Convex integration in SPDEs

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based on a joint works with R. Zhu and X. Zhu

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$$\begin{aligned} \partial_t u + \operatorname{div}(u \otimes u) + \nabla p &= \Delta u + f \\ \operatorname{div} u &= 0 \end{aligned} \qquad x \in \mathbb{T}^3, t \in (0,T)$$

- f irregular, possibly dependent on u, e.g. $f = G(u)\frac{\mathrm{d}W}{\mathrm{d}t}$
- a Brownian motion W – additional randomness variable $\omega \in \Omega$, i.e. $u = u(t, x, \omega)$
- adaptedness: $u(t)$ measurable wrt $\sigma(W(s); 0 \leqslant s \leqslant t)$ (or a bigger σ-field $\mathcal{F}_t$)
- f being a space-time white noise: $f \in B^{-5/2-}_{\infty,\infty}$ $\mathbb{P}$-a.s.
- hope that a noise can help with the well-posedness issue

- damping – no explosion with large probability (Röckner, Zhu, Zhu '14)

$$G(u) = \alpha u$$

- transport noise – no explosion for the vorticity form with large probability (Flandoli, Luo '19)

$$G(\xi) \circ \mathrm{d}W = \sigma \cdot \nabla \xi \circ \mathrm{d}W$$

- Feller and strong Feller property: smoothing wrt the initial condition as opposed to continuous dependence (Da Prato, Debussche '03, Flandoli, Romito '08, Zhu, Zhu '14)

$$G(u) = G$$

sufficiently nondegenerate

Pathwise uniqueness *Two solutions u_1, u_2 on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with the same initial condition coincide pathwise:*

$$\mathbb{P}(u_1(t) = u_2(t) \text{ for all } t \in [0, T]) = 1.$$

- law of a solution – pushforward measure of $u: (\Omega, \mathbb{P}) \to \mathcal{T}$ on the space of trajectories $\mathcal{T}$

Uniqueness in law *The probability laws of any two solutions u_1, u_2 defined possibly on different probability spaces and starting from the same initial law coincide:*

$$\mathrm{Law}[u_1] = \mathrm{Law}[u_2].$$

- W and $-W$ have the same law

Yamada–Watanabe–Engelbert's theorem *For a certain class of S(P)DEs the following are equivalent:*

- *pathwise uniqueness,*
- *uniqueness in law and existence of a pathwise solution.*

Non-uniqueness in law

$$\begin{aligned} \mathrm{d}u + [\mathrm{div}\,(u \otimes u) + \nabla p]\mathrm{d}t &= \Delta u \mathrm{d}t + G(u)\mathrm{d}W \\ \mathrm{div}\, u &= 0 \end{aligned} \qquad x \in \mathbb{T}^3, t \in (0,T)$$

- either additive $G(u)\mathrm{d}W = G\mathrm{d}W$, linear multiplicative $G(u)\mathrm{d}W = u\mathrm{d}W$ or nonlinear cylindrical $G(u)\mathrm{d}W = g(\langle u, \varphi\rangle)\mathrm{d}W$ noise

Idea:

1. **convex integration** similar to Buckmaster–Vicol
   - existence of probabilistically strong and analytically weak solutions
   - possible up to a stopping time τ (to control the noise uniformly in ω)
   - they behave badly – energy not decreasing
2. **probabilistic extension** to $[0, \infty)$
   - connect to a Leray probabilistically weak solution obtained by compactness arguments
3. **comparison** with a Leray probabilistically weak solution starting from the same $u(0)$

$$\begin{aligned} \mathrm{d}u + [\operatorname{div}(u\otimes u) + \nabla p]\mathrm{d}t &= \Delta u \mathrm{d}t + G\mathrm{d}W \\ \operatorname{div} u &= 0 \end{aligned} \qquad x \in \mathbb{T}^3, t \in (0,T)$$

- follow the approach of Buckmaster–Vicol, use intermittent jets
- apart from w_{q+1}^p, w_{q+1}^c, w_{q+1}^t we introduce a stochastic corrector w_{q+1}^s: let

$$\mathrm{d}z = \Delta z \mathrm{d}t + G\mathrm{d}W\,, \qquad z_q = \mathbb{P}_{\leqslant \lambda_{q+1}^{\alpha/8}} z, \qquad z_\ell = z_q *_{t,x} \varphi_\ell,$$

$$w_{q+1}^s = z_{q+1} - z_\ell$$

- then

$$u_{q+1} = u_\ell + w_{q+1}^p + w_{q+1}^c + w_{q+1}^t + w_{q+1}^s, \qquad u_\ell = u_q *_{t,x} \varphi_\ell$$

- stopping time to control the noise uniformly in ω

$$\tau = \inf\{t \geqslant 0; \|z(t)\|_{H^{1-\delta}} \geqslant L\} \wedge \inf\left\{t \geqslant 0; \|z\|_{C_t^{1/2-2\delta}L^2} \geqslant L\right\} \wedge L$$

- adaptedness – needed for the extension of solutions

- let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geqslant 0}, \mathbb{P})$ and a Brownian motion W be given

H., Zhu, Zhu '19 *Let $T>0$, $K>1$ and $\kappa\in(0,1)$ be given. There exists $\gamma\in(0,1)$ and a $\mathbb{P}$-a.s. strictly positive stopping time τ satisfying*

$$\mathbb{P}(\tau \geqslant T) > \kappa$$

such that the following holds true:

- *There exists an $(\mathcal{F}_t)_{t\geqslant 0}$-adapted process u which belongs to $C([0,\tau]; H^\gamma)$ $\mathbb{P}$-a.s. and is an analytically weak solution.*
- *In addition, for the additive noise case*

$$\|u(T)\|_{L^2_x} > K\|u(0)\|_{L^2_x} + K(T\,\mathrm{tr}(GG^*))^{1/2}$$

  on the set $\{\tau \geqslant T\}$.
- *Corresponding failure of the energy inequality in the case of linear multiplicative and non-linear cylindrical noise.*

Leray probabilistically weak solutions obtained by compactness:

- exist for every given $u_0 \in L^2_{\mathrm{div}}$ but not on a given probability space

Non-uniqueness of Markov solutions

- probabilistic version of the semiflow property:
  - future states depend only upon the present state, not the past
  - knowledge of the whole past provides no more useful information than knowing the present state only
- follows from uniqueness
- abstract selection procedure by Krylov (minimizing a sequence of functionals)
- properties: stability, shift and concatenation (disintegration and reconstruction)

Idea:

- non-uniqueness in a class of solutions where Markov selection is possible
- relaxed energy inequality

$$E^p(t) := \|x(t)\|_{L^2}^{2p} + 2p\int_0^t \|x(r)\|_{L^2}^{2p-2}\|x(r)\|_{H^\gamma}^2 \mathrm{d}r - (C_{p,1} + C_{p,2}C_G)\int_0^t \|x(r)\|_{L^2}^{2p-2}\mathrm{d}r$$

is an almost sure supermartingale

H., Zhu, Zhu '21 *Let* $\bar{e} \geqslant \underline{e} > 4$ *and* $\tilde{e} > 0$ *be given. Then there exist* $\gamma \in (0,1)$ *and a* $\mathbb{P}$*-a.s. strictly positive stopping time* τ *such that the following holds true:*

For every $e\colon [0,1] \to [\underline{e}, \infty)$ *belonging to* C_b^1 *with* $\|e\|_{C^0} \leqslant \bar{e}$ *and* $\|e'\|_{C^0} \leqslant \tilde{e}$*, there exist a deterministic initial value* u_0 *and a probabilistically strong and analytically weak solution* $u \in C([0,\tau]; H^\gamma)$ $\mathbb{P}$*-a.s. satisfying*

$$\mathrm{esssup}_{\omega\in\Omega} \sup_{t\in[0,\tau]} \|u(t)\|_{H^\gamma} < \infty,$$

and for $t \in [0,\tau]$

$$\|u(t)\|_{L^2}^2 = e(t).$$

- applied with $e(t) = c_0 + c_1 t$
- solutions to the SPDE with deterministic energy

Existence and non-uniqueness of global-in-time probabilistically strong solutions

- non-uniqueness for any prescribed random initial condition in L^2
- no control of the energy
- extension of convex integration solutions by other convex integration solutions

H., Zhu, Zhu '21 *There exists an $\mathbb{P}$-a.s. strictly positive arbitrarily large stopping time τ, such that for any initial condition $u_0 \in L^2_\sigma$ $\mathbb{P}$-a.s. the following holds true:*

There exists an $(\mathcal{F}_t)_{t\geq 0}$-adapted process u which belongs to $L^p(0,\tau;L^2)\cap C([0,\tau],W^{\frac{1}{2},\frac{31}{30}})$ $\mathbb{P}$-a.s. for all $p\in[1,\infty)$ and is an analytically weak solution with $u(0)=u_0$.

There are infinitely many such solutions u.

Thanks for your attention!