INFINITE-DIMENSIONAL REPRESENTATIONS OF ALGEBRAS

The aim of these lecture notes is to give an example-driven introduction to a class of modules called *pure-injective* modules, as well as the techniques that allow us to study them systematically. In particular, we will focus on modules over a K-algebra A, where K is a field. Every A-module has an in-built K-vector space structure and this approach will give us access to many interesting examples of modules where this underlying K-vector space is *infinite-dimensional*. These are known as infinite-dimensional modules or infinite-dimensional representations.

Studying infinite-dimensional modules in general contains some obvious challenges. It is not easy to make use of the underlying linear algebra – for example, we can prove that any K-vector space has a basis (see Example 1.11) but, since the proof is not constructive, it can be difficult to identify a basis in a given example. It is therefore useful to study K-subspaces of modules, called *finite matrix subgroups* or *pp-definable subgroups*, that are controlled by some finite data. The pure-injective modules are those that behave well with respect to these K-subspaces.

The isomorphism classes of indecomposable pure-injective modules form the underlying set of a topological space, known as the *Ziegler spectrum*. The final part of these notes contains an account of the basic properties of this topological space in the case where A is a finite-dimensional algebra. We will end by making use of these basic properties to characterise finite representation type in terms of the Ziegler spectrum.

We do not assume any prior knowledge of representation theory or category theory. A significant majority of references approaching these topics make use of categorical techniques, often focusing on the connection with functor categories. The idea of these lectures is to demonstrate the usefulness of a more computational viewpoint, with the hope of making this topic accessible to a broad audience.

1. Algebras and Modules

This section is dedicated to examples of K-algebras and modules over them. This will pave the way to Sections 2 and 3, which contain some concrete examples of infinite-dimensional modules that will allow us to illustrate the definitions and results covered in the later sections.

1.1. K-algebras.

Definition 1.1. A **K-algebra** is a K-vector space A with a K-bilinear multiplication $A \times A \to A$, $(a, b) \mapsto ab$ such that there exists an element $1 \in A$ (called the **unit**) such that $1a = a = a1$.

From now on, we will use A to denote an arbitrary K-algebra.

Example 1.2. *The one-dimensional vector space K with multiplication given by the field multiplication and the unit given by the multiplicative identity in the field.*

Example 1.3. *Consider the set*

$$K[X] := \{k_0 + k_1X + k_2X^2 + \cdots + k_nX^n \mid n \geq 0, k_i \in K \text{ for } 0 \leq i \leq n\}$$

of polynomials with one free variable X and with coefficients in K. This is a K-vector space with a countably infinite basis $\{1, X, X^2, X^3, \dots\}$ and we define multiplication in $K[X]$ to be the usual multiplication of polynomials. The element 1 is the unit.

Example 1.4. *Consider the following finite directed graph Q (in this context Q is known as a **quiver**).*

$$\alpha \circlearrowright \bullet^1 \xrightarrow{\beta} \bullet^2 \circlearrowleft \delta$$

Let KQ be the vector space with basis given by the paths in Q (including a path of length zero for each vertex denoted by e_i for each vertex i). That is, the elements of KQ are formal K-linear combinations of elements of the set

$$\boldsymbol{Pa} := \{e_1, e_2, \beta, \alpha^n, \delta^m, \beta\alpha^n, \delta^m\beta, \delta^m\beta\alpha^n \mid n, m \in \mathbb{N}\}.$$

If p, q are paths in $\boldsymbol{Pa}$, then we define their product $p \cdot q$ to be the concatenation of the paths if this is possible and 0 otherwise. Extending this product K-linearly allows us to define a multiplication in KQ. There is a unit element given by $e_1 + e_2$. This algebra is called the ***path algebra*** *of Q.*

Example 1.5. *Consider the algebra KQ described in Example 1.4 and the ideal I generated by the set $\rho := \{\delta\beta\alpha, \alpha^2, \delta^2\}$. This is an example of an admissible ideal of KQ (see* [1, Def. II.2.1]*) and a pair (Q, I) consisting of a quiver and an admissible ideal of KQ is called a* ***bound quiver****.*

The quotient algebra KQ/I is called the ***path algebra of the bound quiver*** *(Q, I). Note that the underlying K-vector space of the quotient algebra KQ/I has the following basis:*

$$\{e_1, e_2, \beta, \alpha, \beta, \beta\alpha, \delta\beta\}.$$

To learn more about general path algebras of bound quivers see [1].

1.2. **Modules over a K-algebra.**

Definition 1.6. Let A be a K-algebra. Then a **(left) A-module** is a K-vector space M with an A-action, that is, a K-bilinear map $A \times M \to M$, $(a, m) \mapsto am$ such that, for any $a, b \in A$ and any $m \in M$, we have that $(ab)m = a(bm)$ and $1m = m$.

Unless otherwise specified, the terminology "A-module", will mean "left A-module".

Example 1.7. *The definition of an K-module coincides with the definition of a K-vector space.*

Example 1.8. *Consider the algebra $K[X]$ defined in Example 1.3. By definition, a $K[X]$-module M is a K-vector space together with a $K[X]$-action. Let $p = k_0 + k_1X + k_2X^2 + \cdots + k_nX^n$ be an arbitrary element of $K[X]$. Then, for any element $m \in M$, we have that*

$$pm = (k_0 + k_1X + k_2X^2 + \cdots + k_nX^n)m = k_0m + k_1(Xm) + k_2(X^2m) + \cdots + k_n(X^nm).$$

It follows that the action of $K[X]$ is determined by the K-vector space structure of M as well as the K-linear endomorphism $\Phi \colon M \to M$ given by $m \mapsto Xm$. Conversely, a K-vector space M together with a K-linear endomorphism Φ uniquely determines a $K[X]$-module. In other words, we can view $K[X]$-modules as representations of the one-loop quiver.

$$M \circlearrowleft \Phi$$

Example 1.9. *Consider the bound quiver (Q, I) given in Example 1.5. Modules over the path algebra KQ/I are determined by* ***representations of the bound quiver*** *(Q, I) (see, for example,* [1, Thm. III.1.6]*). That is, a pair of vector spaces U_1 and U_2, together with K-linear maps U_α, U_β and U_δ arranged in the following configuration*

$$U_\alpha \circlearrowright U_1 \xrightarrow{U_\beta} U_2 \circlearrowleft U_\delta$$

such that $U_\alpha^2 = 0$, $U_\delta^2 = 0$ and $U_\delta U_\beta U_\alpha = 0$.

Given such a representation, we define a KQ/I-module with underlying vector space $U_1 \oplus U_2$. Since every element of KQ/I is a K-linear combination of elements of the set $\{e_1, e_2, \beta, \alpha, \beta, \beta\alpha, \delta\beta\}$, it is enough to specify the action of these basis elements.

- *The action of e_1 is given by the matrix* $\begin{pmatrix} \mathrm{id}_{U_1} & 0 \\ 0 & 0 \end{pmatrix}$
- *The action of e_2 is given by the matrix* $\begin{pmatrix} 0 & 0 \\ 0 & \mathrm{id}_{U_2} \end{pmatrix}$
- *The action of α is given by the matrix* $\begin{pmatrix} U_\alpha & 0 \\ 0 & 0 \end{pmatrix}$
- *The action of β is given by the matrix the matrix* $\begin{pmatrix} 0 & 0 \\ U_\beta & 0 \end{pmatrix}$
- *The action of δ is given by the matrix* $\begin{pmatrix} 0 & 0 \\ 0 & U_\delta \end{pmatrix}$

The action of the remaining two paths in the basis ($\beta\alpha$ and $\delta\beta$) is given by the composition of the relevant matrices

1.3. **Infinite-dimensional A-modules.**

Definition 1.10. Let A be a K-algebra. An A-module M is called **finite-dimensional** if the underlying vector space of M is finite-dimensional. An A-module that is not finite-dimensional is called **infinite-dimensional**.

There is a very broad and well-developed body of research devoted to the study of finite-dimensional modules over finite-dimensional algebras. In these lecture notes, however, we will look at certain classes of infinite-dimensional modules. These modules arise naturally in the representation theory of algebras and in future lectures we will begin to explore some of the rich theory surrounding infinite-dimensional pure-injective modules. Before we enter into this framework, we will consider how infinite-dimensional modules over the algebras given in Section 1.1 look.

Example 1.11. *By definition, any finite-dimensional K-module V of dimension $n \in \mathbb{N}$ has a basis with n elements in it. It follows from this that V is isomorphic to K^n.*

Now let us consider an infinite-dimensional K-module W. It is well-known that, despite not being finite-dimensional, the vector space W has a basis. Let us sketch an argument to prove this claim. Let $\mathcal{L}$ be the set of linearly independent sets contained in W ordered by inclusion. It is clear that, for any chain $\mathcal{L}_1 \subset \mathcal{L}_2 \subset \mathcal{L}_3 \subset \dots$ in $\mathcal{L}$, the union $\bigcup_{i\geq 1} \mathcal{L}_i$ is an upper bound in $\mathcal{L}$. Thus we may apply Zorn's lemma to obtain a maximal linearly independent set $\mathcal{M}$. If $\mathcal{M}$ does not span W, then choose an element $w \in W \setminus \mathrm{Span}(\mathcal{M})$. The set $\mathcal{M} \cup \{w\}$ is linearly independent, contradicting the maximality of $\mathcal{M}$. We have shown that $\mathcal{M}$ spans W and so is a basis. It follows that W is isomorphic to the direct sum $K^{(\mathcal{M})}$ of copies of K indexed by the set $\mathcal{M}$.

From this perspective, the infinite-dimensional K-modules are not much more interesting than the finite-dimensional ones. This kind of behaviour is typical of semi-simple rings (in fact, K is even a simple ring); see [6, Sec. 1.2] for more information about this family of rings.

Example 1.12. *In Example 1.8, we saw that a representation (M, Φ) uniquely determines a $K[X]$-module. This is an infinite-dimensional $K[X]$-module if and only if M is an infinite-dimensional K-vector space.*

Example 1.13. *In Example 1.9, we saw that KQ/I-modules are determined by representations of the quiver with relations. Such a representation corresponds to an infinite-dimensional KQ/I-module if and only if the K-vector space $U_1 \oplus U_2$ is infinite-dimensional.*

1.4. **Homomorphisms between A-modules.**

Definition 1.14. Let M and N be A-modules. Then an **A-homomorphism** is a K-linear map $f\colon M \to N$ such that, for any $a \in A$ and $m \in M$, we have that $f(am) = af(m)$.

Example 1.15. *The definition of a K-homomorphism coincides with the definition of a K-linear map.*

Example 1.16. *Let M and N be $K[X]$-modules and suppose $\Phi\colon M \to M$ and $\Psi\colon N \to N$ are the K-linear endomorphisms determined by the actions of X on M and N respectively (see Example 1.8). Then a K-linear map $f\colon M \to N$ is a $K[X]$-homomorphism if and only if, for any $m \in M$ and $p = k_0 + k_1X + k_2X^2 + \cdots + k_nX^n \in K[X]$, we have that*

$$k_0f(m) + k_1Xf(m) + \cdots + k_nX^nf(m) = pf(m) = f(pm) = k_0f(m) + k_1f(Xm) + \cdots + k_nf(X^nm)$$

if and only if we have $Xf(m) = f(Xm)$ for all $m \in M$, i.e. $\Phi \circ f = f \circ \Psi$.

Example 1.17. *Consider representations*

$$U_\alpha \circlearrowright U_1 \xrightarrow{U_\beta} U_2 \circlearrowleft U_\delta \qquad V_\alpha \circlearrowright V_1 \xrightarrow{V_\beta} V_2 \circlearrowleft V_\delta$$

of (Q, I). We saw in Example 1.9 that these representations determine KQ/I-modules with underlying vector spaces $U_1 \oplus U_2$ and $V_1 \oplus V_2$ respectively. It follows from the definition that a K-linear map $\begin{pmatrix} a & b \\ c & d \end{pmatrix} : U_1 \oplus U_2 \to V_1 \oplus V_2$ is a KQ/I-homomorphism if and only if we have that $b = 0 = c$ and the following diagrams commute:

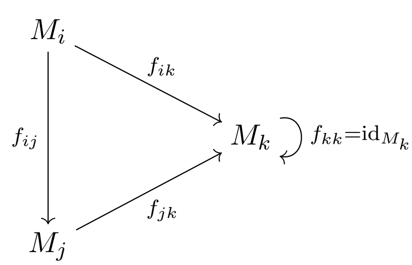
$$\begin{array}{ccc} U_1 & \xrightarrow{a} & V_1 \\ {\scriptstyle U_\alpha}\downarrow & & \downarrow{\scriptstyle V_\alpha} \\ U_1 & \xrightarrow[a]{} & V_1 \end{array} \qquad \begin{array}{ccc} U_1 & \xrightarrow{a} & V_1 \\ {\scriptstyle U_\beta}\downarrow & & \downarrow{\scriptstyle V_\beta} \\ U_2 & \xrightarrow[d]{} & V_2 \end{array} \qquad \begin{array}{ccc} U_2 & \xrightarrow{d} & V_2 \\ {\scriptstyle U_\delta}\downarrow & & \downarrow{\scriptstyle V_\delta} \\ U_2 & \xrightarrow[d]{} & V_2 \end{array}$$

2. Direct Limits

In this section we will introduce direct limits as a means to build examples of infinite-dimensional A-modules. The notion of a direct limit allows you to build a new module out of a given family of modules. The word "direct" refers to the fact that the family of modules must form a direct system, i.e. there are A-homomorphisms between the modules that satisfy the next definition.

Definition 2.1. A **directed set** is a nonempty set I with a reflexive and transitive binary relation $\leq$ such that, for every $i, j \in I$, there exists $k \in I$ such that $i \leq k$ and $j \leq k$.

Definition 2.2. Let I be a directed set. A collection of A-modules $\{M_i \mid i \in I\}$ together with a collection of A-homomorphisms $\{f_{ij}\colon M_i \to M_j \mid i, j \in I, i \leq j\}$ is called a **direct system** of A-modules if $f_{ii} = \mathrm{id}_{M_i}$ for all $i \in I$ and $f_{jk}f_{ij} = f_{ik}$ for all $i \leq j \leq k$.



Remark 2.3. *The direct limit (as we define it in Definition 2.4 below) satisfies a universal property that means it is isomorphic to the colimit of the diagram* $\mathcal{F} = \{f_{ij} : M_i \to M_j \mid i, j \in I, i \leq j\}$ *in the category of* A*-modules. See, for example,* [11, Sec. IV.8] *for more details.*

We will usually refer only to the set $\{f_{ij} : M_i \to M_j \mid i, j \in I, i \leq j\}$ of morphisms as a direct system of A-modules since the existence of the modules $\{M_i \mid i \in I\}$ is implied by this. Define an equivalence relation on the disjoint union $\bigsqcup_{i\in I} M_i$ by declaring that $m_i \sim m_j$ whenever $m_i \in M_i$, $m_j \in M_j$ and there exists $k \geq i, j$ such that $f_{ik}(m_i) = f_{jk}(m_j)$.

Definition 2.4. The **direct limit** $\varinjlim_I M_i$ of a direct system $\mathcal{F} = \{f_{ij} : M_i \to M_j \mid i, j \in I, i \leq j\}$ is the A-module given by the set $\bigsqcup_{i\in I} M_i/\sim$ with the unique A-module operations such that, for every $k \in I$, the canonical map $M_k \to \varinjlim_I M_i$ is an A-homomorphism.

Remark 2.5. *An explicit description of the operations defining the* A*-module structure of* $\varinjlim_I M_i$ *can be found in* [11, Sec. I.5].

2.1. **Examples of direct limits.** Next we will introduce some examples of infinite-dimensional modules that arise as direct limits of finite-dimensional modules.

Remark 2.6. *It is important to observe that a direct limit is not necessarily an infinite-dimensional module. For example, if we take any finite-dimensional module* M*, then we can define a direct system* $\{id_{M_{nm}} : M_n \to M_m \mid M_n \cong M, M_m \cong M \text{ for all } n \leq m\}$ *where the associated directed set is* $I = \mathbb{N}$*. Then the direct limit* $\varinjlim_I M_i$ *is isomorphic to* M *and hence is finite-dimensional.*

Example 2.7. *Consider the* K*-algebra* $K[X]$ *given in Example 1.3 and let* $k \in K$*. For each* $n \in \mathbb{N}$*, consider the vector space* K^n *and the* K*-linear endomorphism given by the Jordan block*

$$J_{k,n} = \begin{pmatrix} k & 1 & 0 & \cdots & 0 & 0 \\ 0 & k & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ldots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & k & 1 \\ 0 & 0 & 0 & \cdots & 0 & k \end{pmatrix}.$$

In Example 1.8, we saw that this defines a $K[X]$*-module which we will denote by* $M_{k,n}$*. The* $(n+1 \times n)$*-matrix*

$$\begin{pmatrix} & I_n & \\ 0 & \cdots & 0 \end{pmatrix}$$

(where I_n *is the* $(n \times n)$*-identity matrix) defines a* $K[X]$*-homomorphism* $f_n : M_{k,n} \to M_{k,n+1}$*. We consider the directed set* $\mathbb{N}$ *and the direct system*

$$\{g_{nm} := f_{m-1} \circ \cdots \circ f_n : M_{k,n} \to M_{k,m} \mid n < m \text{ in } \mathbb{N}\} \cup \{g_{nn} := I_n \mid n \in \mathbb{N}\}.$$

The direct limit $M_{k,\infty} := \varinjlim_{\mathbb{N}} M_{k,n}$ *is called the* k***-Prüfer module*** *over* $K[X]$*.*

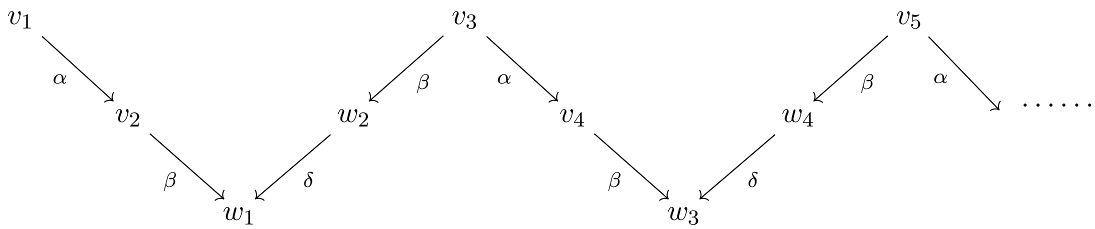
Proposition 2.8. *The* k*-Prüfer module* $M_{k,\infty}$ *is isomorphic to the module with underlying vector space* $K^{(\mathbb{N})}$ *and with the action of* X *given by the* K*-linear endomorphism* $J_{k,\infty} : K^{(\mathbb{N})} \to K^{(\mathbb{N})}$ *defined by* $(k_n)_{n\in\mathbb{N}} \mapsto (kk_n + k_{n+1})_{n\in\mathbb{N}}$*.*

Proof. Let $\Phi : \varinjlim_{\mathbb{N}} M_{k,n} \to \varinjlim_{\mathbb{N}} M_{k,n}$ denote the K-linear endomorphism induced by the action of X on $\varinjlim_{\mathbb{N}} M_{k,n}$. Consider the map

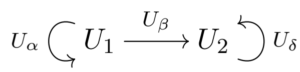
$$h : \varinjlim_{\mathbb{N}} M_{k,n} \to K^{(\mathbb{N})}$$

that takes an equivalence class $[(k_i)_{i=1}^m]$ with $(k_i)_{i=1}^m \in M_{k,m}$ to the element $(k'_i)_{i\in\mathbb{N}}$ in $K^{(\mathbb{N})}$ with $k'_i := k_i$ for $i \leq m$ and $k'_i = 0$ for $i > m$. It is straightforward to check that h is a well-defined $K[X]$-isomorphism. □

Example 2.9. *Consider the periodic sequence $z = (\dots \beta^{-1}\delta^{-1}\beta\alpha\beta^{-1}\delta^{-1}\beta\alpha\beta^{-1}\delta^{-1}\beta\alpha)$ of arrows in (Q, I) and their formal inverses. We may represent this sequence in the following diagram:*

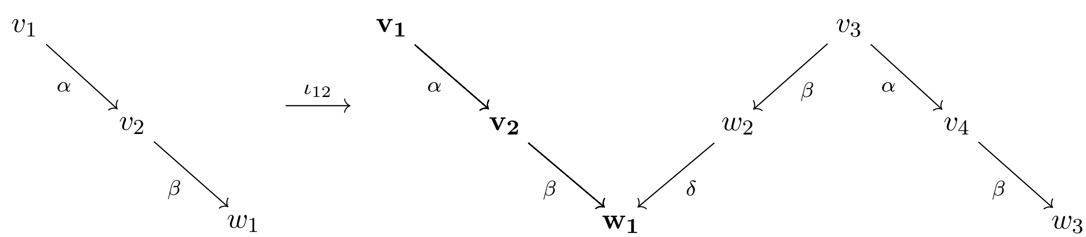


The labels on the starting and ending points of the arrows correspond to basis elements of a representation



*of (Q, I), which is defined as follows. Define U_1 to be the K-vector space with basis $\{v_i \mid i \in \mathbb{N}\}$ and define U_2 to be the K-vector space with basis $\{w_i \mid i \in \mathbb{N}\}$. Intuitively, we think of the labels on the arrows in the above diagram as corresponding to the K-linear maps that make up this representation. More precisely, define $U_\alpha \colon U_1 \to U_1$ to be the K-linear map that takes v_i to v_{i+1} when i is odd and takes v_i to zero when i is even. Define $U_\beta \colon U_1 \to U_2$ to be the K-linear map that take v_1 to zero and v_i to w_{i-1} when $i \geq 2$. Define $U_\delta \colon U_2 \to U_2$ to take w_i to zero when i is odd and takes w_i to w_{i-1} when i is even. The module $M(z)$ determined by this representation is called an **infinite string module** over KQ/I.*

The module described in Example 2.9 arises as the direct limit of a direct system of finite-dimensional modules. For each $n \in \mathbb{N}$, consider the submodule $M(z_n)$ of $M(z)$ spanned by the basis elements $\{v_i \mid 1 \leq i \leq 2n\} \cup \{w_i \mid 1 \leq i \leq 2n-1\}$. The module $M(z_n)$ therefore has an underlying vector space that is isomorphic to $K^{2n} \oplus K^{2n-1}$. We will represent a typical element of $M(z_n)$ by $((k_i)_{i=1}^{2n}, (l_i)_{i=1}^{2n-1})$. Consider the direct system of canonical inclusions denoted by $\{\iota_{nm} \colon M(z_n) \to M(z_m) \mid n \leq m\}$. For example, the inclusion ι_{12} is represented by the following diagram



where the bold text indicates the image of ι_{12}.

Proposition 2.10. *The string module $M(z)$ described in Example 2.9 is isomorphic to $\varinjlim_{\mathbb{N}} M(z_n)$.*

Proof. Consider the map

$$h \colon \varinjlim_{\mathbb{N}} M(z_n) \to M(z)$$

that takes an equivalence class $[((k_i)_{i=1}^{2n}, (l_i)_{i=1}^{2n-1})]$ to the element $((k'_i)_{i\in\mathbb{N}}, (l'_i)_{i\in\mathbb{N}})$ where $k'_i = k_i$ for $1 \leq i \leq 2n$; $k'_i = 0$ for $i > 2n$; $l'_i = l_i$ for $1 \leq i \leq 2n-1$ and $l'_i = 0$ for $i > 2n-1$. It is straightforward to check that h is a well-defined KQ/I-isomorphism. $\square$

3. Duality

In this section we will describe a duality, induced by the usual K-vector space duality, that will allow us to construct new (left) A-modules from right A-modules.

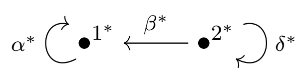
3.1. **Right A-modules and the opposite algebra.** The definition of a right A-module is analogous to Definition 1.6 with A acting on the right instead of the left. Another way of viewing right A-modules is as left A^{op}-modules, where A^{op} is the opposite algebra. This perspective will be useful for us when computing examples in this section.

Definition 3.1. Let A be a K-algebra with multiplication $A \times A \to A$ sending (a, b) to ab. Define the **opposite algebra** A^{op} to be K-algebra with the same underlying vector space as A and K-bilinear multiplication $*\colon A^{op} \times A^{op} \to A^{op}$ sending (a, b) to $a * b := ba$.

Suppose M is a left A^{op}-module. Then, by definition, M is a K-vector space with an A^{op}-action $\cdot\colon A^{op} \times M \to M$ such that, for any $a, b \in A^{op}$ and any $m \in M$, we have that $(a * b) \cdot m = a \cdot (b \cdot m)$ ans $1 \cdot m = m$. We can then define a right A-action $M \times A \to M$ on M to be $ma := a \cdot m$ for all $m \in M$ and $a \in A$. Then we have $m(ab) = m(b * a) = (b * a) \cdot m = b \cdot (a \cdot m) = (a \cdot m)b = (ma)b$ and $m1 = 1 \cdot m = m$. We have shown that any left A^{op}-module determines a right A-module. A similar argument yields the converse statement.

Example 3.2. *Both K and $K[X]$ are commutative algebras and so they coincide with their opposite algebra. In particular, every right module over K or $K[X]$ is also a left module and vice versa.*

Example 3.3. *Consider the algebra KQ/I from Example 1.5. The opposite algebra $(KQ/I)^{op}$ is given by the path algebra of the opposite quiver Q^**



with relation $\rho^* := \{\alpha^*\beta^*\delta^*, (\alpha^*)^2, (\delta^*)^2\}$. That is, take the K-vector space KQ^* with basis

$$\mathbf{Pa}^* := \{e_1^*, e_2^*, \alpha^*, \beta^*, \delta^*, \alpha^*\beta^*, \beta^*\delta^*, \alpha^*\beta^*\delta^*\}$$

together with the multiplication induced by concatenation of paths. Then $(KQ/I)^{op} = KQ^*/I^*$ where I^* is the ideal of KQ^* generated by ρ^*. By an analogous argument to the one given in Example 1.9, the left KQ^*/I^*-modules (i.e. right KQ/I-modules) are given by representations of (Q^*, I^*).

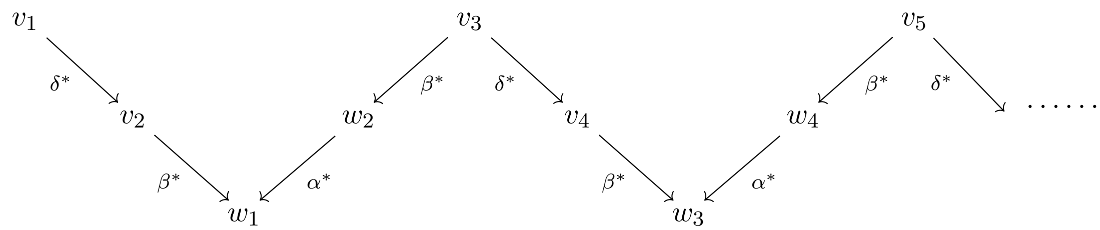
3.2. **Dual modules.** We know that any right A-module has an underlying K-vector space structure and so we may consider the dual K-vector space. The following definition yields a canonical way of equipping the dual K-vector space with a left A-module structure.

Definition 3.4. Let A be a K-algebra and let M be a right A-module (equivalently a left A^{op}-module). Then the K-**dual** M^* of M is defined to be the left A-module consisting of the usual K-dual vector space M^* and the A-action $A \times M^* \to M^*$ given by $(a, f) \mapsto af$ where $(af)(m) = f(ma)$ for each $m \in M$.

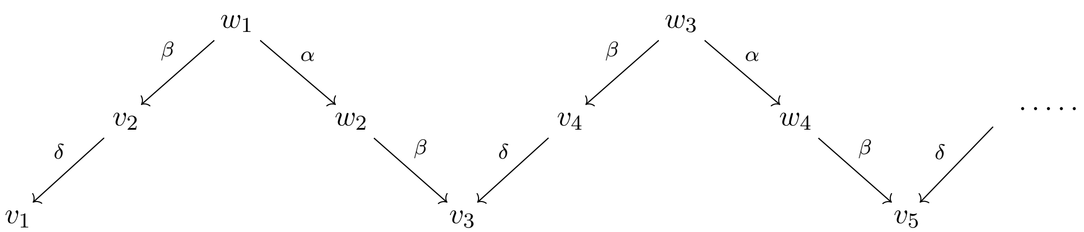
Example 3.5. *Consider a K-vector space V with basis $\mathcal{B}$. We have already observed that $V \cong K^{(\mathcal{B})}$. Then the dual K-module coincides with the dual K-vector space, which is given by the direct product $K^{\mathcal{B}}$ of copies of K indexed by $\mathcal{B}$.*

Example 3.6. *Let $k \in K$ and consider the k-Prüfer module $M_{k,\infty}$ as a right $K[X]$-module. Then the dual module $M_{k,\infty}^*$ is called the k-**adic module** over $K[X]$ and will be denoted $M_{k,-\infty}$. Moreover, $M_{k,-\infty}$ is isomorphic to the module with underlying vector space given by $K^{\mathbb{N}}$ and with the action of X given by the endomorphism $J_{k,\infty}\colon K^{\mathbb{N}} \to K^{\mathbb{N}}$ defined by $(k_n)_{n\in\mathbb{N}} \mapsto (kk_n + k_{n-1})_{n\in\mathbb{N}}$ where k_0 is defined to be zero.*

Example 3.7. *Consider the infinite string module over KQ^*/I^* given by periodic sequence* $y^* = (\dots(\beta^*)^{-1}(\alpha^*)^{-1}\beta^*\delta^*(\beta^*)^{-1}(\alpha^*)^{-1}\beta^*\delta^*(\beta^*)^{-1}(\alpha^*)^{-1}\beta^*\delta^*)$, *represented by the following diagram:*



Then the left KQ^*/I^*-module $M(y^*)$ can be considered as a right KQ/I-module. The ***dual infinite string module*** *$M(y^*)^*$ can be described explicitly as follows. Take the periodic sequence of dual arrows $y = (\dots\beta\alpha\beta^{-1}\delta^{-1}\beta\alpha\beta^{-1}\delta^{-1}\beta\alpha\beta^{-1}\delta^{-1})$ represented by the following diagram:*



Define a representation

$$V_\alpha \circlearrowright V_1 \xrightarrow{V_\beta} V_2 \circlearrowleft V_\delta$$

of (Q, I) as follows. Define V_1 to be the K-vector space $K^{\mathbb{N}}$ where we label $w_i := (0, \dots, 0, 1, 0, \dots)$ with 1 in the ith position. Define V_2 to be the K-vector space $K^{\mathbb{N}}$ and, similarly, we label $v_i := (0, \dots, 0, 1, 0, \dots)$ with 1 in the ith position.. The labels on the arrows correspond to the K-linear maps that make up this representation. Define $V_\alpha \colon V_1 \to V_1$ to be the K-linear map that takes w_i to w_{i+1} when i is odd and takes w_i to zero when i is even. Define $V_\beta \colon V_1 \to V_2$ to be the K-linear map that takes w_i to v_{i+1} for all $i \in \mathbb{N}$. Define $V_\delta \colon V_2 \to V_2$ to take v_i to zero when i is odd and takes v_i to v_{i-1} when i is even.

4. Finite Matrix Subgroups and Pp-definable Subgroups

4.1. **Finite matrix subgroups.** In our setting, a matrix subgroup of an A-module M is a K-subspace of the underlying K-vector space of M that can be realised as the trace of an element $x \in L$ in M for some A-module L. In this section we introduce the notion of a finite matrix subgroup, which is a matrix subgroup where the module L is finitely presented.

Definition 4.1. An A-module L is called **finitely presented** if there exist $n, m \in \mathbb{N}$ such that $L \cong A^n/\mathrm{im}(\Phi)$ where $\Phi \colon A^m \to A^n$ and $\mathrm{im}(\Phi)$ denotes the image of Φ. Note that Φ can be represented by a $(m \times n)$-matrix $P = (a_{ji})$ with $a_{ji} \in A$ for $1 \leq i \leq n$ and $1 \leq j \leq m$ such that $\Phi \colon \begin{pmatrix} a_1 & \dots & a_m \end{pmatrix} \mapsto \begin{pmatrix} a_1 & \dots & a_m \end{pmatrix} P$.

Example 4.2. *If A is a finite-dimensional algebra, then the finitely presented modules coincide with the finite-dimensional modules. Since a finitely presented module is a quotient of the finite-dimensional module A^n, it must be a finite-dimensional module itself. Conversely, if M is an n-dimensional module, then the K-basis is also an A-generating set, so we may define an epimorphism $\Psi \colon A^n \to M$. Then the kernel* $\ker(\Psi)$ *of Ψ is finite-dimensional and so, by the same argument, there* *exists an epimorphism $\Omega\colon A^m \to \ker(\Psi)$. Then the composition $\Phi := \iota \circ \Omega$, where $\iota\colon \ker(\Psi) \to A^n$ is the canonical embedding, is the desired presentation of M.*

Definition 4.3. Let M be an A-module and let L be a finitely presented A-module. For a fixed $l \in L$, consider the K-subspace $H_{(L,l)}(M) := \{f(l) \mid f\colon L \to M \text{ an } A\text{-homomorphism}\}$. A K-subspace of this form is called a **finite matrix subgroup** of M.

Remark 4.4. *Finite matrix subgroups are usually defined in the context of modules over a ring (that is not necessarily a K-algebra). In that more general setting the set $H_{(L,l)}(M)$ is a subgroup of the underlying abelian group structure of M. This is why $H_{(L,l)}(M)$ is called a finite matrix* subgroup *rather than a finite matrix* subspace*.*

Example 4.5. *Consider the KQ/I-module $M(z)$ from Example 2.9 and the finite-dimensional submodule $M(z_2)$ with the underlying vector space spanned by $\{v_1, v_2, v_3, v_4, w_1, w_2, w_3\}$. Then the finite matrix subgroup $H_{M(z_2),w_2}(M(z))$ is the K-subspace spanned by*

$$\{w_2\} \cup \{w_{2n-1} \mid n \in \mathbb{N}\}.$$

This is witnessed by the fact that there are the following KQ/I-homomorphisms from $M(z_2)$ to $M(z)$:

- *The embedding $f_0\colon M(z_2) \to M(z)$ given by $v_i \mapsto v_i$ and $w_j \mapsto w_j$ for $i \in \{1,2,3,4\}$ and $j \in \{1,2,3\}$. Then $f_0(w_2) = w_2$.*
- *For each $n \in \mathbb{N}$, we have a KQ/I-homomorphism $f_n\colon M(z_2) \to M(z)$ given by $v_3 \mapsto v_{2n}$, $w_2 \mapsto w_{2n-1}$, $v_i \mapsto 0$ and $w_j \mapsto 0$ for $i \in \{1,2,4\}$ and $j \in \{1,3\}$. Then $f_n(w_2) = w_{2n-1}$.*

It is an interesting exercise to prove that these KQ/I-homomorphisms $\{f_n \mid n \geq 0\}$ form a basis for the K-vector space of KQ/I-homomorphisms from $M(z_2)$ to $M(z)$. Alternatively, we may apply the more general theorem proved in [2, Sec. 1.4].

Remark 4.6. *In general, a finite matrix subgroup $H_{L,l}(M)$ is not an A-submodule of M. Indeed, if we take the KQ/I-module $Ae_1 = \{a \in A \mid \exists b \in A \text{ such that } a = be_1\}$. Then $H_{Ae_1,e_1}(M(z))$ coincides with the K-subspace spanned by the set $\{v_n \mid n \text{ odd}\}$. This is not an KQ/I-submodule of $M(z)$ since, for example, $w_2 = \beta v_3$ is not contained in $H_{Ae_1,e_1}(M(z))$.*

4.2. **Pp-definable subgroups.** The notion of a pp-definable subgroup comes from the area of logic called model theory. They are the sets of elements of a module that realise a given positive primitive formula in the language of A-modules. We will not put too much emphasis on the model theoretic perspective in these lectures, however, if you are interested in this subject you can read more in [7].

Definition 4.7. Let M be an A-module and let $\sum_{i=1}^{n} a_{ji}x_i = 0$ where $1 \leq j \leq m$ be a finite A-linear system. That is, the symbols x_i denote free variables and $a_{ji} \in A$ for each $1 \leq i \leq n$ and $1 \leq j \leq m$. Note that the system depends on a $(m \times n)$-matrix $P = (a_{ji})$ with $a_{ji} \in A$ for $1 \leq i \leq n$ and $1 \leq j \leq m$. Consider the K-subspace

$$\phi_P(M) := \{u_1 \in M \mid \text{ there exist } u_2, \ldots, u_n \in M \text{ such that } \sum_{i=1}^{n} a_{ji}u_i = 0 \text{ for all } 1 \leq j \leq m\}.$$

A K-subspace of this form is called a **pp-definable subgroup** of M.

Remark 4.8. *The symbol ϕ_P refers the first order formula*

$$\exists x_2 \ldots \exists x_n \left(\sum_{i=1}^{n} a_{1i}x_i = 0 \ \wedge \cdots \wedge \ \sum_{i=1}^{n} a_{mi}x_i = 0 \right)$$

that should be read as "there exist x_2 up to x_n such that $\sum_{i=1}^{n} a_{1i}x_i = 0$ and $\sum_{i=1}^{n} a_{2i}x_i = 0 \ldots$" and so on up to m. The notation $\phi_P(M)$ is then used for the solution set of this formula in M.

That is, the set of elements u_1 in M such that, when we replace x_1 with u_1, the statement above in quotation marks is true. Note that this coincides with what is written in Definition 4.7.

In parallel to Remark 4.4, we observe that, if we made this definition for a module M over a general ring, then $\phi_P(M)$ would form a subgroup of the underlying abelian group structure of M. In our setting, this is even a K-subspace.

Example 4.9. *Consider the following system of KQ/I-linear equations:*

$$e_1 x_1 = 0$$

$$\delta x_1 - x_2 = 0$$

$$\beta x_3 - x_2 = 0$$

$$\alpha x_4 - x_3 = 0$$

$$\beta x_4 = 0.$$

We have the corresponding matrix

$$P = \begin{pmatrix} e_1 & 0 & 0 & 0 \\ \delta & -1 & 0 & 0 \\ 0 & -1 & \beta & 0 \\ 0 & 0 & -1 & \alpha \\ 0 & 0 & 0 & \beta \end{pmatrix}.$$

Then the pp-definable subgroup $\Phi_P(M(z))$ of $M(z)$ is the K-subspace spanned by the set

$$\{w_2\} \cup \{w_{2n-1} \mid n \in \mathbb{N}\}.$$

The pp-definable subgroup $\Phi_P(M(y^*)^*)$ of $M(y^*)^*$ is given by the set of (possibly infinite) K-linear combinations of the set $\{v_{2n-1} \mid n \in \mathbb{N}\}$.

If we look at Example 4.5 and Example 4.9, then we find that $H_{M(z_2),w_2}(M(z)) = \Phi_P(M(z))$. In the next proposition we will show that the set of finite matrix subgroups of an A-module M coincides with the set of pp-definable subgroups of M.

Proposition 4.10. *Let U be a K-subspace of an A-module M. The following statements are equivalent.*

1. *There exists a finitely presented A-module L and $l \in L$ such that $U = H_{L,l}(M)$.*
2. *There exists a finite A-linear system determined by a matrix P such that $U = \phi_P(M)$.*

Proof. Consider an $(m \times n)$-matrix $P = (a_{ji})$ with $a_{ji} \in A$ for $1 \leq i \leq n$ and $1 \leq j \leq m$. We have already observed that P determines both an A-linear system $\sum_{i=1}^{n} a_{ji} x_i = 0$ where $1 \leq j \leq m$ and a finitely presented module $L := A^n/\mathrm{im}(\Phi)$ where $\Phi\colon A^m \to A^n$ is the A-homomorphism $\begin{pmatrix} a_1 & \dots & a_m \end{pmatrix} \mapsto \begin{pmatrix} a_1 & \dots & a_m \end{pmatrix} P$. We fix the following notation. For each $1 \leq j \leq m$, let $d_j := (0 \dots 010 \dots 0)$ be the element of A^m with 1 in the jth position and zeroes elsewhere. For each $1 \leq i \leq n$, define $e_i := (0 \dots 010 \dots 0)^T$ to be the element of A^n with 1 in the ith position and zeroes elsewhere and let $l_i := \pi(e_i)$ where $\pi\colon A^n \to L$ is the canonical quotient morphism. Observe that $\Phi(d_j) = (a_{j1} \dots a_{jn})^T = \sum_{i=1}^{n} a_{ji} e_i$.

We will show that $H_{L,l_1}(M) = \phi_P(M)$ for all A-modules M. First we show that $H_{L,l_1}(M) \subseteq \phi_P(M)$ so let $u_1 = f(l_1)$ for some A-homomorphism $f\colon L \to M$. Set $u_i := f(l_i)$ for $1 < i \leq n$. Then, for each $1 \leq j \leq m$, we have that

$$\sum_{i=1}^{n} a_{ji} f(l_i) = f(\sum_{i=1}^{n} a_{ji} l_i) = f\pi(\sum_{i=1}^{n} a_{ji} e_i) = f\pi\Phi(d_j) = 0$$

since $\pi\Phi = 0$. We therefore have that $u_1 \in \phi_P(M)$. Next we show the other inclusion $\phi_P(M) \subseteq H_{L,l_1}(M)$. Let $u_1 \in \phi_P(M)$ and consider elements $u_2, \ldots, u_n \in M$ that satisfy the A-linear equations. The assignment $e_i \mapsto u_i$ extends uniquely to an A-homomorphism $f' \colon A^n \to M$. As

$$f'\Phi(d_j) = f'(\sum_{i=1}^{n} a_{ji}e_i) = \sum_{i=1}^{n} a_{ji}u_i = 0,$$

there exists a unique A-homomorphism $f \colon L \to M$ such that $f\pi = f'$. In particular, we have that $u_1 = f'(e_1) = f(l_1)$ so $u_1 \in H_{L,l_1}(M)$ as desired. □

5. Pure Submodules and Pure-injective Modules

The aim of this next section is to define and give examples of the modules that give the points of the Ziegler spectrum. The first definition is that of a pure submodule. These are the submodules that respect the pp-definable (equivalently the finite matrix) subgroups.

Definition 5.1. Let L and M be A-modules such that $L \subseteq M$ is an A-submodule. Then L is a **pure submodule** of M if $\phi_P(L) = \phi_P(M) \cap L$ for all $(m \times n)$-matrices P with entries in A. A monomorphism $f \colon L \to M$ such that $\mathrm{im}(f) \subseteq M$ is a pure submodule is called a **pure monomorphism**.

Example 5.2. *Consider Example 3.7 and consider the submodule $U_1 \oplus U_2$ of $M(y^*)^*$ spanned (as a K-vector space) by the elements $\{v_n \mid n \in \mathbb{N}\} \cup \{w_m \mid m \in \mathbb{N}\}$ and $U_1 \oplus U_2 \subseteq V_1 \oplus V_2$ be the canonical embedding. This is a pure submodule of $M(y^*)^*$.*

A useful characterisation of a pure monomorphism makes use of the duality defined Definition 3.4. Notice that, for any A-homomorphism $g \colon M \to N$, the usual K-linear map $g^* \colon N^* \to M^*$ induced by K-vector space duality is an A^{op}-homomorphism. It is well-known that a monomorphism $g \colon M \to N$ is pure if and only if there exists an A^{op}-homomorphism $h \colon M^* \to N^*$ such that $g^* \circ h = \mathrm{id}_{M^*}$. To see a proof of this, as well as other characterisations of pure monomorphisms, see [3, Lem. 2.19].

Example 5.3. *For any A-module M, the morphism $\delta_M \colon M \to M^{**}$ given by $m \mapsto \mathrm{ev}_m$ where $ev_m(f) = f(m)$ for all $f \in M^*$ is a pure monomorphism. This follows from the discussion preceding this example since $\delta_M^* \circ \delta_{M^*} = \mathrm{id}_{M^*}$.*

Definition 5.4. A non-zero A-module N is called **pure-injective** if, for every pure monomorphism $f \colon N \to M$, there exists an A-homomorphism $g \colon M \to N$ such that $gf = \mathrm{id}_N$.

Example 5.5. *It follows from Example 5.3 that every pure-injective module is a direct summand of a dual module. It turns out that this, in fact, characterises pure-injective modules (see* [3, Thm. 2.27]*). In particular, any dual module is pure-injective.*

For any finite-dimensional A-module M, we have that $M \cong M^{**}$. It therefore follows that finite-dimensional A-modules are pure-injective.

Example 5.6. *The modules defined in Examples 2.7, 2.9, 3.6 and 3.7 are pure-injective modules. The fact that the infinite string module (Example 2.9) and the dual infinite string module (Example 3.7) are pure-injective is proved in* [10]. *The k-Prüfer module (Example 2.7) is an injective $K[X]$-module and so clearly it is also pure-injective. The k-adic module (Example 3.6) is a dual module and so it is pure-injective by Example 5.5.*

6. The Ziegler Spectrum

In this section we introduce a topological space called the Ziegler spectrum. The points of the space are the indecomposable pure-injective modules.

Definition 6.1. A non-zero A-module M is called **indecomposable** if, whenever $M \cong N \oplus L$, either $L = 0$ or $N = 0$.

Remark 6.2. *The collection of isomorphism classes of indecomposable pure-injective A-modules has cardinality at most $2^{\kappa+\aleph_0}$ where κ is the cardinality of A. In particular, the isomorphism classes of indecomposable pure-injective modules form a set, which we denote by Zg_A.*

The elements of Zg_A are isomorphism classes $[N]$ but we will drop the square brackets and refer instead to the representative N as a **point** of Zg_A. If N is a finite-dimensional module we refer to it as a **finite-dimensional point**. Similarly, if N is infinite-dimensional then we refer to N as a **infinite-dimensional point** of Zg_A.

The Ziegler topology on Zg_A can be defined in many different ways (see, for example, [8, Ch. 5.1]). In these lecture notes we will define the topology in terms of pp pairs.

Definition 6.3. Let (P, Q) be a pair of matrices with entries in A (possibly of different sizes). We will call (P, Q) a **pp pair** if $\phi_P(M) \subseteq \phi_Q(M)$ for all A-modules M.

According to Proposition 4.10, a pair (P, Q) of matrices with entries in A determines a pair of pointed finitely presented modules (L, l) and (N, n) such that $\phi_P(M) = H_{L,l}(M)$ and $\phi_Q(M) = H_{N,n}(M)$ for all A-modules M. Clearly this means that (P, Q) is a pp pair if and only if $H_{L,l}(M) \subseteq H_{N,n}(M)$ for all A-modules M.

Definition 6.4. Let Zg_A be the set of isomorphism classes of indecomposable pure-injective A-modules. We call a set $\mathcal{U} \subseteq \mathrm{Zg}_A$ **basic open** if there exists a pp pair (P, Q) such that

$$\mathcal{U} = \{M \in \mathrm{Zg}_A \mid \phi_P(M) \subsetneq \phi_Q(M)\}.$$

Denote the basic open set corresponding to a pp pair (P, Q) by (ϕ_P/ϕ_Q).

Recall that a topological space Z is called **quasi-compact** if, whenever $Z = \bigcup_{i\in I} U_i$ for U_i open sets, we have that there is a finite subset $F \subseteq I$ such that $Z = \bigcup_{i\in F} U_i$. In other words, any open cover of Z has a finite subcover

Theorem 6.5 (Ziegler, 1984)**.** *The basic open sets form a base of a topology on Zg_A and, moreover, the basic open sets are quasi-compact. This topological space is called the* ***Ziegler spectrum*** *of A.*

The proof of the above theorem is originally due to Ziegler and is contained in his landmark paper [12] on the model theory of modules. A more algebraic proof was given later by Herzog using functor categories [4]. See also Krause [5]. Unfortunately, both the model theoretic and more algebraic arguments require material that is beyond the scope of these lectures and so we do not prove the theorem here.

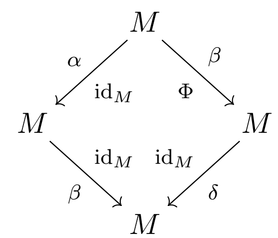
Corollary 6.6. *The Ziegler spectrum of A is a quasi-compact topological space.*

Proof. By the theorem, it suffices to show that there is a pp pair (P, Q) such that $\mathrm{Zg}_A = (\phi_P/\phi_Q)$. If we take P to be the (1×1)-matrix 0 and Q to be the (1×1)-matrix 1, then $\phi_P(M) = M$ and $\phi_Q(M) = 0$ for any module M. Thus $\mathrm{Zg}_A = (\phi_P/\phi_Q)$ is quasi-compact. □

In Example 6.8 we will describe the points of the Ziegler spectrum of the algebra KQ/I introduced in Example 1.5. In order to do this we describe a way of building a representation of the bound quiver (Q, I) from $K[X]$-modules. Recall from Example 1.8 that each $K[X]$-module is determined by a K-vector space M and a K-linear endomorphism $\Phi\colon M \to M$. Given such a pair (M, Φ), we may define a representation of (Q, I) as follows:

$$W_\alpha \circlearrowleft W_1 \xrightarrow{W_\beta} W_2 \circlearrowright W_\delta$$

where both W_1 and W_2 are isomorphic to $M \oplus M$ and the K-linear maps are given by $W_\alpha := \begin{pmatrix} 0 & 0 \\ \mathrm{id}_M & 0 \end{pmatrix}$, $W_\beta := \begin{pmatrix} \Phi & 0 \\ 0 & \mathrm{id}_M \end{pmatrix}$ and $W_\delta := \begin{pmatrix} 0 & 0 \\ \mathrm{id}_M & 0 \end{pmatrix}$. A KQ/I-module of this kind is known as a **band module** because it can be visualised as follows:

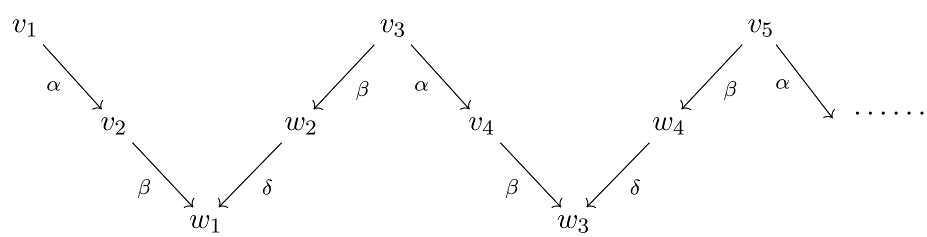


For every $K[X]$-module M, we will denote the corresponding band module over KQ/I by $\mathbf{Ba}(M)$.

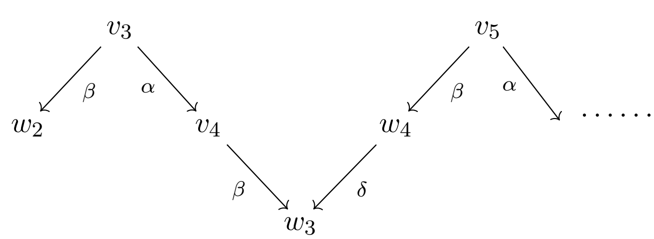
Remark 6.7. *The assignment $M \mapsto \mathbf{Ba}(M)$ extends to a functor from the category of $K[X]$-modules to the category of KQ/I-modules.*

Example 6.8. *Let K be an algebraically closed field. The following is a complete list of the points of the Ziegler spectrum of KQ/I. This classification can be found in* [9]*:*

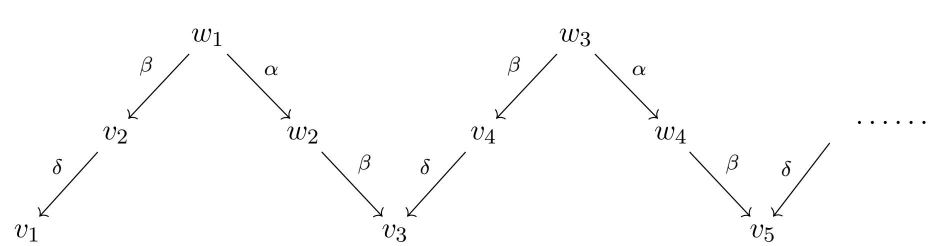
- *The finite-dimensional KQ/I-modules; see Example 5.5.*
- *The infinite string module $M(z)$ described in Example 2.9 corresponding to the sequence*



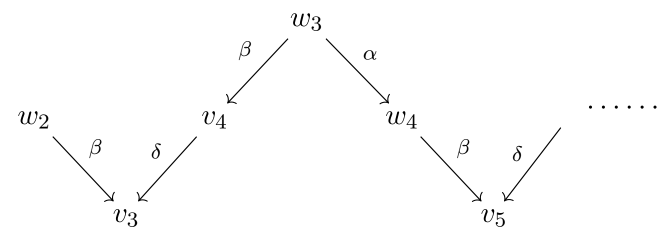
- *The submodule of $M(z)$ spanned by $\{w_i \mid i \geq 2\} \cup \{v_j \mid j > 2\}$. This module is the infinite string module $M(w)$ associated to the sequence*



- *The dual infinite string module $M(y^*)^*$ described in Example 3.7 corresponding to the dual sequence $y^* = (\ldots(\beta^*)^{-1}(\alpha^*)^{-1}\beta^*\delta^*(\beta^*)^{-1}(\alpha^*)^{-1}\beta^*\delta^*)$. The module $M(y^*)^*$ can be visualised as*



- *The submodule of $M(y^*)^*$ consisting of elements of the form $\sum_{n\geq 2} k_{n+1}v_{n+1} + l_n w_n$ where $k_n, l_n \in K$. This module is the dual string module $M(x^*)^*$ where $x^* = (\dots(\beta^*)^{-1}(\alpha^*)^{-1}\beta^*\delta^*(\beta^*)^{-1})$ that can be visualised as*



- *The following band modules:*
  - *For each $0 \neq k \in K$, the module $\mathbf{Ba}(M_{k,\infty})$ where $M_{k,\infty}$ is the k-Prüfer module described in Example 2.7.*
  - *For each $0 \neq k \in K$, the module $\mathbf{Ba}(M_{k,-\infty})$ where $M_{k,-\infty}$ is the k-adic module described in Example 3.6.*
  - *The module $\mathbf{Ba}(K(X))$ where $K(X)$ denotes the field of rational functions.*

7. Representation-finite Finite-dimensional Algebra and the Ziegler Spectrum

The finite-dimensional A-modules satisfy the following well-known decomposition theorem known as the Krull-Remak-Schmidt Theorem. See, for example, [1, Thm. 4.19].

Theorem 7.1. *Let M be a finite-dimensional A-module. Then $M \cong \bigoplus_{i=1}^{n} M_i$ where M_i is an indecomposable module for each $1 \leq i \leq n$. Moreover, this decomposition is unique up to isomorphism and reordering of the direct summands.*

This starting point suggests that, if we wish to know about the finite-dimensional A-modules, then we should attempt to understand the indecomposable ones. By Example 5.5, the indecomposable finite-dimensional modules form a subset $\mathcal{U}_0 \subseteq \mathrm{Zg}_A$ of the Ziegler spectrum of A.

In this section we will consider the case where A is a finite-dimensional K-algebra and consider the topological properties of the subset $\mathcal{U}_0$. We will make use of the following three important results about finite-dimensional modules over finite-dimensional algebras.

Proposition 7.2 ([8, Cor. 5.3.36, Cor. 5.3.37, Thm. 5.1.12])**.** *Let A be a finite-dimensional algebra.*

(1) *The set $\mathcal{U}_0$ of finite-dimensional points in Zg_A is **dense** in Zg_A. In other words, the closure $\overline{\mathcal{U}_0}$ of $\mathcal{U}_0$ is equal to Zg_A.*
(2) *The finite-dimensional points in Zg_A are **isolated**. In other words, the set $\{M\}$ is an open set for all $M \in \mathcal{U}_0$.*
(3) *The finite-dimensional points in Zg_A are **closed**. In other words, the set $\{M\}$ is a closed set for all $M \in \mathcal{U}_0$.*

Using these three facts, together with what we have learned in the previous sections, we can now prove the final theorem of the course, that characterises when the set $\mathcal{U}_0$ only has finitely many elements. A finite-dimensional algebra A such that the set $\mathcal{U}_0 \subseteq \mathrm{Zg}_A$ is finite is said to have **finite-representation type**.

Theorem 7.3. *Let A be a finite-dimensional algebra. The following statements are equivalent.*

(1) *The algebra A has finite-representation type.*

(2) *The Ziegler spectrum* Zg_A *only has finitely many points.*
(3) *The Ziegler spectrum* Zg_A *does not contain any infinite-dimensional points.*

Proof. First we show that **(1)** and **(2)** are equivalent: The implication **(2)** implies **(1)** is immediate because $\mathcal{U}_0 \subseteq \mathrm{Zg}_A$. Suppose that **(1)** holds. Then $\mathcal{U}_0 = \bigcup_{M \in \mathcal{U}_0} \{M\}$ is a finite union of closed sets by Proposition 7.2**(3)** and therefore $\mathcal{U}_0$ is a closed set. In particular, we have that $\mathcal{U}_0 = \overline{\mathcal{U}_0} = \mathrm{Zg}_A$ by Proposition 7.2**(1)**. We have shown that Zg_A is a finite set and so **(2)** holds.

Next we show that **(1)** and **(3)** are equivalent: Note that it was shown in the above paragraph that, if **(1)** holds, then $\mathcal{U}_0 = \mathrm{Zg}_A$, i.e. **(3)** holds. To show the converse, suppose that **(3)** holds. Then $\mathcal{U}_0 = \mathrm{Zg}_A$ and so $\mathcal{U}_0$ is a quasi-compact topological space by Corollary 6.6. We have that $\mathcal{U}_0 = \bigcup_{M \in \mathcal{U}_0} \{M\}$ is an open cover by Proposition 7.2**(2)** and clearly this open cover does not have a proper subcover. Since $\mathcal{U}_0$ is quasi-compact, it follows that $\mathcal{U}_0$ is a finite set and so **(1)** holds. □

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