Complete integrability of the Benjamin–Ono equation on the multi-soliton manifolds

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The Benjamin–Ono equation on the line

The Benjamin–Ono (BO) equation on the line reads as

$$(\mathrm{BO}) \quad \partial_t u = \mathrm{H}^{\mathbb{R}} \partial_x^2 u - \partial_x(u^2), \quad (t,x) \in \mathbb{R} \times \mathbb{R}, \tag{1.1}$$

where $u = u(t,x) \in \mathbb{R}$, $\mathrm{H}^{\mathbb{R}}$ denotes the Hilbert transform on $\mathbb{R}$,

$$\widehat{\mathrm{H}^{\mathbb{R}} f}(\xi) = -i \mathrm{sign}(\xi) \hat{f}(\xi), \quad \forall f \in L^2(\mathbb{R}) := L^2(\mathbb{R}, \mathbb{C}), \tag{1.2}$$

where $\mathrm{sign}(0) = 0$ and $\mathrm{sign}(\pm\xi) = \pm 1$, $\forall \xi > 0$.

- Derived by Benjamin (1967) and Ono (1975), equation (1.1) describes the evolution of weakly nonlinear internal long waves in a two-layer fluid s.t. the lower layer is infinite.
- Scaling and translation invariances of (1.1) : if $u = u(t,x)$ is a solution, so is $u_{c,y} : (t,x) \mapsto cu(c^2 t, c(x-y))$. It satisfies the L^1-critical property : $\int_{\mathbb{R}} |u_{c,y}(t,x)| \mathrm{d}x = \int_{\mathbb{R}} |u(c^2 t, x)| \mathrm{d}x$.

Formal Hamiltonian formalism

The BO equation (1.1) can be written in Hamiltonian form

$$\partial_t u = \partial_x \nabla_u E(u), \quad E(u) = \frac{1}{2}\langle |\mathrm{D}|u, u\rangle_{H^{-\frac{1}{2}}, H^{\frac{1}{2}}} - \frac{1}{3}\int_{\mathbb{R}} u^3, \quad (1.3)$$

where $\mathrm{D} = -i\partial_x$, $X_E(u) = \partial_x \nabla_u E(u)$ is the Hamiltonian vector field of E with respect to the Gardner bracket

$$\{F, G\}(u) := \langle \partial_x \nabla_u F(u), \nabla_u G(u)\rangle_{L^2(\mathbb{R})}, \quad (1.4)$$

for any sufficiently regular functionals F and G. Under appropriate conditions, the Gardner bracket is the Poisson bracket of the following symplectic form on some appropriate manifolds

$$\omega(h_1, h_2) = \langle h_1, \partial_x^{-1} h_2\rangle_{L^2}, \ \forall h_k \in \mathcal{W} := \partial_x \left(H^1(\mathbb{R}, \mathbb{R})\right). \quad (1.5)$$

Moreover, $\mathcal{W} = \{u \in L^2(\mathbb{R}, \mathbb{R}) : \int_{\mathbb{R}} \frac{|\hat{u}(\xi)|^2}{|\xi|^2} \mathrm{d}\xi < +\infty\}$.

Well-posedness results

The problem of global (resp. local) well-posedness of the BO equation (1.1) in the Sobolev space $H^s(\mathbb{R},\mathbb{R})$, $\forall s \geq 0$, has been studied by many mathematicians, we show only the major results :

- GWP in $H^3(\mathbb{R},\mathbb{R})$: Saut (1979) ;
- LWP in $H^s(\mathbb{R},\mathbb{R})$: Iorio (1986) for $s > \frac{3}{2}$; Ponce (1991) for $s = \frac{3}{2}$; Koch–Tzvetkov (2003) for $s > \frac{5}{4}$; Kenig–Koenig (2003) for $s > \frac{9}{8}$;
- GWP in $H^1(\mathbb{R},\mathbb{R})$: Tao (2004) ;
- GWP in $H^{\frac{1}{4}}(\mathbb{R},\mathbb{R})$: Burq–Planchon (2008) ;
- GWP in $L^2(\mathbb{R},\mathbb{R})$: Ionescu–Kenig (2007), Molinet–Pilod (2012) and Ifrim–Tataru (2019).

Szegő projector on the line

Let $\Pi^{\mathbb{R}} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ denote the Szegő projector, defined by

$$\Pi^{\mathbb{R}} := \frac{\mathrm{Id}_{L^2(\mathbb{R})} + i\mathrm{H}^{\mathbb{R}}}{2} \Leftrightarrow \widehat{\Pi^{\mathbb{R}} f}(\xi) = 1_{[0,+\infty)}(\xi)\hat{f}(\xi), \quad \forall f \in L^2(\mathbb{R}).$$

We define $L^2_+(\mathbb{R}) := \Pi^{\mathbb{R}}(L^2(\mathbb{R}))$ and $H^s_+(\mathbb{R}) := H^s(\mathbb{R}) \bigcap L^2_+(\mathbb{R})$, $\forall s \geq 0$. If $u : t \in \mathbb{R} \mapsto u(t) \in H^2(\mathbb{R}, \mathbb{R})$ solves the BO equation (1.1) and $w : t \in \mathbb{R} \mapsto w(t) := \Pi^{\mathbb{R}}(u(t)) \in H^2_+(\mathbb{R})$, then equation (1.1) reads as a special quadratic filtered Schrödinger (NLS–Szegő) equation on the space $H^2_+(\mathbb{R})$:

$$i\partial_t w - \partial_x^2 w + i\partial_x(w^2 + 2\Pi^{\mathbb{R}}(|w|^2)) = 0, \; (t,x) \in \mathbb{R} \times \mathbb{R}. \quad (1.6)$$

We refer to Sun (2019, 2020) to see the long time and asymptotic behavior of cubic or quintic NLS–Szegő equations. Moreover,

$$2\mathrm{Re}|_{L^2_+(\mathbb{R})} = (\Pi^{\mathbb{R}}_{|L^2(\mathbb{R},\mathbb{R})})^{-1}.$$

Hardy spaces

Let the upper half-plane and the lower half-plane be denoted by $\mathbb{C}_+ = \{z \in \mathbb{C} : \mathrm{Im} z > 0\}$, $\mathbb{C}_- = \{z \in \mathbb{C} : \mathrm{Im} z < 0\}$ respectively. For every $p \in (0, +\infty]$, let $\mathbb{H}^p(\mathbb{C}_+)$ denote the Hardy space on $\mathbb{C}_+$, i.e. $\mathbb{H}^p(\mathbb{C}_+) = \{g \in \mathrm{Hol}(\mathbb{C}_+) : \|g\|_{\mathbb{H}^p(\mathbb{C}_+)} < +\infty\}$, where

$$\|g\|_{\mathbb{H}^p(\mathbb{C}_+)} = \sup_{y>0} \left(\int_{\mathbb{R}} |g(x+iy)|^p \mathrm{d}x \right)^{\frac{1}{p}}, \text{ if } p \in (0, +\infty); \quad (1.7)$$

$\|g\|_{\mathbb{H}^\infty(\mathbb{C}_+)} = \sup_{z\in\mathbb{C}_+} |g(z)|$. A function $g \in \mathbb{H}^\infty(\mathbb{C}_+)$ is called an inner function if $\lim_{y\to 0^+} |g(x+iy)| = 1$, $\forall x \in \mathbb{R}$. When $p = 2$, the Paley–Wiener theorem yields the identification between $L^2_+(\mathbb{R})$ and $\mathbb{H}^2(\mathbb{C}_+)$:

$$L^2_+(\mathbb{R}) = \{f \in L^2(\mathbb{R}) : \mathrm{supp} \hat{f} \subset [0, +\infty)\} = \mathbb{H}^2(\mathbb{C}_+).$$

Solitary waves of the BO equation

A smooth solution $u = u(t,x)$ is called a solitary wave of the BO equation if $u(t,x) = c\mathcal{R}(c(x-y-ct))$, where $c > 0$, $y \in \mathbb{R}$ and $\mathcal{R} \in C^\infty(\mathbb{R})$ solves the following non local elliptic equation

$$\mathrm{H}^{\mathbb{R}}\mathcal{R}' + \mathcal{R} - \mathcal{R}^2 = 0, \qquad \mathcal{R}(x) > 0 \tag{1.8}$$

- The complete classification of solitary waves are given by Benjamin (1967), Ono (1975) for the explicit formula and by Amick–Toland (1991) and Frank–Lenzmann (2013) for the uniqueness statement.
- The unique (up to translation) solution of equation (1.8) is given by the following formula

$$\mathcal{R}(x) = \frac{2}{1+x^2}, \qquad \forall x \in \mathbb{R}. \tag{1.9}$$

Explicit formula of the solution

If $u : t \in \mathbb{R} \mapsto u(t) \in C^\infty(\mathbb{R}, \mathbb{R})$ solves the BO equation on the line $\partial_t u = \mathrm{H}^{\mathbb{R}} \partial_x^2 u - \partial_x(u^2)$ such that $u(0,x) = \frac{2\eta_1}{(x-x_1)^2+\eta_1^2}$, for some $x_1 \in \mathbb{R}$ and $\eta_1 > 0$, then we have the following explicit formula of solutions

$$u(t,x) = \frac{2\eta_1}{(x-(x_1+\eta_1^{-1}t))^2+\eta_1^2}, \quad \forall (t,x) \in \mathbb{R}^2. \tag{1.10}$$

Problem : Can we find the explicit formula of some general solutions of the BO equation ?

We introduce the notion of complete integrability in Liouville–Arnold sense in the domain of Hamiltonian dynamical system in order to solve this problem.

Action–angle coordinates

We introduce 'complete integrability' in the sense of existence of action–angle coordinates. Given $(\mathcal{M}_{2N}, \Lambda)$ a symplectic manifold of dimension $2N$, consider the following Hamiltonian system

$$\partial_t u(t) = X_H(u(t)), \quad u : t \in \mathbb{R} \mapsto u(t) \in \mathcal{M}_{2N}, \tag{1.11}$$

where $H : \mathcal{M}_{2N} \to \mathbb{R}$ is smooth and X_H denotes the Hamiltonian vector field of H with respect to the symplectic form Λ. Assume that there exist an open subset $\mathcal{Y}_{2N} \subset \mathbb{R}^{2N}$ and a diffeomorphism $\Phi : u \in \mathcal{M}_{2N} \mapsto (I_1(u), \cdots I_N(u); \gamma_1(u), \cdots \gamma_N(u)) \in \mathcal{Y}_{2N}$, such that if u solves system (1.11), then

$$\partial_t(I_j \circ u)(t) = 0, \quad \partial_t(\gamma_j \circ u)(t) = \mathrm{k}_j = \mathrm{const},$$

where the function $\mathrm{k}_j : \mathcal{M}_{2N} \to \mathbb{R}$ depends only on $I_1, I_2, \cdots, I_N$. Then Φ introduces global action–angle coordinates of system (1.11), i.e. the coordinates $\{I_j\}_{1 \le j \le N}$ are referred to as actions and $\{\gamma_j\}_{1 \le j \le N}$ as angles.

Goal

The main goal is to establish global action–angle coordinates for the BO equation $\partial_t u = \mathrm{H}^{\mathbb{R}} \partial_x^2 u - \partial_x(u^2)$ when restricted to every multi-soliton manifold $\mathcal{U}_N$, which is given by

$$\mathcal{U}_N := \{u \in L^2(\mathbb{R}, \mathbb{R}) : u(x) = \sum_{j=1}^{N} \frac{2\eta_j}{(x - x_j)^2 + \eta_j^2}, x_j \in \mathbb{R}, \eta_j > 0\}.$$

The presentation is divided to 3 parts :

- The BO equation on the real analytic symplectic manifold $\mathcal{U}_N$ is a globally well-posed Hamiltonian dynamical system.
- We state the main results on global action–angle coordinates, the explicit formula and the asymptotic approximation.
- Outlining the construction of global action–angle coordinates by using the Lax pair structure of the BO equation and the inverse spectral transform.

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Real analytic structure of $\mathcal{U}_N$

A function of the form

$$u(x) = \sum_{j=1}^{N} \frac{2\eta_j}{(x-x_j)^2+\eta_j^2} = -2\mathrm{Im}\frac{Q_u'(x)}{Q_u(x)}, \qquad (2.1)$$

is called an N-soliton, where $\eta_j > 0$ and $x_j \in \mathbb{R}$, $\forall j = 1, 2, \cdots, N$,

$$Q_u(X) := \prod_{j=1}^{N}(X - x_j + i\eta_j) \qquad (2.2)$$

is called the characteristic polynomial of u.

Proposition (Topological and real analytic structure)

Equipped with the subspace topology of $L^2(\mathbb{R},\mathbb{R})$, the subset $\mathcal{U}_N$ is a simply connected, real analytic, embedded submanifold of the $\mathbb{R}$-Hilbert space $L^2(\mathbb{R},\mathbb{R})$ and $\dim_{\mathbb{R}} \mathcal{U}_N = 2N$.

Symplectic structure of $\mathcal{U}_N$

Given $u \in \mathcal{U}_N$, since $\int_{\mathbb{R}} u = 2\pi N$, the tangent space to $\mathcal{U}_N$ at u, denoted by $\mathcal{T}_u(\mathcal{U}_N)$, is included in an auxiliary $\mathbb{R}$-vector space denoted by $\mathcal{T} := \{h \in L^2(\mathbb{R}, (1+x^2)\mathrm{d}x) : h(\mathbb{R}) \subset \mathbb{R}, \int_{\mathbb{R}} h = 0\}$. So $\mathcal{T} = \mathcal{W} \bigcap L^2(\mathbb{R}, x^2\mathrm{d}x) \subset \mathcal{W}$ by the Hardy's inequality, where $\mathcal{W} = \partial_x \left(H^1(\mathbb{R}, \mathbb{R})\right)$. Recall $\boldsymbol{\omega}$ given by (1.5) : $\forall h_1, h_2 \in \mathcal{W}$,

$$\boldsymbol{\omega}(h_1, h_2) = \int_{\mathbb{R}} \frac{i\hat{h}_1(\xi)\overline{\hat{h}_2(\xi)}}{2\pi\xi}\mathrm{d}\xi = -\mathrm{Im}\int_0^{+\infty} \frac{\hat{h}_1(\xi)\overline{\hat{h}_2(\xi)}}{\pi\xi}\mathrm{d}\xi.$$

Proposition (Symplectic structure)

Endowed with the closed non-degenerate 2-form $\omega : u \in \mathcal{U}_N \mapsto \omega_u = \boldsymbol{\omega}$ i.e. $\omega_u(h_1, h_2) = \frac{i}{2\pi}\int_{\mathbb{R}} \frac{\hat{h}_1(\xi)\overline{\hat{h}_2(\xi)}}{\xi}\mathrm{d}\xi$, $\forall h_1, h_2 \in \mathcal{T}_u(\mathcal{U}_N)$, the real analytic manifold $(\mathcal{U}_N, \omega)$ is a symplectic manifold.

Hamiltonian formalism and Invariance of $\mathcal{U}_N$

For every smooth function $f : \mathcal{U}_N \to \mathbb{R}$, its Hamiltonian vector field $X_f \in \mathfrak{X}(\mathcal{U}_N)$ is given by $X_f : u \in \mathcal{U}_N \mapsto \partial_x \nabla_u f(u) \in \mathcal{T}_u(\mathcal{U}_N)$, where $\nabla_u f(u)$ denotes its Fréchet derivative : $\mathrm{d}f(u)(h) = \langle h, \nabla_u f(u)\rangle_{L^2}$, for every $h \in \mathcal{T}_u(\mathcal{U}_N)$. Then the BO equation on the N-soliton manifold $(\mathcal{U}_N, \omega)$ can be written in Hamiltonian form

$$\partial_t u(t) = X_E(u(t)), \qquad u : t \in \mathbb{R} \mapsto u(t) \in \mathcal{U}_N, \tag{2.3}$$

where $E(u) := \frac{1}{2}\langle |\mathrm{D}|u, u\rangle_{L^2(\mathbb{R})} - \frac{1}{3}\int_{\mathbb{R}} u^3$ with $\mathrm{D} = -i\partial_x$, $\forall u \in \mathcal{U}_N$.

Proposition (S. 2020, Global well-posedness of BO on $\mathcal{U}_N$)

For every N-soliton $u_0 \in \mathcal{U}_N$, there exists a unique smooth function $u : t \in \mathbb{R} \mapsto u(t) \in \mathcal{U}_N$ solving the BO equation (1.3) with initial datum $u(0) = u_0$.

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Main result

Let $\Omega_N := \{(r_1, r_2, \cdots, r_N) \in \mathbb{R}^N : r_1 < r_2 < \cdots < r_N < 0\}$ denote the subset of actions and $\nu = \sum_{j=1}^{N} \mathrm{d}r_j \wedge \mathrm{d}\alpha_j$ denotes the canonical symplectic form on $\Omega_N \times \mathbb{R}^N$.

Theorem (S. 2020, Global action–angle coordinates on $\mathcal{U}_N$)

There exists a real analytic diffeomorphism $\Phi_N : \mathcal{U}_N \to \Omega_N \times \mathbb{R}^N$ such that Φ_N preserves the symplectic structure i.e. $\Phi_N^* \nu = \omega$ and

$$E \circ \Phi_N^{-1}(r_1, r_2, \cdots, r_N; \alpha_1, \alpha_2, \cdots, \alpha_N) = -\frac{1}{2\pi} \sum_{j=1}^{N} |r_j|^2.$$

Here, the 2-form $\Phi_N^* \nu$ denotes the pullback of the symplectic form ν by Φ_N, i.e. $(\Phi_N^* \nu)_u (h_1, h_2) = \nu_{\Phi_N(u)}(\mathrm{d}\Phi_N(u)(h_1), \mathrm{d}\Phi_N(u)(h_2))$, for every $h_1, h_2 \in \mathcal{T}_u(\mathcal{U}_N)$ and for every $u \in \mathcal{U}_N$.

Poisson brackets

The Poisson bracket of two smooth functions $f, g : \mathcal{U}_N \to \mathbb{R}$ is defined by $\{f, g\} : u \in \mathcal{U}_N \mapsto \omega_u(X_f(u), X_g(u)) \in \mathbb{R}$, so it coincides with the Gardner bracket in (1.4),

$$\{f, g\}(u) = \langle \partial_x \nabla_u f(u), \nabla_u g(u) \rangle_{L^2(\mathbb{R})}.$$

Let $\Phi_N : u \mapsto (I_1(u), I_2(u), \cdots, I_N(u); \gamma_1(u), \gamma_2(u), \cdots, \gamma_N(u))$ denote the action–angle map $\Phi_N : \mathcal{U}_N \to \Omega_N \times \mathbb{R}^N$ constructed in the preceding theorem. Its symplectomorphism property $\Phi_N^* \nu = \omega$ is equivalent to the following Poisson bracket characterization :

$$\{I_j, I_k\} = 0, \quad \{I_j, \gamma_k\} = 1_{j=k}, \quad \{\gamma_j, \gamma_k\} = 0 \quad \text{on} \quad \mathcal{U}_N,$$

$1 \leq j, k \leq N$; which is also equivalent to the following tangent vector correspondence $\mathrm{d}\Phi_N(u) : \mathcal{T}_u(\mathcal{U}_N) \to \mathcal{T}_{\Phi_N(u)}(\Omega_N \times \mathbb{R}^N)$,

$$\mathrm{d}\Phi_N(u) : X_{I_k}(u) \mapsto \frac{\partial}{\partial \alpha_k}\Big|_{\Phi_N(u)}, \quad \mathrm{d}\Phi_N(u) : X_{\gamma_k}(u) \mapsto -\frac{\partial}{\partial r_k}\Big|_{\Phi_N(u)}.$$

Global action–angle coordinates

Since the energy functional in (1.3) of the BO equation (1.1) reads as $E(u) = -\frac{1}{2\pi}\sum_{j=1}^{N} I_j(u)^2$, $\forall u \in \mathcal{U}_N$, we have the following

$$\{I_j, E\}(u) = \mathrm{d}E(u)(X_{I_j}(u)) = \frac{\partial}{\partial \alpha_j}\Big|_{\Phi_N(u)}(E \circ \Phi_N^{-1}) = 0,$$

$$\mathrm{k}_j(u) = \{E, \gamma_j\}(u) = -\frac{\partial}{\partial r_j}\Big|_{\Phi_N(u)}(E \circ \Phi_N^{-1}) = \frac{I_j(u)}{\pi}.$$

The frequency $\mathrm{k}_j = -\frac{I_j}{\pi} : \mathcal{U}_N \to (0, +\infty)$ is a linear function of the action I_j. So the motions of angles are completely decoupled.

Corollary (Linearization of the BO equation (1.1))

Let $u : t \in \mathbb{R} \mapsto u(t) \in \mathcal{U}_N$ solve the BO equation (1.1), *then*

$$\partial_t \left(I_j \circ u\right)(t) = 0, \quad \partial_t \left(\gamma_j \circ u\right)(t) = \mathrm{k}_j(u(t)) = -\frac{I_j}{\pi}(u(t)),$$

$\forall t \in \mathbb{R}$. *As a consequence,* $\Phi_N^{-1}(\{\mathrm{r}\} \times \mathbb{R}^N)$ *is a Lagrangian submanifold that is invariant under the flow of* (1.1), $\forall \mathrm{r} \in \Omega_N$.

$$
\begin{array}{ccc}
u_0 \in \mathcal{U}_N & \xrightarrow{\quad \Phi_N \quad} & (I_j(u_0);\gamma_j(u_0))_{1\le j\le N} \\
 & & \Big\downarrow \text{linear flow} \\
u(t) \in \mathcal{U}_N & \xleftarrow{\quad \Phi_N^{-1} \quad} & (I_j(u(t));\gamma_j(u(t)))_{1\le j\le N}
\end{array}
$$

Corollary (Linearization of the BO equation (1.1))

Let $u : t \in \mathbb{R} \mapsto u(t) \in \mathcal{U}_N$ solve the BO equation (1.1), then

$$\partial_t \left(I_j \circ u\right)(t) = 0, \quad \partial_t \left(\gamma_j \circ u\right)(t) = \mathrm{k}_j(u(t)) = -\frac{I_j}{\pi}(u(t)).$$

So $I_j \circ u(t) = I_j \circ u(0)$ and $\gamma_j \circ u(t) = \gamma_j \circ u(0) + (\mathrm{k}_j \circ u(0))t$, $\forall t \in \mathbb{R}$. Moreover, $\Phi_N^{-1}(\{\mathrm{r}\} \times \mathbb{R}^N)$ is a Lagrangian submanifold that is invariant under the flow of (1.1), $\forall \mathrm{r} \in \Omega_N$.

Inverse spectral formula

Let Q_u be the characteristic polynomial of the N-soliton $u \in \mathcal{U}_N$, i.e. $u = -2\mathrm{Im}\frac{Q_u'}{Q_u}$. The inverse spectral matrix

$$M : u \in \mathcal{U}_N \mapsto (M_{kj}(u))_{1\leq j,k\leq N} \in \mathbb{C}^{N\times N}$$

associated to $\Phi_N : u \mapsto (I_1(u), \cdots, I_N(u); \gamma_1(u), \cdots, \gamma_N(u))$ is

$$M_{kj}(u) := \begin{cases} \frac{2\pi i}{I_k(u)-I_j(u)}\sqrt{\frac{I_k(u)}{I_j(u)}}, & \text{if} \quad j \neq k, \\ \gamma_j(u) + \frac{\pi i}{I_j(u)}, & \text{if} \quad j = k. \end{cases} \tag{3.1}$$

Proposition (Inverse spectral formula)

The polynomial Q_u is the characteristic polynomial of the matrix $M(u) \in \mathbb{C}^{N\times N}$ defined by (3.1), so $u(x) = -2\mathrm{Im}\frac{\frac{\mathrm{d}}{\mathrm{d}x}\det(x-M(u))}{\det(x-M(u))}$, $\forall u \in \mathcal{U}_N$. The translation–scaling parameters of u are eigenvalues with corresponding multiplicities of the matrix $M(u)$.

Explicit formula of multi-soliton solutions

Let $u : t \in \mathbb{R} \mapsto u(t) \in \mathcal{U}_N$ solve the BO equation on the line

$$\text{(BO)} \quad \partial_t u = \mathrm{H}^{\mathbb{R}} \partial_x^2 u - \partial_x(u^2), \quad (t,x) \in \mathbb{R} \times \mathbb{R},$$

then $\forall (t,x) \in \mathbb{R} \times \mathbb{R}$, we have the following explicit formula

$$u(t,x) = 2\mathrm{Im}\langle \left(M(u_0) - (x + \tfrac{t}{\pi}\mathfrak{V}(u_0))\right)^{-1} X(u_0), Y(u_0)\rangle_{\mathbb{C}^N},$$

where the inner product of $\mathbb{C}^N$ is $\langle X, Y\rangle_{\mathbb{C}^N} = X^T \overline{Y}$; the matrix

$$\mathfrak{V}(u) = \begin{pmatrix} I_1(u) & & & \\ & I_2(u) & & \\ & & \ddots & \\ & & & I_N(u) \end{pmatrix} \in \mathbb{C}^{N\times N}, \quad \forall u \in \mathcal{U}_N;$$

$M(u)$ is given by (3.1) and $X(u), Y(u) \in \mathbb{C}^N$ are defined by

$$\sqrt{2\pi} X(u)^T = (\sqrt{|I_1(u)|}, \sqrt{|I_2(u)|}, \cdots, \sqrt{|I_N(u)|}),$$

$$\sqrt{2\pi}^{-1} Y(u)^T = (\sqrt{|I_1(u)|^{-1}}, \sqrt{|I_2(u)|^{-1}}, \cdots, \sqrt{|I_N(u)|^{-1}}).$$

Asymptotic approximation

Corollary (Constant velocity and scaling at ∞)

Recall that $\mathcal{R}(x) = \frac{2}{1+x^2}$. *Let* $u : t \in \mathbb{R} \mapsto u(t) \in \mathcal{U}_N$ *solve the BO equation* (1.1) *such that* $u(0) = u_0 \in \mathcal{U}_N$. *We define a function* $u_\infty(t,x) = u_\infty(t,x;u_0) := \sum_{j=1}^N \mathcal{R}_{\mathrm{k}_j(u_0)}(x - \gamma_j(u_0) - \mathrm{k}_j(u_0)t)$, *where* $\mathcal{R}_c(x) = c\mathcal{R}(cx)$ *and* $\mathrm{k}_j := -\frac{I_j}{\pi}$. *Then we have*

(i) *for any* $R > 0$, $\lim_{t\to\pm\infty} \|u(t) - u_\infty(t)\|_{L^2(-R,R)} = 0$;

(ii) *for any* $x \in \mathbb{R}$, $\lim_{t\to\pm\infty} \frac{u(t,x)}{u_\infty(t,x)} = 1$.

When $t \to \pm\infty$, the N-soliton solutions of the BO equation (1.1) can be approximated asymptotically by the superposition of N solitons such that the j th soliton which starts from the point $\gamma_j(u_0)$, moves with constant velocity $\mathrm{k}_j(u_0)$ and constant scaling parameter $\mathrm{k}_j(u_0)$, see also Matsuno (1984), Case (1978, 1979).

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Outline of the proof

The action variables $\{\frac{I_j(u)}{2\pi}\}_{j=1}^N$ are the eigenvalues of Lax operator $L_u : h \in H^1_+(\mathbb{R}) \mapsto -i\partial_x h - \Pi^{\mathbb{R}}(uh) \in L^2_+(\mathbb{R})$ of the BO equation, discovered by Nakamura and Bock–Kruskal (1979). The simplicity, finiteness and sign of each $I_j(u)$ are given by Wu (2016).

The angle variables $\{\gamma_j(u)\}_{j=1}^N$ are constructed from the analysis of the shift semi-group $(S(\eta)^*)_{\eta\geq 0}$ acting on the Hardy space $L^2_+(\mathbb{R})$, which is defined by $S(\eta)f = e_\eta f$ with $e_\eta(x) = e^{i\eta x}$.

The local diffeomorphism property of $\Phi_N : \mathcal{U}_N \to \Omega_N \times \mathbb{R}^N$ is established by introducing the following generating functional : $\mathcal{H}_\lambda(u) = \langle (L_u+\lambda)^{-1}\Pi^{\mathbb{R}}u, \Pi^{\mathbb{R}}u\rangle_{L^2}$, $\forall u \in \mathcal{U}_N, \lambda \in \mathbb{R}\backslash\sigma(-L_u)$. Then the Lax pair formulation holds : $\mathrm{d}L(u)(X_{\mathcal{H}_\lambda}(u)) = [B_u^\lambda, L_u]$, for some bounded anti-symmetric operator $B_u^\lambda \in \mathfrak{B}(L^2_+(\mathbb{R}), L^2_+(\mathbb{R}))$. So $2\pi\{\lambda_j, \gamma_k\}(u) = \delta_{kj}$ and $\{\lambda_k, \lambda_j\}(u) = 0$.

Outline of the proof

The real analyticity of Φ_N is given by classical perturbation theory for linear operators, see Kato (1966), Reed–Simon (1978), etc. The diffeomorphism property is obtained by Hadamard's global inverse function theorem. Then the symplectomorphism property of the action–angle map $\Phi_N : (\mathcal{U}_N, \omega) \to (\Omega_N \times \mathbb{R}^N, \nu)$ is obtained by restricting the 2-form $\omega - \Phi_N^* \nu$ to a Lagrangian submanifold $\Lambda_N := \bigcap_{j=1}^N \gamma_j^{-1}(0) \subset \mathcal{U}_N$. Similar arguments can be found in

- Kappeler–Pöschel (2003) for the Korteweg–de Vries equation on $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$;
- Gérard–Grellier (2010) and Pocovnicu (2011) for the cubic Szegő equation on $\mathbb{T}$ and on $\mathbb{R}$, respectively ;
- Grébert–Kappeler (2014) for the cubic defocusing Schrödinger equation on $\mathbb{T}$;
- Gérard–Kappeler (2019) for the Benjamin–Ono equation on $\mathbb{T}$.

Space-periodic regime

The N-gap potential manifold for the BO equation on the torus $\mathbb{T} := \mathbb{R}/2\pi\mathbb{Z}$ in Gérard–Kappeler (2019) is defined by

$$U_N^{\mathbb{T}} = \{v = 2\mathrm{Re}h \in L^2(\mathbb{T}, \mathbb{R}) : h(y) = -e^{iy}\frac{\mathfrak{Q}'(e^{iy})}{\mathfrak{Q}(e^{iy})}, \mathfrak{Q} \in \mathbb{C}_N^+[X]\},$$

where $\mathbb{C}_N^+[X]$ consists of all monic polynomial $\mathfrak{Q} \in \mathbb{C}[X]$ of degree N, whose roots are in the annulus $\mathscr{A} := \{z \in \mathbb{C} : |z| > 1\}$. The restriction of Birkhoff coordinates in Gérard–Kappeler (2019) to $U_N^{\mathbb{T}}$ becomes the global action–angle coordinates of space-periodic BO equation when restricted to $U_N^{\mathbb{T}}$. The union $\bigcup_{N\geq 0} U_N^{\mathbb{T}}$ is dense in $L^2_{r,0}(\mathbb{T}) = \{v \in L^2(\mathbb{T}, \mathbb{R}) : \int_{\mathbb{T}} v = 0\}$ and $U_N^{\mathbb{T}}$ is a connected, real analytic symplectic submanifold of $L^2_{r,0}(\mathbb{T})$.

Proposition (S. 2020, Universal covering)

The N-soliton manifold $\mathcal{U}_N = \mathcal{U}_N^{\mathbb{R}}$ is the universal covering of the manifold $U_N^{\mathbb{T}}$. So $U_N^{\mathbb{T}}$ is mapped real bi-analytically onto $\mathcal{U}_N^{\mathbb{R}}/\mathbb{Z}$.

Contents

Other related work

We refer to Saut (2019) for an excellent account of old and recent results about the BO equation.

- The multi-phase solutions are constructed by finite zone integration in Dobrokhotov–Krichever (1991) ; they have also established an inversion formula for multi-phase solutions.
- The traveling waves of the BO equation both on $\mathbb{R}$ and on $\mathbb{T}$ are completely classified by Amick–Toland (1991).
- Some orbital stability and asymptotic stability results about N-solitons : Kenig–Martel ; Gustafson–Takaoka–Tsai (2009), Neves–Lopes, Matsuno (2006), etc.
- The Cauchy initial-value problem for a positive initial data of the BO equation on the line in the zero-dispersion limit is studied in Miller–Xu (2010).

Other related work

- The direct and inverse scattering transform is formulated by Ablowitz–Fokas (1983), Coifman–Wickerhauser (1990), Kaup–Matsuno (1998), Miller–Wetzel (2016), Wu (2016, 2017).
- Continuous family of conservation laws controlling every H^s-norm : See Talbut (2019) in the case $-\frac{1}{2} < s < 0$ and Besov norms ; See Killip–Vișan–Zhang (2018), Koch–Tataru (2018), Koch–Liao (2020) for KdV, mKdV, NLS, GP, etc.)
- $\forall s > -\frac{1}{2}$, the Birkhoff coordinates of the BO equation on $\mathbb{T}$ can be extended to $H^s_{r,0}(\mathbb{T}) := \{v \in H^s(\mathbb{T}, \mathbb{R}) : \int_{\mathbb{T}} v = 0\}$ and their global real analyticity in $H^s_{r,0}(\mathbb{T})$ is proven. Moreover, Tao's gauge transform in the space-periodic regime can be interpreted as a high frequency approximation of the Birkhoff coordinates of BO equation on $\mathbb{T}$, up to scaling. See the series of works by Gérard–Kappeler–Topalov (2020-2021).

Perspective

- Large data inverse scattering transform.
- The soliton resolution conjecture of the BO equation.
- The interaction of the multi-soliton part and the scattering part of a general solution of the BO equation on the line.
- Spectral analysis of the Lax operator L_u in the setting $u \in L^2(\mathbb{R}, \mathbb{R})$.
- The relation between the solutions of the space non-periodic BO equation on $\mathcal{U}_N$ and the solutions of the space-periodic BO equation on $U_N^{\mathbb{T}}$.

Thank you for your attention !