \Omega \\ \mathbf{n}\cdot\nabla u = \lambda_i & \text{on each } \Gamma_i \end{cases}$$

with $\lambda_i := \mathbf{n}\cdot\nabla u_i$.

- For each edge PML Ω_i:

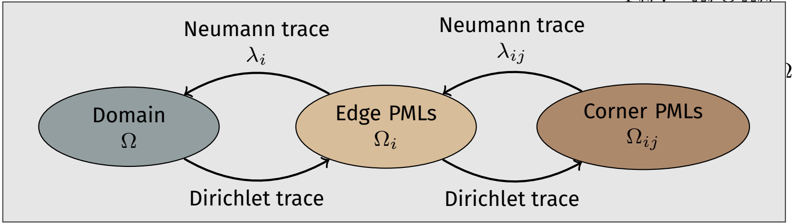
$$\begin{cases} -\nabla\cdot(\mathbb{D}\,\nabla u_i) - E k^2 u_i = 0 & \text{in } \Omega_i \\ \mathbf{n}\cdot(\mathbb{D}\,\nabla u_i) = 0 & \text{on } \Gamma_i^{\text{ext}} \\ u_i = u & \text{on } \Gamma_i \\ \mathbf{n}\cdot(\mathbb{D}\,\nabla u_i) = \lambda_{ij} & \text{on each } \Gamma_{ij} \end{cases}$$



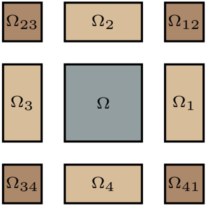
$\Gamma_i := \Omega \cup \Omega_i$

$\Gamma_{ij} := \Omega_i \cup \Omega_{ij}$

-



The PML as a DtN operator — Decomposed problem [2/3]



Variational formulation with Lagrange multipliers

Find $u \in H^1(\Omega),\ u_i \in H^1(\Omega_i),\ u_{ij} \in H^1(\Omega_{ij})$

$\lambda_i \in H^{-1/2}(\Gamma_i),\ \lambda_{ij} \in H^{-1/2}(\Gamma_{ij})$ with $i, j = 1, \dots, 4$

such that

- For the domain Ω, each edge PML Ω_i and each corner PML Ω_{ij}:

$$\int_\Omega \nabla u\, \overline{\nabla v}\, \mathrm{d}\Omega - \int_\Omega k^2\, u\, \overline{v}\, \mathrm{d}\Omega - \int_{\Gamma_i} \lambda_i\, \overline{v}\, \mathrm{d}\Gamma = \int_\Omega f\, \overline{v}\, \mathrm{d}\Omega$$

$$\int_{\Omega_i} \nabla u_i\, \mathbb{D}\, \overline{\nabla v_i}\, \mathrm{d}\Omega - \int_{\Omega_i} k^2\, E\, u_i\, \overline{v_i}\, \mathrm{d}\Omega + \int_{\Gamma_i} \lambda_i\, \overline{v_i}\, \mathrm{d}\Gamma - \sum \int_{\Gamma_{ij}} \lambda_{ij}\, \overline{v_i}\, \mathrm{d}\Gamma = 0$$

$$\int_{\Omega_{ij}} \nabla u_{ij}\, \mathbb{D}\, \overline{\nabla v_{ij}}\, \mathrm{d}\Omega - \int_{\Omega_{ij}} k^2\, E\, u_{ij}\, \overline{v_{ij}}\, \mathrm{d}\Omega + \sum \int_{\Gamma_{ij}} \lambda_{ij}\, \overline{v_{ij}}\, \mathrm{d}\Gamma = 0$$

  for all $v \in H^1(\Omega)$, for all $v_i \in H^1(\Omega_i)$ and for all $v_{ij} \in H^1(\Omega_{ij})$.

- For each interface domain/PML Γ_i and each interface PML/PML Γ_{ij}:

$$\int_{\Gamma_i} (u_i - u)\, \overline{\mu_i}\, \mathrm{d}\Gamma = 0 \qquad\qquad \int_{\Gamma_{ij}} (u_{ij} - u_i)\, \overline{\mu_{ij}}\, \mathrm{d}\Gamma = 0$$

  for all $\mu_i \in H^{-1/2}(\Gamma_i)$ and for all $\mu_{ij} \in H^{-1/2}(\Gamma_{ij})$.

Variational formulation with Lagrange multipliers

Find $(u_{\text{all}}, \lambda_{\text{all}}) \in \mathcal{U} \times \mathcal{L}$ with

$$\begin{cases} a(u_{\text{all}}, v_{\text{all}}) + \overline{b(v_{\text{all}}, \lambda_{\text{all}})} = l(v_{\text{all}}) \\ b(u_{\text{all}}, \mu_{\text{all}}) = 0 \end{cases}$$

for all $(v_{\text{all}}, \mu_{\text{all}}) \in \mathcal{U} \times \mathcal{L}$.

Spaces:

$$\mathcal{U} := H^1(\Omega) \times \cdots \times H^1(\Omega_i) \times \cdots \times H^1(\Omega_{ij}) \times \cdots$$
$$\mathcal{L} := H^{-1/2}(\Gamma_i) \times \cdots \times H^{-1/2}(\Gamma_{ij}) \times \cdots$$

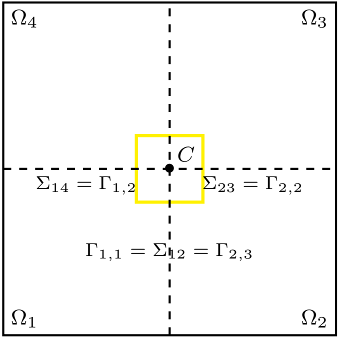
Forms:

$$a(u_{\text{all}}, v_{\text{all}}) := \int_\Omega (\nabla u \cdot \nabla \overline{v} - k^2 u \overline{v}) \, \mathrm{d}\Omega + \sum_{\Omega_i} \int_{\Omega_i} \cdots + \sum_{\Omega_{ij}} \int_{\Omega_{ij}} \cdots$$

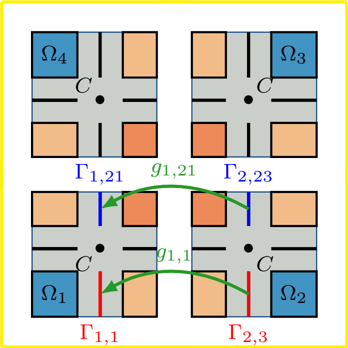
$$b(u_{\text{all}}, \mu_{\text{all}}) := \sum_{\Gamma_i} \int_{\Gamma_i} (u - u_i) \overline{\mu_i} \, \mathrm{d}\Gamma + \sum_{\Gamma_{ij}} \int_{\Gamma_{ij}} (u_i - u_{ij}) \overline{\mu_{ij}} \, \mathrm{d}\Gamma$$

$$l(v_{\text{all}}) := \int_\Omega f \overline{v} \, \mathrm{d}\Omega.$$

Domain partition with 4 subdomains



Data transfer at the cross point
(domain-PML and PML-PML)



Variational formulation of one subdomain Ω_I

Find $(u_{I,\text{all}}, \lambda_{I,\text{all}}) \in \mathcal{U}_I \times \mathcal{L}_I$ with

$$\begin{cases} a(u_{I,\text{all}}, v_{I,\text{all}}) + \overline{b(v_{I,\text{all}}, \lambda_{I,\text{all}})} = l(v_{I,\text{all}}) \\ b(u_{I,\text{all}}, \mu_{I,\text{all}}) = 0 \end{cases}$$

for all $(v_{I,\text{all}}, \mu_{I,\text{all}}) \in \mathcal{U}_I \times \mathcal{L}_I$.

Spaces:

$$\mathcal{U}_I := H^1(\Omega_I) \times \cdots \times H^1(\Omega_{I,i}) \times \cdots \times H^1(\Omega_{I,ij}) \times \cdots$$

$$\mathcal{L}_I := H^{-1/2}(\Gamma_{I,i}) \times \cdots \times H^{-1/2}(\Gamma_{I,ij}) \times \cdots$$

Forms:

$$a(u_{I,\text{all}}, v_{I,\text{all}}) := \int_{\Omega_I} (\nabla u_I \cdot \nabla \overline{v_I} - k^2 u_I \overline{v_I}) \, \mathrm{d}\Omega_I + \sum_{\Omega_{I,i}} \int_{\Omega_{I,i}} \cdots + \sum_{\Omega_{I,ij}} \int_{\Omega_{I,ij}} \cdots$$

$$b(u_{I,\text{all}}, \mu_{I,\text{all}}) := \sum_{\Gamma_i} \int_{\Gamma_i} (u_I - u_{I,i}) \overline{\mu_{I,i}} \, \mathrm{d}\Gamma + \sum_{\Gamma_{I,ij}} \int_{\Gamma_{I,ij}} (u_{I,i} - u_{I,ij}) \overline{\mu_{I,ij}} \, \mathrm{d}\Gamma$$

$$l(v_{I,\text{all}}) := \int_{\Omega_I} f \overline{v_I} \, \mathrm{d}\Omega + \sum_{\Gamma_{I,i}} \int_{\Gamma_{I,i}} g_{I,i} \overline{v_I} \, \mathrm{d}\Gamma + \sum_{\Gamma_{I,ij}} \int_{\Gamma_{I,ij}} g_{I,ij} \overline{v_{I,i}} \, \mathrm{d}\Gamma$$

Update formula:

$$g_{I,i}^{\ell+1} = -g_{J,i'}^{\ell} + 2\lambda_{J,i'}^{\ell}$$

$$g_{I,ij}^{\ell+1} = -g_{J,i'j'}^{\ell} + 2\lambda_{J,i'j'}^{\ell}$$

where J is neighbouring subdomain with $\Gamma_{I,i} = \Gamma_{J,i'}$ and $\Gamma_{I,ij} = \Gamma_{J,i'j'}$

DDM iterative process
The PML as a DtN operator
Finite element discretization
Numerical results

Variational formulation

Find $(u_{\mathsf{all}}, \lambda_{\mathsf{all}}) \in \mathcal{U} \times \mathcal{L}$ with

$$\begin{cases} a(u_{\mathsf{all}}, v_{\mathsf{all}}) + \overline{b(v_{\mathsf{all}}, \lambda_{\mathsf{all}})} = l(v_{\mathsf{all}}) \\ b(u_{\mathsf{all}}, \mu_{\mathsf{all}}) = 0 \end{cases}$$

for all $(v_{\mathsf{all}}, \mu_{\mathsf{all}}) \in \mathcal{U} \times \mathcal{L}$.

Algebraic system

Finite element discretization leads to the saddle-point problem:

$$\begin{bmatrix} \mathsf{A} & \mathsf{B}^\top \\ \mathsf{B} & 0 \end{bmatrix} \begin{bmatrix} \mathsf{u}_{\mathsf{all}} \\ \mathsf{l}_{\mathsf{all}} \end{bmatrix} = \begin{bmatrix} \mathsf{f}_{\mathsf{all}} \\ 0 \end{bmatrix}$$

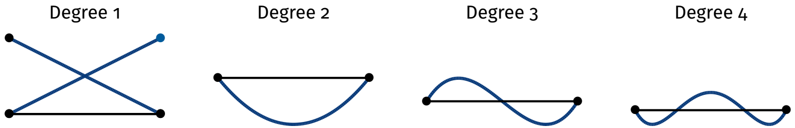
where only B depend on the discretization of the Lagrange multipliers.

Finite element discretizations

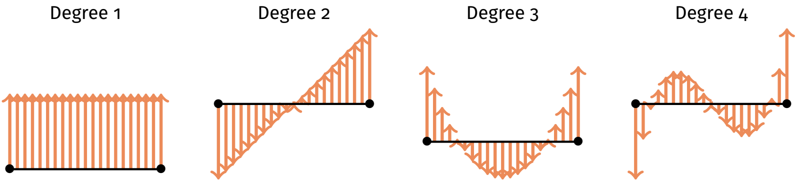
Solution u_{all} Continuous basis functions

Multipliers μ_{all} $\begin{cases} (C) \; H^1\text{-conforming basis functions } \textit{(continuous)} \\ (D) \; \text{Projection on } \mathbf{n} \text{ of } \mathbf{H}(\mathrm{div})\text{-confrom. basis func. } \textit{(discontinuous)} \end{cases}$

Hierarchical H^1-conforming basis functions (C)



Projection on $\mathbf{n}$ of hierarchical $\mathbf{H}(\mathrm{div})$-conforming basis functions (D)



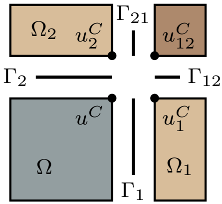
FE discretization — Comparison of FE bases for multipliers [1/2]

Issue with the continuous basis functions (C)

At each corner, relations are linearly dependent:

$$\begin{cases} u^C - u_1^C = 0 \\ u^C - u_2^C = 0 \\ u_1^C - u_{12}^C = 0 \\ u_2^C - u_{12}^C = 0 \end{cases}$$

$\Longrightarrow$ B is not surjective ; system not solvable

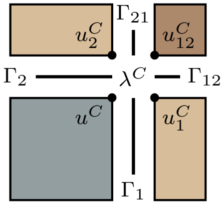


Strategies

- Additional Lagrange multiplier λ_C to break the dependency

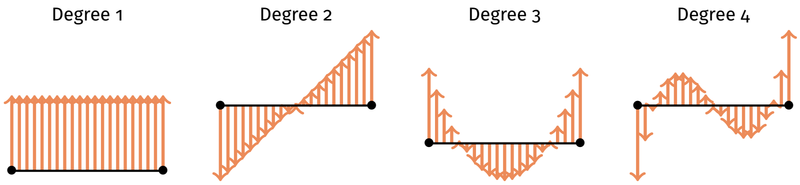
$$\begin{cases} u^C - u_1^C + \lambda^C = 0 \\ u^C - u_2^C - \lambda^C = 0 \\ u_1^C - u_{12}^C + \lambda^C = 0 \\ u_2^C - u_{12}^C - \lambda^C = 0 \\ \lambda_1^C - \lambda_2^C + \lambda_{12}^C - \lambda_{21}^C = 0 \end{cases}$$

  Strategy used for FETI *[e.g. Toselli-Widlund 2005]*



- Penalization *[Boffi-Brezzi-Fortin 2005]*: $\begin{bmatrix} \mathsf{A} & \mathsf{B}^\top \\ \mathsf{B} & \tau\mathsf{M} \end{bmatrix} \begin{bmatrix} \mathsf{u}_{\mathsf{all}} \\ \mathsf{l}_{\mathsf{all}} \end{bmatrix} = \begin{bmatrix} \mathsf{f}_{\mathsf{all}} \\ 0 \end{bmatrix}$

  However, continuity at interfaces not exactly enforced, *e.g.* $u^C - u_1^C = \tau\lambda_1^C$.

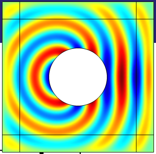


Issues with the discontinuous basis functions (D)

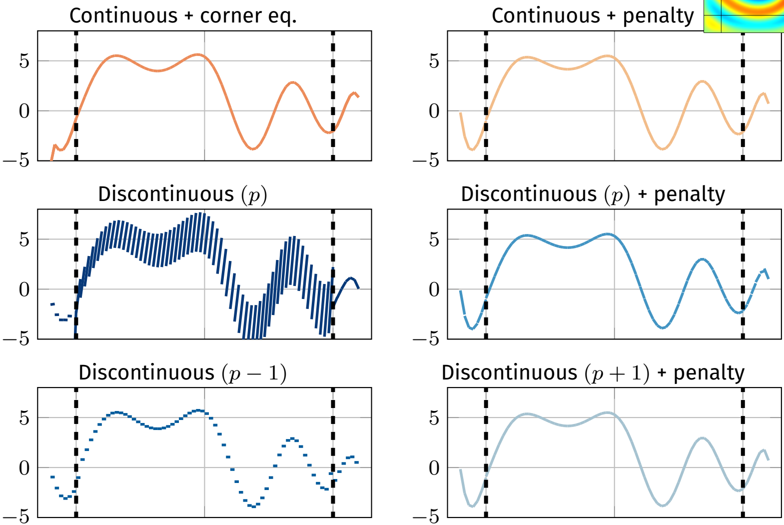
- If degree(multipliers) $=$ degree(sol) $\Rightarrow$ stability issues
- If degree(multipliers) $\leq$ degree(sol) $\Rightarrow$ underdeterm. system ; solution not exact
- If degree(multipliers) $>$ degree(sol) $\Rightarrow$ overdeterm. system ; system not solvable

Strategies

- Different polynomial degrees for solution u and the Lagrange multipliers λ
- Penalization $(\tau = 0.002\, h^2)$

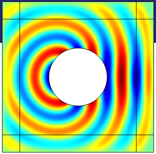


Neumann traces along the top interface

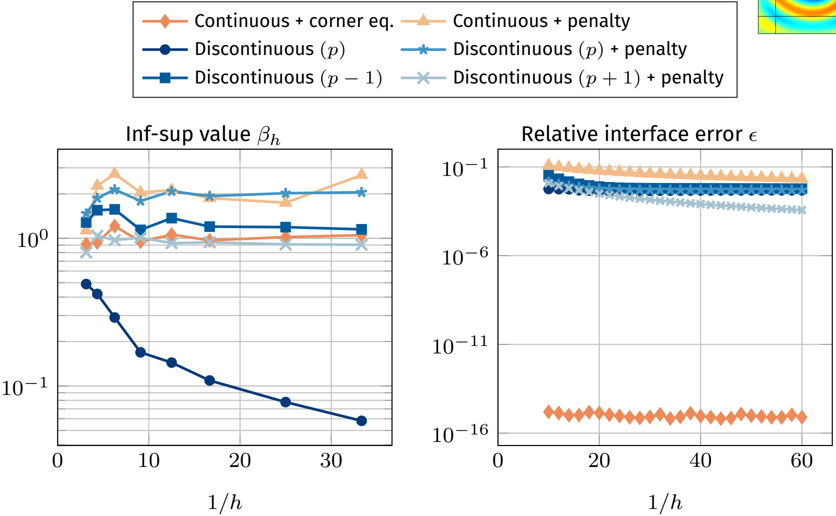


Parameters: $k = 4\pi$, $h \approx 1/30$, P2 elements — $p =$ degree for the solution

FE discretization — Numerical experiment [2/2]



Inf-sup test *[Chapelle-Bathe 1993]* **+ Interface error**



Interface error = L2 norm of the jump on the solution at the interfaces

Chapelle-Bath test: $\beta = \min_h \beta_h$

DDM iterative process
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Numerical results — Scattering benchmark

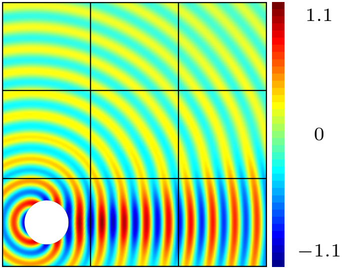
Benchmark: Scattering of a plane wave by a disk

- Exterior condition: PML with $\delta_{\text{PML}} = 6h$ and $\sigma = \sigma_h$
- Interface condition: PML with $\delta_{\text{PML}} = 6h$ and $\sigma = \sigma_h$ *tests with other params ...*

$$\sigma_q(x) = \sigma^\star x^2/\delta_{\text{PML}}^2 \quad \sigma_h(x) = 1/(\delta_{\text{PML}} - x) \quad \sigma_{hs}(x) = 1/(\delta_{\text{PML}} - x) - 1/\delta_{\text{PML}}$$

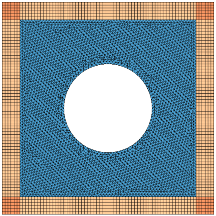
- GMRES iterative solver — `GmshFEM` and `GmshDDM` codes

Numerical solution + Domain partition



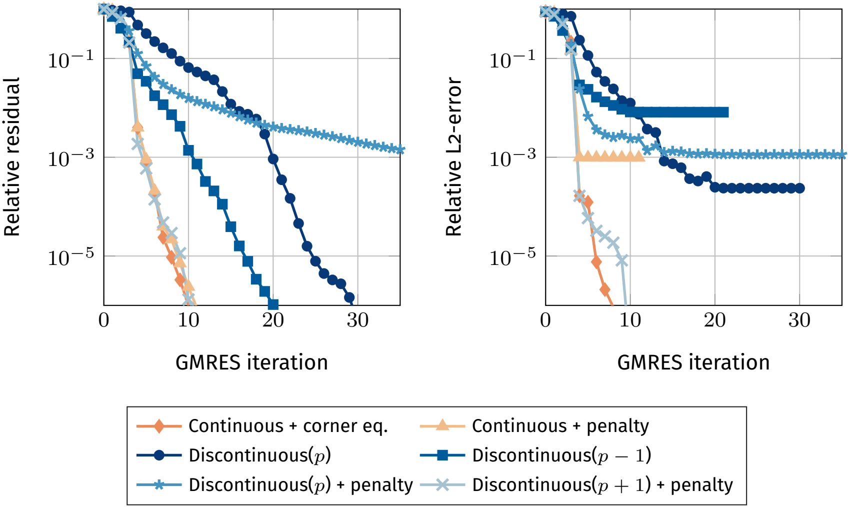
$k = 4\pi$ — P2 or P3
15 points per wavelength

Mesh of one subdomain
with the surrounding PMLs



Triangular elements *(subdomain)*
Square elements *(extruded PMLs)*

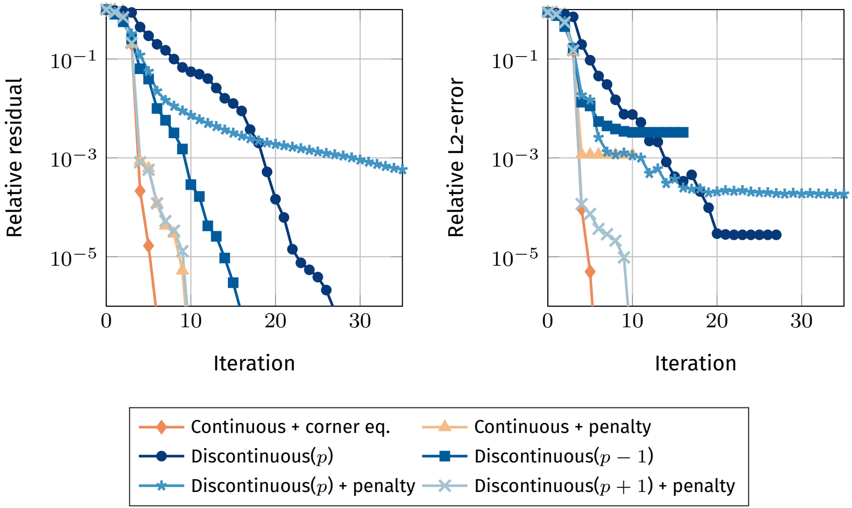
Comparison of discretizations of the Lagrange multipliers [P2 elements]



Error L2-error: DDM solution compared to reference numerical solution.
Same discretization used for interface and exterior PMLs.

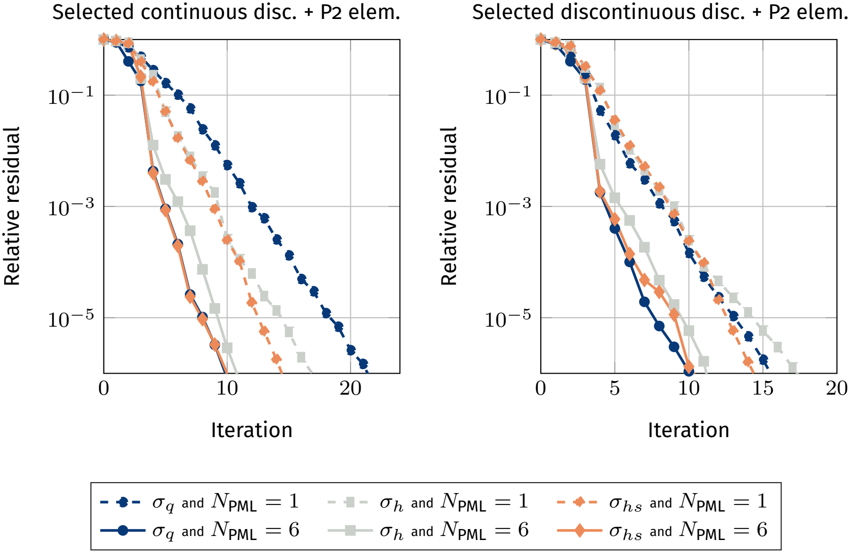
Numerical results — Comparison of discretizations [2/2]

Comparison of discretizations of the Lagrange multipliers [P4 elements]

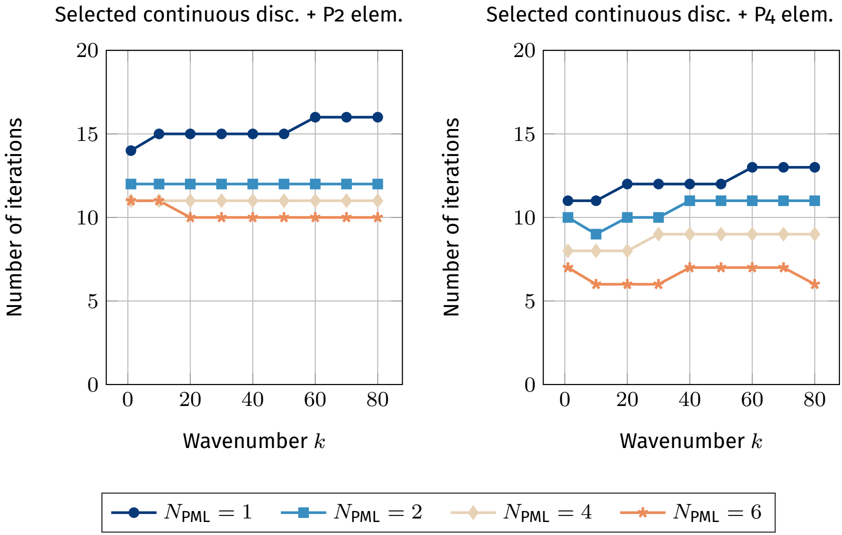


Error L2-error: DDM solution compared to reference numerical solution.
Same discretization used for interface and exterior PMLs.

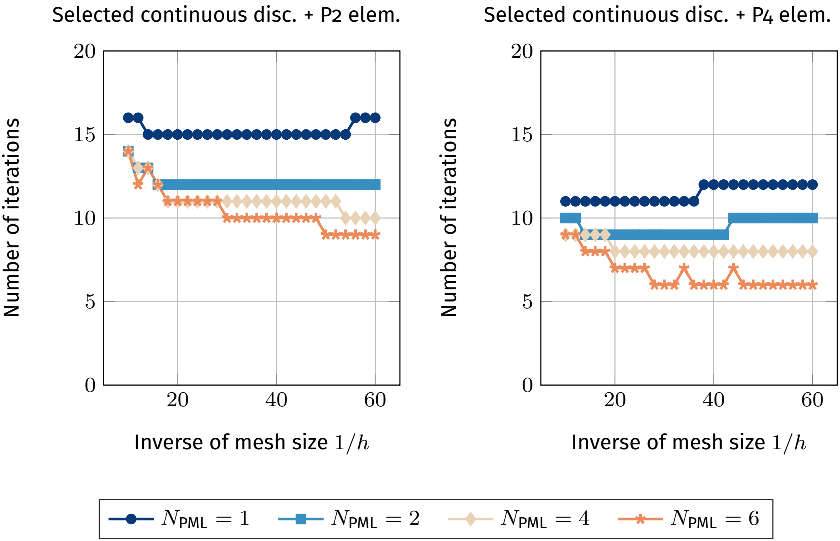
Influence of the absorption functions σ and the layer thickness $\delta_{\mathrm{PML}} = N_{\mathrm{PML}} h$



Influence of wavenumber k with points per wavelength $= 15$



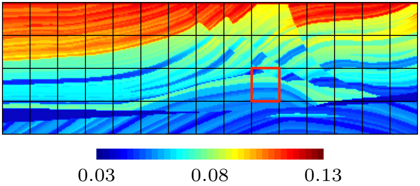
Influence of characteristic mesh cell size h with $k = 4\pi$



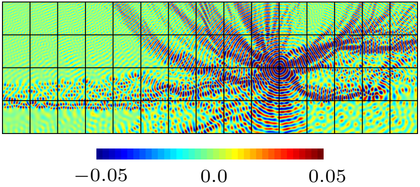
Benchmark: Marmousi benchmark

- Exterior condition: PML with $\delta_{\text{PML}} = 6h$ and $\sigma = \sigma_h$
- Interface condition: PML with $\delta_{\text{PML}} = h, 2h, 3h$ and $\sigma = \sigma_h$
- GMRES iterative solver — `GmshFEM` and `GmshDDM` codes

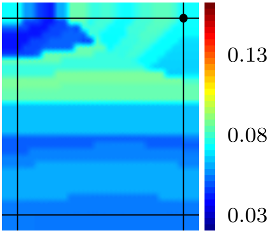
Wavenumber $k(\mathbf{x})$ $[f = 10\text{Hz}]$



Numerical solution



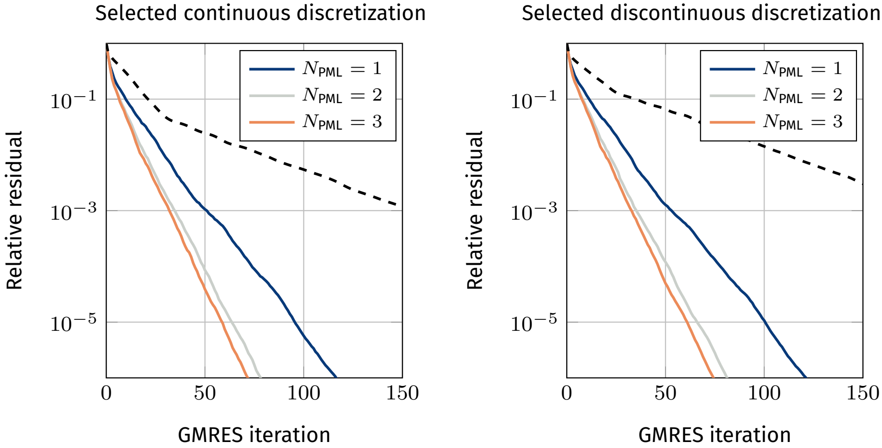
Subdomain with PMLs



Square elements [Q1]

$h \approx 10\text{m}$

Residual history



Dashed line corresponds to basic impedance transmission $(\mathcal{T} = -\imath k)$
Same discretization used for interface and exterior PMLs.

DDM iterative process
The PML as a DtN operator
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Non-overlapping substructuring DDM with PML transmission for checkerboard partitions with cross points

Summary

- → Strategy based on the weak coupling of PMLs with Lagrange multipliers
- → Two discretizations for the multipliers *("continuous" and "discontinuous")*
- → Numerical experiments:
  - Compare two discretizations for the multipliers
  - Study the PML parameters *(absorption function and thickness)*
  - Behavior depending on k, h and heterogeneous media

Royer, Geuzaine, Béchet, M. (2022). A non-overlapping DDM with PML transmission conditions for the Helmholtz equation. CMAME, 395, 115006 + `GmshFEM` / `GmshDDM` codes

Personnel comments on DDM with transmission conditions based on non-reflecting boundary techniques *(PML, HABC, ...)*

- One has to deal with the limitations of these techniques *(practical / theoretical)*
- These approaches require specific solvers / limited theoretical background
- These approaches incorporate physics in the solver
- Mostly developed/tested for standard CG/FD methods ...

  ... a lot to do for DG, HDG, Trefftz methods

A *non-overlapping DDM* with *PML transmission conditions* and *checkerboard partitions* for Helmholtz problems

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