# The Rayleigh-Taylor instability in the Boussinesq approximation

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joint work with József Kolumbán

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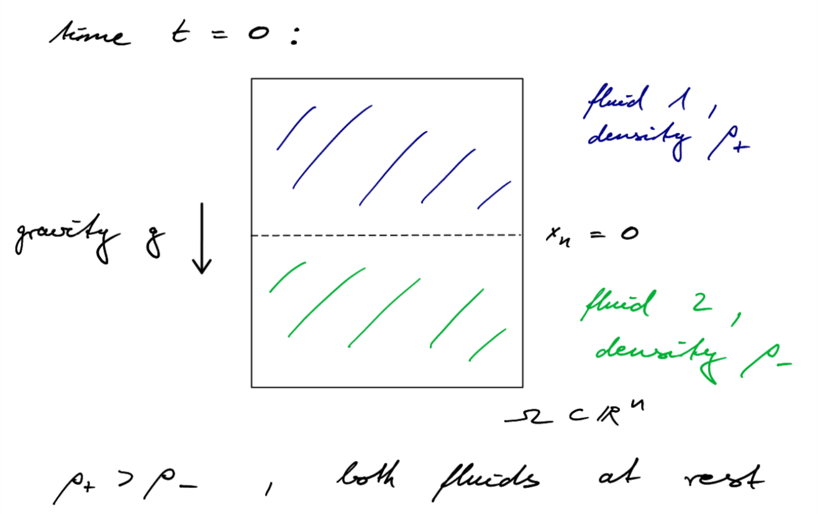
# Introduction



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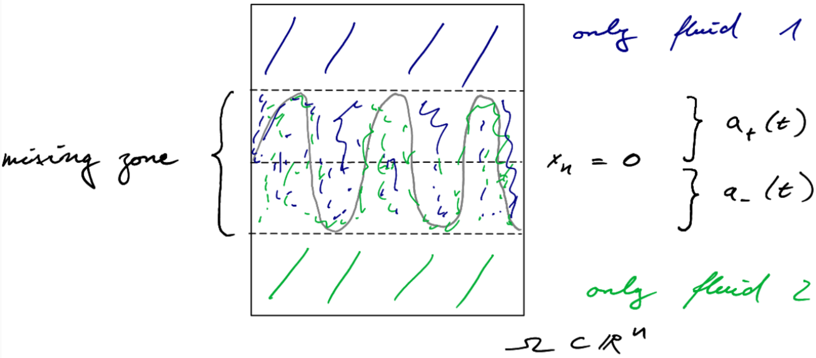


( not necessarily $a_+(t) = a_-(t)$ )

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- experiments and numerical simulations:

$$a_\pm(t) = \alpha_\pm \mathcal{A} g t^2,$$

where $\mathcal{A} = \frac{\rho_+ - \rho_-}{\rho_+ + \rho_-}$ is the Atwood number and $\alpha_\pm > 0$ a constant, see surveys: Abarzhi (2010); Boffetta, Mazzino (2017); Zhou (2017)

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- particular experiment by Ramaprabhu, Andrews (2003): cold and hot water with $\mathcal{A} \approx 7.5 \cdot 10^{-4} \Rightarrow \alpha_{\pm} \approx 0.07$

# The mathematical model

- on $\Omega \times [0, T)$, $\Omega \subset \mathbb{R}^n$ bounded domain, $T > 0$ consider the *inhomogenous incompressible Euler equations*

$$\begin{cases} \partial_t(\rho v) + \mathrm{div}(\rho v \otimes v) + \nabla p = -\rho g e_n \\ \partial_t \rho + \mathrm{div}(\rho v) = 0 \\ \mathrm{div} v = 0 \end{cases}$$

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- no-penetration boundary condition: $v \cdot \vec{n} = 0$ on $\partial\Omega \times [0, T)$
- initial data: $\rho(x, 0) = \rho_0(x)$ and $v(x, 0) = v_0(x)$ with $\mathrm{div} v_0 = 0,\ v_0 \cdot \vec{n} = 0$

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- Boussinesq approximation for $\mathcal{A} = \frac{\rho_+ - \rho_-}{\rho_+ + \rho_-}$ small:

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- allows to normalize $\rho_- \to -1$ and $\rho_+ \to +1$
- can be seen as a system in-between inhomogeneous Euler and incompressible porous media equation (IPM) (replace acceleration $\partial_t v + \operatorname{div} v \otimes v$ by velocity $v$)

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Convex integration for IPM:

- Córdoba, Faraco, Gancedo 2011; Székelyhidi 2012; Förster, Székelyhidi 2018; Castro, Faraco, Mengual 2019 & 2021; Noisette, Székelyhidi 2020; Mengual 2020; Hitruhin, Lindberg 2021; Castro, Córdoba, Faraco 2021

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Convex integration for non-two-phase ($\rho_0 \in \mathcal{C}^2 \cap L^\infty$) Boussinesq (with Coriolis force and dissipation for $\rho$):

- Chiodaroli-Michálek 2017

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- consider Boussinesq system on $\Omega = (0,1)^{n-1} \times (-1,1)$ with initial data

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- weak admissibility: $E(t) \leq E_0$ for a.e. $t \in (0, T)$, where

$$E(t) = \int_\Omega \frac{1}{2} |v(x,t)|^2 + \rho(x,t) g \mathcal{A} x_n \, dx,$$

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- Rayleigh (1883) and Taylor (1950) showed that it is linearly unstable

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- for any $t \in (0,T)$ and any open ball $B \subset \{\, x \in \Omega : x_n \in (-a(t), a(t)) \,\}$ there holds

$$\int_B 1 - \rho(x,t)\, dx \cdot \int_B \rho(x,t) - (-1)\, dx > 0.$$

"turbulent mixing at every time slice"

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- no condition on $\mathcal{A}$ (but Boussinesq approximation only reasonable for small $\mathcal{A}$)

# Boussinesq system as differential inclusion

- Recall

$$\begin{cases} \partial_t v + \operatorname{div}(v \mathring{\otimes} v) + \nabla \left( p + \frac{2}{n} |v|^2 \right) = -\rho g \mathcal{A} e_n \\ \partial_t \rho + \operatorname{div}(\rho v) = 0 \\ \operatorname{div} v = 0, \quad \rho \in \{ -1, 1 \} \ a.e. \end{cases}$$

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- in contrast to inhomogeneous Euler: transformation to accelerated domain not possible, and not needed

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- **Differential inclusion:**

$z = (\rho, v, m, \sigma)$ solves (*) & takes pointwise a.e. values in $K$

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Explicit relaxation: $z$ belongs to interior of $K^{co}_{(x,t)} = K^{\Lambda}_{(x,t)}$ iff

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- $\lambda_{max}\left(\frac{v\otimes v - \rho(m\otimes v + v\otimes m) + m\otimes m}{1-\rho^2} - \sigma\right) < e_0(x,t) + \rho e_1(x,t)$

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**Convex integration Thm.:** $\exists$ subsolution $\Rightarrow \exists\ \infty$-many solutions which are close in weak $L^2$-topology.

# Boussinesq system as differential inclusion

Explicit relaxation: $z$ belongs to interior of $K^{co}_{(x,t)} = K^{\Lambda}_{(x,t)}$ iff

- $\rho \in (-1, 1)$
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Note that there is freedom in the choice of $e_0$, $e_1$ (affecting the kinetic energy of the induced solutions).

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Ansatz for subsolutions:

- one-dimensional: $\bar{z}(x,t) = \bar{z}(x_n,t)$, $\bar{v}(x,t) = \bar{v}_n(x_n,t)e_n$, $\bar{m}(x,t) = \bar{m}_n(x_n,t)e_n$

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  with $f : [-1,1] \to [-1,1]$, $f(\pm 1) = \pm 1$,
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- here $e_0, e_1$ viewed as parameters, have to satisfy hull inequalities

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- cf. low Atwood number experiment: $a(t) = 0.07 g\mathcal{A}t^2$

# Thank you!