Advection-diffusion equations with rough coefficients: weak solutions and vanishing viscosity

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Introduction

Given $\boldsymbol{b}\colon (0,T)\times\mathbb{T}^d \to \mathbb{R}^d$ on $\mathbb{T}^d := \mathbb{R}^d/\mathbb{Z}^d$ with $\operatorname{div}\boldsymbol{b} = 0$, we want to study the *transport/continuity* equation

$$\begin{cases} \partial_t u + \boldsymbol{b}\cdot\nabla u = 0 & \text{in } (0,T)\times\mathbb{T}^d \\ u|_{t=0} = u_0 & \text{in } \mathbb{T}^d, \end{cases} \tag{TE}$$

where $u_0\colon \mathbb{T}^d \to \mathbb{R}$ is a given initial datum.

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- Method of characteristics: if $\boldsymbol{b} \in L^1_t W^{1,p}_x$, for some $p \geq 1$, we define the **Lagrangian solution**

$$u^{\mathrm{L}}(t,x) := u_0(\boldsymbol{X}(t,\cdot)^{-1}(x)),$$

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  being $\boldsymbol{X}$ the Regular Lagrangian Flow of $\boldsymbol{b}$.
- If $\boldsymbol{b} \in L^1_t W^{1,p}_x$ and $u_0 \in L^q_x$ with $1/p + 1/q \leq 1$, then u^{L} is the *unique distributional* solution in $L^\infty_t L^q_x$ to (CE) [DiPerna-Lions '89].

Taming non-uniqueness

Outside DiPerna-Lions' regime, there are several ill-posedness results, obtained via *convex integration schemes* [Modena-Székelyhidi, Modena-Sattig, Bruè-Colombo-De Lellis, Cheskidov-Luo et al. '18 - '21].

Pivotal problem

Tame this non-uniqueness phenomenon, establishing **selection criteria**.

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Tame this non-uniqueness phenomenon, establishing **selection criteria**.

Inspired by convervation laws, we consider the **vanishing viscosity scheme**, i.e. study for $\varepsilon > 0$

$$\begin{cases} \partial_t v_\varepsilon + \operatorname{div}(\boldsymbol{b} v_\varepsilon) = \varepsilon \Delta v_\varepsilon & \text{in } (0,T) \times \mathbb{T}^d \\ v_\varepsilon|_{t=0} = v_{0,\varepsilon} & \text{in } \mathbb{T}^d \end{cases} \tag{VV$_\varepsilon$}$$

and understand compactness/convergence of $(v_\varepsilon)_\varepsilon$ as $\varepsilon \downarrow 0$. This naturally leads to the study of **advection-diffusion equations**.

Advection-diffusion equation

Given a vector field $\boldsymbol{b} \colon [0, T] \times \mathbb{T}^d \to \mathbb{R}^d$ on $\mathbb{T}^d := \mathbb{R}^d / \mathbb{Z}^d$, we thus study

$$\begin{cases} \partial_t v + \operatorname{div}(v\boldsymbol{b}) = \Delta v & \text{in } (0, T) \times \mathbb{T}^d \\ v|_{t=0} = v_0 & \text{in } \mathbb{T}^d, \end{cases} \tag{ADE}$$

where $\operatorname{div} \boldsymbol{b} = 0$ and $v_0 \colon \mathbb{T}^d \to \mathbb{R}$ is a given initial datum. In this talk, we address two peculiar aspects:

1. multiple **notions of solutions** (based on their **regularity**) can be given for (ADE) and, contrary to (TE), the presence of the Laplacian allows one to obtain well-posedness results even **without weak differentiability** of $\boldsymbol{b}$;

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1. multiple **notions of solutions** (based on their **regularity**) can be given for (ADE) and, contrary to (CE), the presence of the Laplacian allows one to obtain well-posedness results even **without weak differentiability** of $\boldsymbol{b}$;
2. **ill-posedness results via convex integration** are available also **for** (ADE) and should thus be taken into account.

Advection-diffusion equation: notions of solutions

Let us assume $\boldsymbol{b} \in L^1_t L^p_x$ with $\operatorname{div} \boldsymbol{b} = 0$ and $v_0 \in L^q$.

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1. If $1/p + 1/q \leq 1$ we can give a *distributional* definition of solution. Existence results for these solutions are easily obtained from energy estimates;
2. if, in addition, $p \geq 2$ and $q \geq 2$, then one can find *parabolic solutions*, i.e. distributional solutions with $u \in L^2_t H^1_x$. Existence follows again from energy estimates. Furthermore, in this regime, parabolic solutions are unique (classical commutators' estimate);
3. if $\boldsymbol{b} \in L^1_t W^{1,1}_x$ and $v_0 \in L^\infty$, then there always exists a unique parabolic solution [Le Bris-Lions '03].

Advection-diffusion equation: a regularity result for distributional solutions

Convex integration [Modena-Sattig '20]

There exists a divergence-free vector field $\boldsymbol{b} \in L^\infty_t L^2_x$ for which (ADE) admits infinitely many distributional solutions $v \in L^\infty_t L^2_x$ (the parabolic one being unique).

It is important to understand if there is a condition that guarantees the *parabolic regularity* (therefore uniqueness) of a distributional solution.

B.-Ciampa-Crippa '21

Let $p, q \in [1, \infty)$ such that $1/p + 1/q \leq 1/2$. If $\boldsymbol{b} \in L^2_t L^p_x$ is a divergence-free vector field and $u \in L^\infty_t L^q_x$ is a distributional solution to (ADE), then $u \in L^2_t H^1_x$.

The proof is based on a simple commutators' estimate in $L^2_t H^{-1}_x$.

The selection principle via vanishing viscosity

We now consider Sobolev $\boldsymbol{b}$, with $\operatorname{div} \boldsymbol{b} = 0$. Our starting point was

$$\begin{cases} \partial_t u \;\; + \operatorname{div}(\boldsymbol{b} u \;\;) = 0 & \text{in } (0, T) \times \mathbb{T}^d \\ u \;\; |_{t=0} = u_0 & \text{in } \mathbb{T}^d \end{cases} \tag{TE}$$

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for some approximant $v_{0,\varepsilon}$ of u_0.

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for some approximant $v_{0,\varepsilon}$ of u_0. For each $\varepsilon > 0$, (VV_ε) is parabolically well-posed [LeBris-Lions '03].

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Let $\boldsymbol{b} \in L^1_t W^{1,1}_x$ divergence-free and $u_0 \in L^1$. Let $(v_{0,\varepsilon})_\varepsilon \subset L^\infty$ be *any* sequence of functions such that $v_{0,\varepsilon} \to u_0$ in L^1. Then the *vanishing viscosity* sequence $(v_\varepsilon)_{\varepsilon>0} \subseteq L^\infty_t L^\infty_x \cap L^2_t H^1_x$ of parabolic solutions to (VV_ε) converges in $C([0,T]; L^1(\mathbb{T}^d))$ to the **Lagrangian solution** u^{L} to (CE).

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Remark

This selection principle works also beyond the distributional regime ($\boldsymbol{b} \in L^1_t W^{1,\mathbf{1}}_x$ and $u_0 \in L^{\mathbf{1}}$).

Glimpses of the proofs of the vanishing viscosity scheme

We present two proofs:

1. one is purely Eulerian, based on a duality argument [DiPerna-Lions '89];
2. the other proof is instead Lagrangian in nature, has its roots in stochastic flows and yields *quantitative* rates of convergence of $v^{\varepsilon} \to u^{\mathsf{L}}$. Such rates depend on the form of approximation/the regularity of the initial datum.

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Corollary I

If $u_0 \in H^1 \cap L^\infty$, then there exists a constant $C > 0$, with $C = C(T, p, \|u_0\|_\infty, \|u_0\|_{H^1}, \|\boldsymbol{b}\|_{W^{1,p}})$ s.t.

$$\sup_{t\in(0,T)} \|v_\varepsilon(t,\cdot) - u^{\mathsf{L}}(t,\cdot)\|_{L^2} \le C|\ln \varepsilon|^{-1/2} \qquad \text{as } \varepsilon \to 0.$$

Compare with [Bruè-Nguyen '20].

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Corollary II

There is **no anomalous dissipation**, i.e. if $\boldsymbol{b} \in L^1_t W^{1,1}_x$ and $u_0 \in L^2$

$$\varepsilon \int_0^T \|\nabla v^{\varepsilon}\|^2_{L^2} \, dt \to 0, \qquad \text{as } \varepsilon \to 0.$$

Thank you!