Genericity of wild solutions to the Transport Equation

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Previously...

Theorem (Modena, Székelyhidi '18; Modena, S. '20)

Existence of infinitely many solutions to the incompressible transport equation from same initial data in the class

$$\rho \in C_t^0 L_x^p, \ v \in C_t^0 \left(L_x^{p'} \cap W_x^{1,q} \right) \text{ such that } \ 1/p + 1/q > 1 + 1/d$$

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"Infinitely many" is not very precise, especially for transport:
Existence of one 'wild solution' $\implies$ Existence of infinitely many.

Typicality results in Intermittent Convex Integration

Recall: A is *nowhere dense* iff $\text{int}(\bar{A}) = \emptyset$
A is *meager* iff countable union of nowhere dense sets
A is *residual* iff A^c meager.

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$\Longrightarrow$ "Wild solutions" without regularity are the generic ones, smooth solutions are rare.

Proof by approximation of a locally smooth solution with non-conservative ones **by explicit iteration**.
(Similar approach for dissipative Euler flows in: de Rosa, Tione '19)

Back to the roots: Baire category method

Powerful method for proving "genericity" (and along the way existence)

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Ingredients: Three objects

- A space of good objects (smooth functions) satisfying some inequality/constraint ('subsolutions'): X_0
- A topology on that space and its closure w.r.t. this topology: X
- A functional on X ('energy') which is of Baire-1 class (pointwise limit of continuous maps): $\mathcal{I}$

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1. If $x \in X$ such that $\mathcal{I}(x) = 0$ then x is what we want (solution, has all the desired properties)

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Conclusion: $\{x \in X : x \text{ is what we want}\}$ is residual in X.

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1. If $x \in X$ such that $\mathcal{I}(x) = 0$ then x is what we want (solution, has all the desired properties) – **Directly from functional setup**
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2. If $\mathcal{I}(x) \neq 0$ then $\mathcal{I}$ is discontinuous at x. – **Perturbation statement**

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The setup for incompressible transport

- Energy functional: for an arbitrary smooth positive profile e

$$\mathcal{I}(\rho, v) = \sup_t \left(e(t) - \frac{\|\rho(t)\|_{L^p}^p}{p} - \frac{\|v(t)\|_{L^{p'}}^{p'}}{p'} \right)$$

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- Subsolutions: smooth ρ, v such that there is a smooth R solving the transport-defect equation and for some fixed constant M

$$M\|R(t)\|_{L^1} < e(t) - \frac{\|\rho(t)\|_{L^p}^p}{p} - \frac{\|v(t)\|_{L^{p'}}^{p'}}{p'}$$

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- Topology: for $p \in (1, \infty)$ and some scaling-subcritical $\tilde{p}$

$$L^p_w \times \left(L^{p'}_w \cap W^{1,\tilde{p}} \right) \text{ in space, uniformly in time.}$$

The perturbation proposition for transport

Proposition (cf. Modena, S. '20)

For smooth positive $a(t)$, $\delta > 0$ and any smooth solution (ρ, v, R) of transport-defect there is another smooth solution (ρ_1, v_1, R_1) satisfying

$$a(t) < \frac{1}{p}\|(\rho_1 - \rho)(t)\|_{L^p}^p + \frac{1}{p'}\|(v_1 - v)(t)\|_{L^{p'}}^{p'} < 2a(t) + M\|R(t)\|_{L^1}$$

$$\|(v_1 - v)(t)\|_{W^{1,\tilde{p}}} < \delta$$

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Conclusion:

Theorem (S., Székelyhidi '21+)

The set of solutions to the transport equations with regularity $C_t\left(L^p \times (L^{p'} \cap W^{1,\tilde{p}})\right)$ and energy profile e is residual in X.

The case of 3D Navier-Stokes equations

The same strategy also applies to Navier-Stokes, if $\tilde{p} < \frac{6}{5}$:

Proposition (cf. Burczak, Modena, Székelyhidi '21)

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Theorem (S., Székelyhidi '21+)

Set of solutions to NSE with regularity $C_t\left(L^2 \cap W^{1,\tilde{p}}\right)$ and (kinetic) energy profile e is residual in X.

Comparison: Typicality results for 3D NSE

Theorem (Colombo, de Rosa, Sorella '21)

Set of Leray solutions is nowhere dense in $L^\infty_t L^2_x$ solutions.
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- Solutions without regularity are generic within all $L^\infty L^2$ solutions
- Proof by approximation of a good solution **by explicit iteration**

Theorem (S., Székelyhidi '21+)

Set of solutions with (kinetic) energy profile $t \mapsto e(t) > 0$ is residual in X.

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- Proof **by Baire argument**, no construction of solutions, no iteration

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Thank You for Your attention!