Rigidity and compensated compactness in L^1

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General compensated compactness

Tartar framework: Sequence of maps $u_j \colon \mathbb{R}^d \to \mathbb{R}^m$ with

$$u_j \rightharpoonup u \quad \text{weakly/weakly* in a Sobolev space or space of measures}$$

and

$$\textbf{linear PDE constraint:} \quad \mathscr{A} u_j = 0$$
$$\textbf{nonlinear pointwise constraint:} \quad u_j \in K \subset \mathbb{R}^m$$

Question: Is there unexpected (compensated) compactness in the sequence (u_j), e.g.,

$$u_j \to u \quad \text{strongly}$$

or at least convergence for some (nonlinear) functions of u_j?

Goal: **Compensated compactness theory** with concentrations
($u \in \mathrm{L}^1$ or $u \in \mathscr{M}_{\mathrm{loc}}$; weak* convergence in the sense of measures)

Hidden (or not so hidden) PDEs

PDE constraints for vector measures $\mu \in \mathscr{M}(\mathbb{R}^d;\mathbb{R}^N)$, $\sigma \in \mathscr{M}(\mathbb{R}^d;\mathbb{R}^n)$:

$$\mathscr{A}\mu := \sum_{|\alpha|\le k} A_\alpha \partial^\alpha \mu = \sigma \quad \text{in } \mathscr{D}',$$

where $A_\alpha \in \mathbb{R}^{n\times N}$, $\partial^\alpha = \partial_1^{\alpha_1}\dots\partial_d^{\alpha_d}$ for each multi-index $\alpha = (\alpha_1,\dots,\alpha_d) \in \mathbb{N}_0^d$.

Example 1 – Functions of bounded variation: For $u \in \mathrm{BV}(\Omega;\mathbb{R}^m)$, we have that $Du = (\mu_j^k) \in \mathscr{M}(\Omega;\mathbb{R}^{m\times d})$ satisfies

$$0 = \mathscr{A}(Du) := \operatorname{curl}(Du) := \Big(\partial_j\mu_i^k - \partial_i\mu_j^k\Big)_{i,j=1,\dots,d;\ k=1,\dots,m}.$$

Application: Deformations with jumps and fractal parts.

Examples

Example 2 – Functions of bounded deformation: For $u \in \mathrm{BD}(\Omega)$, we have that $Eu := \frac{1}{2}(Du + Du^T) = (\mu_j^k) \in \mathscr{M}(\Omega, \mathbb{R}^{d\times d}_{\mathrm{sym}})$ satisfies Saint-Venant's compatibility conditions (1864):

$$0 = \mathscr{A}(Eu) := \mathrm{curl}\,\mathrm{curl}\,(Eu) := \left(\sum_{i=1}^{d} \partial_{ik}\mu_i^j + \partial_{ij}\mu_i^k - \partial_{jk}\mu_i^i - \partial_{ii}\mu_j^k \right)_{j,k=1,\dots,d}$$

Application: Displacements (e.g., perfect plasticity).

Example 3 – Normal 1-currents: A vector measure $T \in \mathscr{M}_{\mathrm{loc}}(\mathbb{R}^d;\mathbb{R}^d)$ is called a **normal 1-current** if

$$\partial T := -\operatorname{div} T \in \mathscr{M}_{\mathrm{loc}}(\mathbb{R}^d;\mathbb{R}),$$

thus:

$$\mathscr{A}T := \partial T = \sigma \in \mathscr{M}_{\mathrm{loc}}(\mathbb{R}^d;\mathbb{R}).$$

Application: Dislocations (they are loops within a crystal grain).

Wanted: Fine structure theory

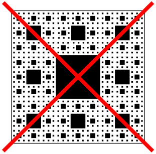
Central question: Restrictions on the *singular part* μ^s of solutions μ of $\mathscr{A}\mu = \sigma$.

$$\mu = g\mathscr{L}^d + \mu^s \quad ?$$

μ^s = jumps, fractals, Cantor measures, ...?

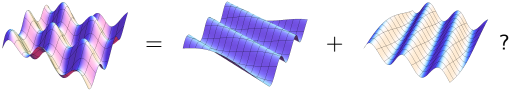
↑

Major goal 1: **Restrictions (rigidity)** on singularities:



Major goal 2: **Fine structure theory** for singularities:

- Shape?
- Dimensions?
- Local structure?



Major goal 3: **Compensated compactness theory**

Rigidity I (polar differential inclusions)

Rigidity in BV

Let $u \in \mathrm{BV}(\Omega;\mathbb{R}^m)$ ($\Omega \subset \mathbb{R}^d$ a Lipschitz domain). At $|Du|$-almost every $x_0 \in \Omega$, a blow-up v satisfies

$$Dv = P_0|Dv|, \qquad \text{where} \qquad P_0 = \frac{D^s u}{|D^s u|}(x_0) \quad \in \mathbb{R}^{m\times d}.$$

Hence: Need to investigate the structure of solutions to

$$Dv = P_0|Dv|, \qquad v \in \mathrm{BV}_{\mathrm{loc}}(\mathbb{R}^d;\mathbb{R}^m),\; P_0 \in \mathbb{R}^{m\times d},\; |P_0| = 1. \qquad (\star)$$

Lemma (Rigidity)

Let $v \in \mathrm{BV}_{\mathrm{loc}}(\mathbb{R}^d;\mathbb{R}^m)$ satisfy $(\star)$. Then:

(i) *If $\operatorname{rank} P_0 \geq 2$, then v is affine.*

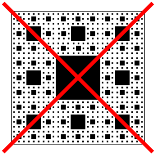
(ii) *If $P_0 = a \otimes \xi$ ($\Leftrightarrow \operatorname{rank} P_0 \leq 1$), then v is one-directional, i.e. there exists $\tilde{v} \in \mathrm{BV}_{\mathrm{loc}}(\mathbb{R})$ such that*

$$v(y) = \tilde{v}(y \cdot \xi)a, \qquad y \in \mathbb{R}^d \text{ a.e.}$$

Theorem ($\mathscr{A}$ = curl, Alberti's Rank-One Theorem 1993)

Let $u \in \mathrm{BV}(\mathbb{R}^d;\mathbb{R}^m)$. Then, for the singular part $D^s u$ of Du:

$$\operatorname{rank}\left(\frac{\mathrm{d}D^s u}{\mathrm{d}|D^s u|}\right) = 1 \qquad |D^s u|\text{-a.e.}$$



Rigidity for symmetrized gradients

Idea: Investigate the structure of solutions to

$$Ev := \frac{1}{2}(Dv + Dv^T) = P_0|Ev|, \qquad v \in \mathrm{BD}_{\mathrm{loc}}(\mathbb{R}^d),\ P_0 \in \mathbb{R}^{d\times d}_{\mathrm{sym}},\ |P_0| = 1.$$

Would like to have (by analogy to BV):

(i) If $P_0 \notin \{\, a \odot b \ :\ a, b \in \mathbb{R}^d \,\}$, then v is affine.

(ii) If $P_0 = a \odot b$ for some $a, b \in \mathbb{R}^d \setminus \{0\}$, then there exists $h_1, h_2 \in \mathrm{BV}_{\mathrm{loc}}(\mathbb{R})$, $v_0 \in \mathbb{R}^d$ and a skew-symmetric matrix $R \in \mathbb{R}^{d\times d}_{\mathrm{skew}}$ such that

$$v(x) = v_0 + h_1(x\cdot b)a + h_2(x\cdot a)b + Rx, \qquad x \in \mathbb{R}^d \text{ a.e.}$$

But: Both assertions are wrong in general!

(i) Take $P_0 = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}$ and $v(x) = \begin{pmatrix} \mathrm{e}^{x_1}\sin(x_2) \\ -\mathrm{e}^{x_1}\cos(x_2) \end{pmatrix}$ (harmonic!).

(ii) Take $P_0 = \begin{pmatrix} 1 & \\ & 0 \end{pmatrix} = \mathrm{e}_1 \odot \mathrm{e}_1$ and $v(x) = \begin{pmatrix} 4x_1^3x_2 \\ -x_1^4 \end{pmatrix}$.

General structure

Question

What is the general structure of a measure $\mu \in \mathscr{M}_{\mathrm{loc}}(\mathbb{R}^d;\mathbb{R}^N)$ *with*

$$\mathscr{A}\mu = 0 \qquad (\text{or } \mathscr{A}\mu = \sigma \in \mathscr{M}(\mathbb{R}^d;\mathbb{R}^n))$$

and

$$\mu = P_0\nu,$$

where $P_0 \in \mathbb{R}^N$ *and* $\nu \in \mathscr{M}^+_{\mathrm{loc}}(\mathbb{R}^d)$*? In particular, when does it hold that*

$$\mu = \tau_1 + \ldots + \tau_N \qquad \text{mod } \mathrm{C}^\infty$$

with $\tau_1, \ldots, \tau_N$ *of a "simple structure"?*

BD case: $N = 2$ (two one-directional parts) – R. '11, De Philippis–R. '17, De Philippis–R. '20

Divergence: Structure of boundaryless currents (e.g. Smirnov's theorem). Also: Garroni & Nesi '04 and Palombaro–Ponsiglione '04, Arroyo-Rabasa '19.

Singular Density Theorem

Let $\mathscr{A}\mu = \sigma$ distributionally for

$$\mathscr{A}\mu := \sum_{|\alpha|\leq k} A_\alpha \partial^\alpha \mu.$$

Principal symbol: $\mathbb{A}^k(\xi) := \sum_{|\alpha|=k} (2\pi\mathrm{i})^k A_\alpha \xi^\alpha$

Tartar wave cone: $\Lambda_{\mathscr{A}} := \bigcup_{|\xi|=1} \ker \mathbb{A}^k(\xi)$

Rigidity/ellipticity: If $\mathscr{A}[P_0\nu] = 0$ with $P_0 \notin \Lambda_{\mathscr{A}}$, then $\nu \in \mathrm{C}^\infty$ ($N = 0$ in Question).

Theorem (*De Philippis & R.* '16)

If $\mu = g\mathscr{L}^d + \mu^s$, then

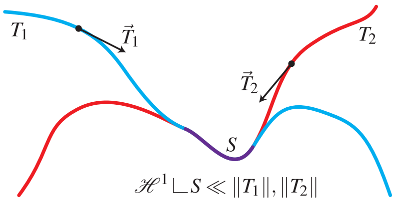
$$\frac{\mathrm{d}\mu^s}{\mathrm{d}|\mu^s|}(x) \in \Lambda_{\mathscr{A}} \qquad \text{for } |\mu^s|\text{-a.e. } x \in \Omega.$$

Corollary (Converse Rademacher Theorem)

Let ν be a positive Radon measure on $\mathbb{R}^d$ such that every Lipschitz function $f : \mathbb{R}^d \to \mathbb{R}$ is differentiable ν-almost everywhere. Then, $\nu \ll \mathscr{L}^d$.

Rigidity II (dimensionality and rectifiability)

Intersections of lines



Question: Given two smooth curves T_1, T_2 in $\mathbb{R}^2$ that intersect on a set S (but do not self-intersect). What do we know about $\vec{T}_1$, $\vec{T}_2$ on the intersection S?

Answer: $\vec{T}_1 \parallel \vec{T}_2$ $\mathscr{H}^1$-almost everywhere on the intersection S ("up to single points").

"The lines do not see the crossing points":
If $\operatorname{span}\{\vec{T}_1, \vec{T}_2\} = \mathbb{R}^2$ on S, then $\mathscr{H}^1(S) = 0$. Actually, $\dim_{\mathscr{H}} S = 0$.

Singularity of ν

Theorem (*De Philippis & R. '16*)

Let $T_1, \dots, T_d$ be normal 1-currents in $\mathbb{R}^d$, i.e.,

$$T_i \in \mathscr{M}_{\text{loc}}(\mathbb{R}^d; \mathbb{R}^d) \quad \textit{with} \quad \operatorname{div} T_i = \sigma_i \in \mathscr{M}_{\text{loc}}(\mathbb{R}^d; \mathbb{R}),$$

and $\nu \in \mathscr{M}^+(\mathbb{R}^d)$ a positive measure with

(i) $\nu \ll \|T_i\|$ *for* $i = 1, \dots, d$*;*

(ii) *for $|\nu|$-a.e. x:* $\operatorname{span}\{\vec{T}_1(x), \dots, \vec{T}_d(x)\} = \mathbb{R}^d$.

Then, $\nu \ll \mathscr{L}^d$.

Proof: Put

$$\mathbf{T} := \begin{pmatrix} T_1 \\ \vdots \\ T_d \end{pmatrix}, \qquad \text{so} \qquad \operatorname{div} \mathbf{T} = \sigma \in \mathscr{M}_{\text{loc}}(\mathbb{R}^d; \mathbb{R}^d).$$

By (ii),

$$\frac{\mathrm{d}\mathbf{T}}{\mathrm{d}|\mathbf{T}|}(x) \notin \Lambda_{\text{div}} = \left\{ M \in \mathbb{R}^{d \times d} \ : \ \det M = 0 \right\} \qquad \text{for } \nu\text{-a.e. } x.$$

Now use $\mathscr{A} = \operatorname{div}$ in the Singular Density Theorem. This gives

$$\frac{\mathrm{d}\mathbf{T}}{\mathrm{d}|\mathbf{T}|}(x) \in \Lambda_{\text{div}} \qquad \text{for } |\mathbf{T}|^s\text{-a.e. } x.$$

Since also $\nu^s \ll |\mathbf{T}|^s$ by (i), we get $\nu^s = 0$. $\square$

Co-cancelling operators

Definition (*van Schaftingen '13*)

The operator $\mathscr{A}$ is called **co-cancelling** if

$$\Lambda^1_{\mathscr{A}} := \bigcap_{\xi \in \mathbb{R}^d \setminus \{0\}} \ker \mathbb{A}^k(\xi) = \{0\}.$$

Example: $\mathscr{A} = \operatorname{div}$ is co-cancelling.

Theorem (*van Schaftingen '13*)

Assume that $\mathscr{A}$ is homogeneous and co-cancelling. If

$$\mathscr{A}(P_0\delta_0) = 0 \qquad \text{for some } P_0 \in \mathbb{R}^m,$$

then $P_0 = 0$.

Example: There is no $P_0 \neq 0$ such that $\operatorname{div}(P_0\delta_0) = 0$.

Corollary

Let $\mathscr{A}\mu = 0$ with $\mathscr{A}$ co-cancelling. If μ is "0-rectifiable", then $\mu = 0$.

Conclusion: Other wave cones might give information about the dimension of μ...

Hierarchy of wave cones

Definition

For $\ell = 1, \dots, d$ we define the ℓ**-dimensional wave cone** as

$$\Lambda^{\ell}_{\mathscr{A}} := \bigcap_{\pi \in \mathrm{Gr}(\ell, d)} \bigcup_{\xi \in \pi \setminus \{0\}} \ker \mathbb{A}^k(\xi),$$

where $\mathrm{Gr}(\ell, d)$ is the Grassmanian of ℓ-planes in $\mathbb{R}^d$.

Inclusions:

$$\Lambda^{1}_{\mathscr{A}} = \bigcap_{\xi \in \mathbb{R}^d \setminus \{0\}} \ker \mathbb{A}^k(\xi) \subset \Lambda^{j}_{\mathscr{A}} \subset \Lambda^{\ell}_{\mathscr{A}} \subset \Lambda^{d}_{\mathscr{A}} = \Lambda_{\mathscr{A}}, \qquad 1 \leq j \leq \ell \leq d.$$

Dimensional estimates

Theorem (*Arroyo-Rabasa, De Philippis, Hirsch & R. '19*)

Let $\mathscr{A}\mu = \sigma$. If $\mathscr{H}^\ell(E) = 0$ for some $\ell \in \{0, \ldots, d\}$, then

$$\frac{\mathrm{d}\mu}{\mathrm{d}|\mu|}(x) \in \Lambda^\ell_{\mathscr{A}} \qquad \text{for } |\mu|\text{-a.e. } x \in E.$$

Remark: For $\ell = d$, this recovers the '16 Singular Density Theorem.

Corollary

Let $\mathscr{A}\mu = \sigma$. Define

$$\ell_{\mathscr{A}} := \max\{\, \ell \in \mathbb{N} \;:\; \Lambda^\ell_{\mathscr{A}} = \{0\}\,\}.$$

Then,

$$\mu \ll \mathscr{H}^{\ell_{\mathscr{A}}}.$$

Remark: For $\ell = 1$, this also improves the result of van Schaftingen '13 ($\dim_{\mathscr{H}} \mu \geq 1$ as opposed to $\dim_{\mathscr{H}} \mu > 0$).

Rectifiability

Define the **upper ℓ-density** of $|\mu|$:

$$\theta^*_\ell(|\mu|)(x) := \limsup_{r\to 0} \frac{|\mu|(B_r(x))}{(2r)^\ell}.$$

Theorem (*Arroyo-Rabasa, De Philippis, Hirsch & R. '19*)

Let $\mathscr{A}\mu = \sigma$ and assume

$$\Lambda^\ell_{\mathscr{A}} = \{0\}.$$

Then, $\mu \llcorner \{\theta^*_\ell(|\mu|) > 0\}$ is concentrated on an ℓ-rectifiable set R and

$$\mu \llcorner R = P(x)\,\mathscr{H}^\ell_x \llcorner R,$$

where

$$P(x_0) \in \bigcap_{\xi \in (T_{x_0} R)^\perp} \ker \mathbb{A}^k(\xi) \qquad \text{for } \mathscr{H}^\ell\text{-a.e. } x_0 \in R \text{ (or } |\mu|\text{-a.e. } x_0 \in R).$$

Here, $T_{x_0}R$ is the the approximate tangent plane to R at x_0.

Remark: Recovers rectifiability results for BV-maps ($\mathscr{A} = \mathrm{curl}$, $\ell_{\mathrm{curl}} = d-1$) and for BD-maps ($\mathscr{A} = \mathrm{curl}\,\mathrm{curl}$, $\ell_{\mathrm{curl}\,\mathrm{curl}} = d-1$).

Rectifiability for divergence constraint

Theorem (*Arroyo-Rabasa, De Philippis, Hirsch & R. '19*)

Let $\operatorname{div}\mu = \sigma$. *Assume that*

$$\operatorname{rank}\left(\frac{\mathrm{d}\mu}{\mathrm{d}|\mu|}(x)\right) \geq \ell \qquad \text{for } |\mu|\text{-a.e. } x.$$

Then, $|\mu| \ll \mathscr{H}^{\ell}$ *and there exist an* ℓ*-rectifiable set* $R \subset U$ *such that*

$$\mu \llcorner \{\theta^*_\ell(|\mu|) > 0\} = P(x)\, \mathscr{H}^{\ell}_x \llcorner R, \qquad \operatorname{rank} P(x) = \ell.$$

Remark: Recovers several known rectifiability criteria for varifolds (Allard '72, Ambrosio–Soner '97, Lin '99, Moser '03, De Philippis–De Rosa–Ghiraldin '18).

Proof: Let $\widetilde{\mu} := (\mu, \sigma)$ and $\mathscr{A}(\widetilde{\mu}) := \operatorname{div}\mu - \sigma$. Then,

$$\Lambda^{\ell}_{\mathscr{A}} = \{\, M \in \mathbb{R}^{d\times d} \;:\; \operatorname{rank} M < \ell \,\}.$$

□

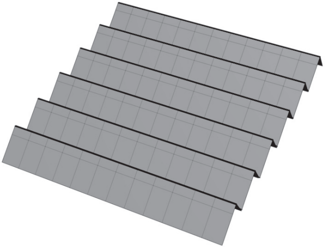
Question

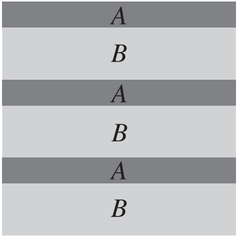
Are these dimensionality/rectifiability results sharp?

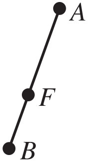
Compensated compactness I: Differential inclusions

Laminates, I

- Let $A, B \in \mathbb{R}^{d\times d}$ with $B - A = a \otimes n := an^T$ for $a, n \in \mathbb{R}^d \setminus \{0\}$.
- Let $\theta \in [0,1]$ and $F := \theta A + (1-\theta)B$.

$u_j =$ 

$\nabla u_j =$ 



- These u_j satisfy the differential inclusion

$$\nabla u_j(x) \in \{A, B\} \quad \text{for a.e. } x \in \Omega$$

and the convergence

$$\nabla u_j \overset{*}{\rightharpoonup} F \quad \text{in } W^{1,\infty}_{\mathrm{loc}}.$$

Ball–James theorem

Theorem (*Ball & James 1987*)

Let $\Omega \subset \mathbb{R}^d$ *be an open, bounded, and connected set and let* $A, B \in \mathbb{R}^{m\times d}$ *with*

$$\operatorname{rank}(A - B) \geq 2.$$

(A) *If* $u \in \mathrm{W}^{1,\infty}(\Omega; \mathbb{R}^m)$ *satisfies the differential inclusion*

$$\nabla u(x) \in \{A, B\} \quad \textit{for a.e. } x \in \Omega,$$

then either $\nabla u \equiv A$ *or* $\nabla u \equiv B$.

(B) *Let* $(u_j) \subset \mathrm{W}^{1,\infty}(\Omega; \mathbb{R}^m)$ *be a norm-bounded sequence such that*

$$\operatorname{dist}(\nabla u_j, \{A, B\}) \to 0 \quad \textit{in measure.}$$

Then, up to extracting a subsequence, either

$$\int_\Omega |\nabla u_j(x) - A| \,\mathrm{d}x \to 0 \qquad \textit{or} \qquad \int_\Omega |\nabla u_j(x) - B| \,\mathrm{d}x \to 0$$

as $j \to \infty$.

Two-state problem

Theorem (*De Philippis, Palmieri & R. '18*)

Let $\Omega \subset \mathbb{R}^d$ *be a domain. Suppose that* $\lambda, \mu \in \mathbb{R}^N$ *with*

$$\lambda - \mu \notin \Lambda_{\mathscr{A}}.$$

(A) *If* $v \in \mathrm{L}^\infty(\Omega;\mathbb{R}^N)$ *is such that*

$$\mathscr{A} v = 0 \qquad \text{and} \qquad v(x) \in \{\lambda, \mu\} \quad \text{for a.e. } x \in \Omega,$$

then either $v \equiv \lambda$ *or* $v \equiv \mu$.

(B) *Let* $(v_j) \subset \mathrm{L}^1(\Omega;\mathbb{R}^N)$ *be a uniformly norm-bounded sequence of maps such that*

$$\mathscr{A} v_j = 0 \qquad \text{and} \qquad \lim_{j\to\infty} \int_\Omega \mathrm{dist}(v_j(x), \{\lambda, \mu\})\,\mathrm{d}x = 0.$$

Then, up to extracting a subsequence, either

$$\int_\Omega |v_j(x) - \lambda|\,\mathrm{d}x \to 0 \qquad \text{or} \qquad \int_\Omega |v_j(x) - \mu|\,\mathrm{d}x \to 0$$

as $j \to \infty$.

Other work: Garroni & Nesi '04 and Palombaro–Ponsiglione '04 ($\mathscr{A} = \mathrm{div}$), Barchiesi '03 (some first-order $\mathscr{A}$), Sorella-Tione '21 (flexibility for 4-state problem).

A conjecture

Conjecture

Suppose $\mu \in \mathscr{M}(\Omega;\mathbb{R}^m)$ *solve*

$$\mathscr{A}\mu = \sigma \quad \textit{in } \mathscr{D}'(\Omega;\mathbb{R}^n).$$

and its polar satisfies

$$\frac{\mathrm{d}\mu}{\mathrm{d}|\mu|}(x) \in K \qquad \textit{for } |\mu|\textit{-a.e. } x \in \Omega$$

with $K \subset (\mathbb{R}^m \setminus \Lambda_{\mathscr{A}}) \cup \{0\}$ *a convex and closed (one-sided) cone. Then, we conjecture that*

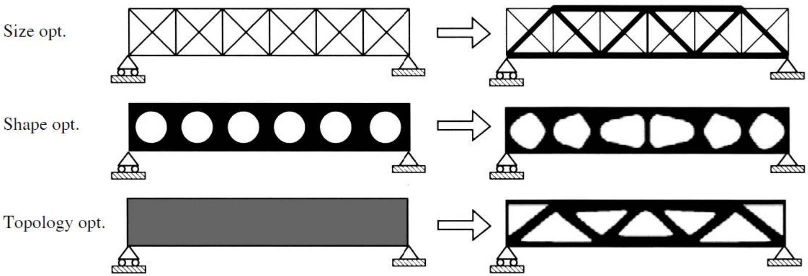
$$\mu \in \mathrm{L}^p_{\mathrm{loc}}(\Omega;\mathbb{R}^m)$$

for every $1 \le p < \frac{d}{d-k}$ *if* $k < d$ *and all* $p \in [1,\infty)$ *otherwise.*

Perturbative first results: Bate & Orponen '20 (for $\mathscr{A} = \mathrm{div}$), Arroyo-Rabasa–De Philippis–Hirsch-R.–Skorobogatova '21.

Compensated compactness II: Shape optimization

Optimal structures



picture from *Gebisa & Lemu 2017 IOP Conf. Ser.: Mater. Sci. Eng. 276 012026*

Objective: Given a bounded domain $\Omega \subset \mathbb{R}^d$ ($d = 2, 3$), find the optimal shape $\omega \subset \Omega$ of prescribed volume $\mathscr{L}^d(\omega) = \varepsilon$ that is maximally strong:

Minimize the (rescaled) compliance $$\min_{\sigma \in \mathrm{L}^2(\omega;\mathbb{R}^{d\times d}_{\mathrm{sym}})} \left\{ \varepsilon \int_\omega j^*(\sigma)\,\mathrm{d}x \;:\; -\operatorname{div}(\sigma \mathbb{1}_\omega) = f \right\}$$

over all shapes $\omega \in \mathscr{A}_\varepsilon := \{ \omega \subset \Omega \;:\; \omega \text{ Lipschitz domain, } \partial\Omega \subset \partial\omega,\ \mathscr{L}^d(\omega) = \varepsilon \}$.

"Light" structures: What happens in the **vanishing-mass limit** $\varepsilon \downarrow 0$?

Bouchitté's conjecture

$$\mathscr{C}_\varepsilon(\mu) := \begin{cases} \min\limits_{\sigma \in \mathrm{L}^2(\mu;\mathbb{R}^{d\times d}_{\mathrm{sym}})} \left\{ \int j^*(\sigma)\, \mathrm{d}\mu \ : \ -\mathrm{div}(\sigma\mu) = f \right\} & \text{if } \mu = \frac{\mathscr{L}^d \llcorner \omega}{\varepsilon} \text{ for } \omega \in \mathscr{A}_\varepsilon, \\ +\infty & \text{otherwise} \end{cases}$$

Conjecture (*Bouchitté '01*)

The **limit compliance** $\overline{\mathscr{C}}$, *for which*

$$\inf_{\omega \in \mathscr{A}_\varepsilon} \mathscr{C}_\varepsilon\left(\frac{\mathscr{L}^d \llcorner \omega}{\varepsilon}\right) \to \inf_{\mu \in \mathscr{M}^1(\overline{\Omega})} \overline{\mathscr{C}}(\mu) \qquad \text{as } \varepsilon \downarrow 0,$$

is given as

$$\overline{\mathscr{C}}(\mu) = \min_{\sigma \in \mathrm{L}^2(\mu;\mathbb{R}^{d\times d}_{\mathrm{sym}})} \left\{ \int \overline{j}^*(\sigma)\, \mathrm{d}\mu \ : \ -\mathrm{div}\,(\sigma\mu) = f \right\},$$

where the **infinitesimal-mass integrand** $\overline{j}^*$ *is defined as the convex conjugate to*

$$\overline{j}(\xi) := \sup_{\substack{\tau \in \mathbb{R}^{d\times d}_{\mathrm{sym}} \\ \det \tau = 0}} \left\{ \xi : \tau - j^*(\tau) \right\}, \qquad \xi \in \mathbb{R}^{d\times d}_{\mathrm{sym}}.$$

Main theorem

Theorem (*Babadjian & Iurlano & R. 2021*)

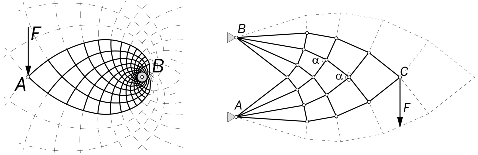
Assume that Ω is a bounded C^2-domain in $\mathbb{R}^2$ or $\mathbb{R}^3$. Then, Bouchitté's vanishing-mass conjecture holds (near a minimum) for the integrand $j(\xi) := \frac{1}{2}|\bullet|^2$, that is,

$$\inf_{\omega \in \mathscr{A}_\varepsilon} \mathscr{C}_\varepsilon\left(\frac{\mathscr{L}^d \llcorner \omega}{\varepsilon}\right) \to \inf_{\mu \in \mathscr{M}^1(\overline{\Omega})} \overline{\mathscr{C}}(\mu) \qquad \text{as } \varepsilon \downarrow 0.$$

Further:

- *Approximate minimizers of $\mathscr{C}_\varepsilon$ converge weakly* to a minimizer of $\overline{\mathscr{C}}$.*
- *Every minimizer of $\overline{\mathscr{C}}$ is the limit of approximate minimizers of $\mathscr{C}_\varepsilon$.*

Corollary: Justification of the theory of Michell trusses (Michell 1904, Prager 1970s)



Previous results: Olbermann '17, '20 (soft constraint)

All other cases of the conjecture: open!

Thank you!

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