Identifying Painlevé equations related to orthogonal polynomials: the geometric approach

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OPSFOTA seminar
24 May 2021

Based on joint work with Anton Dzhamay, Galina Filipuk and Adam Ligęza.
Supported by a LMS Early Career Fellowship.

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   - Determinantal expressions via identification with hierarchies of classical solutions

Painlevé equations related to orthogonal polynomials

The Painlevé differential equations

$$
\begin{aligned}
\mathrm{P_I}: \ & y'' = 6y^2 + t \\
\mathrm{P_{II}}: \ & y'' = 2y^3 + ty + \alpha \\
\mathrm{P_{III}}: \ & y'' = \frac{(y')^2}{y} - \frac{y'}{t} + \alpha\frac{y^2}{t} + \frac{\beta}{t} + \gamma y^3 + \frac{\delta}{y} \\
\mathrm{P_{IV}}: \ & y'' = \frac{1}{2y}\left(y'\right)^2 + \frac{3}{2}y^3 + 4ty^2 + 2(t^2-\alpha)y + \frac{\beta}{y} \\
\mathrm{P_{V}}: \ & y'' = \left(\frac{1}{2y} + \frac{1}{y-1}\right)(y')^2 - \frac{y'}{t} + \frac{(y-1)^2}{t^2}\left(\alpha y + \frac{\beta}{y}\right) + \gamma\frac{y}{t} + \delta\frac{y(y+1)}{y-1} \\
\mathrm{P_{VI}}: \ & y'' = \frac{1}{2}\left(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-t}\right)(y')^2 - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{y-t}\right)y' \\
& \qquad + \frac{y(y-1)(y-t)}{t^2(t-1)^2}\left(\alpha + \beta\frac{t}{y^2} + \gamma\frac{t-1}{(y-1)^2} + \delta\frac{t(t-1)}{(y-t)^2}\right)
\end{aligned}
$$

In each case $y = y(t)$ and $\alpha, \beta, \gamma, \delta$ complex parameters.

Three-term recurrence and recurrence coefficients

Orthonormal polynomials $p_n(x)$ satisfy a *three-term recurrence*:

$$xp_n(x) = a_{n+1}p_{n+1}(x) + b_n p_n(x) + a_n p_{n-1}(x).$$

The coefficients a_n and b_n are usually refered to as the *recurrence coefficients*.

Differential and discrete systems for recurrence coefficients

- Usually it is first shown that the recurrence coefficients $\{a_n, b_n\}$, as functions of the discrete variable n, satisfy, after some change of variables, a system of non-linear difference equations
- As functions of some continuous parameter appearing in the weight, satisfy a Toda-type differential-difference system.
- Combining these, one can obtain a scalar second order (first or higher degree) nonlinear differential equation, which can be very cumbersome.

These systems of differential and difference equations often turn out to be reducible to differential or discrete Painlevé equations (see [Van Assche, 2018] and references therein)

The identification problem

The questions:

Given a second-order nonlinear nonautonomous discrete or differential equation suspected to be transformable to a Painlevé equation, the following questions arise:

- If so, which Painlevé equation?
- What is the change of variables transforming the original system to the standard form?
- How are the parameters from the original system related to those from the Painlevé equation?

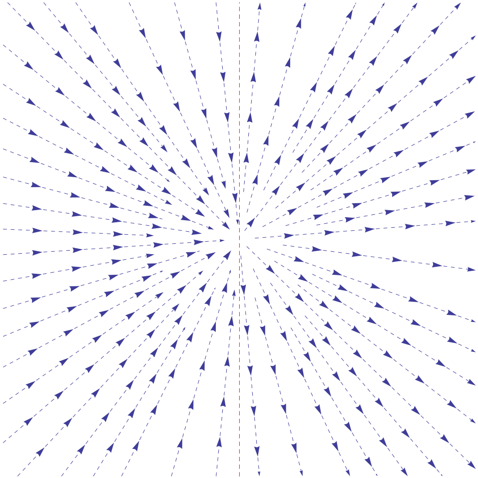
The geometric approach:

Assuming the system is transformable to a Painlevé equation:
If we can construct a space of initial conditions for the system, then we can determine the type of Painlevé equation it is transformable to, and obtain a change of variables from an appropriate identification of this space of initial conditions with that of the standard form.

Geometric theory of differential Painlevé equations: Okamoto's space

Regularisation of ODE systems by blowups: a simple example [Kajiwara et al., 2017]

$$\frac{dx}{dt} = 1 \qquad \frac{dy}{dt} = \frac{y}{x}$$



- At $t_0 \in \mathbb{R}$, any initial condition $(x(t_0), y(t_0)) = (x_0, y_0)$ with $x_0 < 0$ gives a unique solution that can be extended until it reaches the origin.
- The vector field giving the ODE system is not defined (it diverges) on $\{x = 0, y \neq 0\}$.
- On the other side of the origin, any initial condition can similarly be extended forever (in the direction of increasing t)

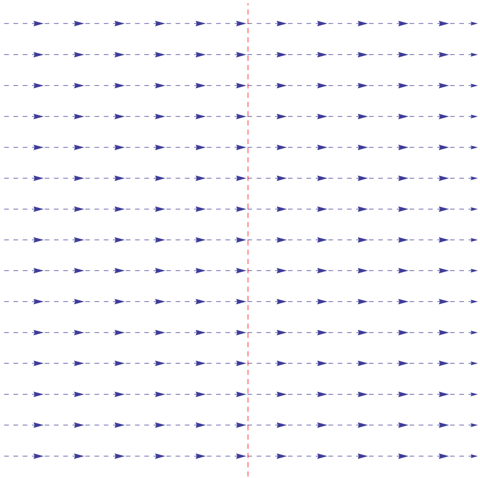
Okamoto's space

Make substitution

$$x = v, \qquad y = uv.$$

- The vector field (u', v') is now regular
- Any initial condition $(u(t_0), v(t_0)) = (u_0, v_0)$ gives a solution curve that can be continued for all $t \in \mathbb{R}$.
- We have *separated* the infinite family of solutions of our original system passing through the origin

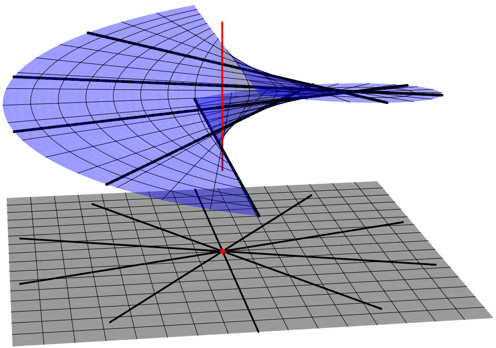
$$\frac{du}{dt} = 0 \qquad \frac{dv}{dt} = 1$$



That was a blowup!

Coordinates:

$$u = \frac{y}{x}, \qquad v = x.$$



Blowup of a point on a surface

- Essentially gluing in a projective line in place of a point.
- The point is replaced by an *exceptional divisor* $E \cong \mathbb{P}^1$.
- After blowing up $p : (x, y) = (x_0, y_0)$, the exceptional line replacing p can be covered by two local affine coordinate charts (u, v) and (U, V) given by

$$u = \frac{y - y_0}{x - x_0}, \quad v = x - x_0,$$
$$U = \frac{x - x_0}{y - y_0}, \quad V = y - y_0.$$

The space of initial conditions

Space of initial conditions for the simple example

- We have a fibre bundle over $\mathbb{R}$, with fibre given by the blown-up surface, with the *inaccessible divisor* $\{x = 0, y \neq 0\}$ removed.
- The flow of the ODE lifted under the blowup gives a foliation of the bundle into disjoint solution curves that are transverse to the fibres.
- Each fibre over $t_0 \in \mathbb{R}$ can be regarded as a *space of initial conditions* for the equation.

Okamoto's space of initial conditions [Okamoto, 1979]

Okamoto discovered that all 6 Painlevé equations admit a similar space: A bundle over the (complex) independent variable space foliated by solution curves transverse to the fibres.

Standard Hamiltonian forms of the Painlevé equations

In constructing the space of initial conditions, Okamoto considered the each Painlevé equation in the form of a non-autonomous Hamiltonian system with polynomial Hamiltonian.

Standard Hamiltonians [Kajiwara et al., 2017]

$$\mathrm{P_I}: H = \frac{1}{2}p^2 - 2q^3 - tq,$$

$$\mathrm{P_{II}}: H = \frac{p^2}{2} - \left(q^2 + \frac{t}{2}\right)p - a_1 q$$

$$\mathrm{P_{III}}: H = \frac{1}{t}\left(p(p-1)q^2 + (a_1+a_2)qp + tp - a_2 q\right)$$

$$\mathrm{P_{IV}}: H = -a_1 p - a_2 q + qp(p-q-t)$$

$$\mathrm{P_V}: H = \frac{1}{t}\left(q(q-1)p(p+t) - (a_1+a_3)qp + a_1 p + a_2 tq\right),$$

$$\text{where} \quad a_0 + a_1 + a_2 + a_3 = 1,$$

$$\mathrm{P_{VI}}: H = \frac{q(q-1)(q-t)}{t(t-1)}\left\{p^2 - \left(\frac{a_0-1}{q-t} + \frac{a_3}{q-1} + \frac{a_4}{q}\right)p\right\} + \frac{a_2(a_1+a_2)(q-t)}{t(t-1)},$$

$$\text{where} \quad a_0 + a_1 + 2a_2 + a_3 + a_4 = 1$$

Construction of Okamoto's space

(Step 1) **Compactify phase space**
Initially the phase space for $\mathrm{P_J}$ is the trivial bundle $\mathbb{C}^2 \times B_\mathrm{J}$, where $B_\mathrm{J} = \mathbb{C} \backslash \{\text{fixed singularities}\}$.
Compactify fibres from $\mathbb{C}^2$ to $\mathbb{CP}^2$ (or $\mathbb{CP}^1 \times \mathbb{CP}^1$ is also fine for $\mathrm{P_{II-VI}}$)

(Step 2) **Resolve singularities through blowups**
In the fibres of the resulting space, identify points of indeterminacies of the vector field (through which infinitely many solutions pass) and blow them up.

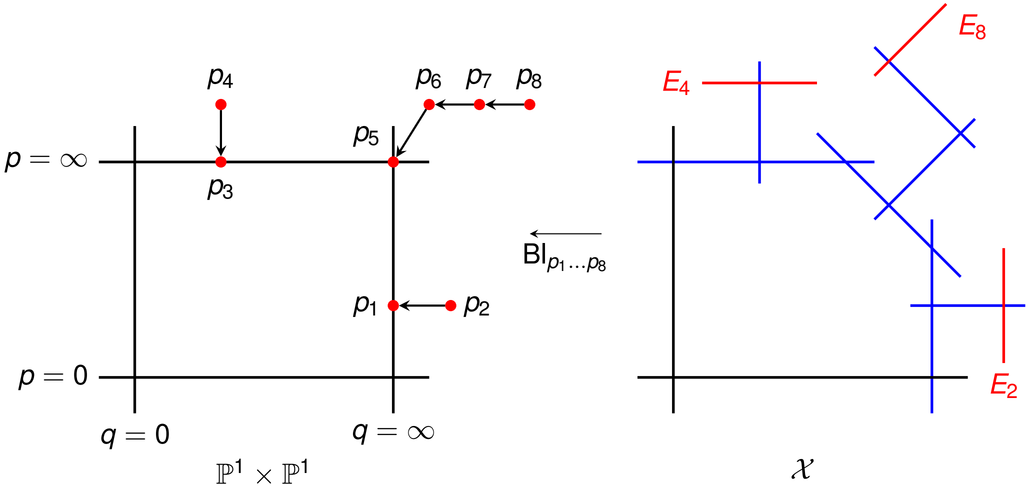
(Step 3) **Remove inaccessible divisors**
After all singularities are resolved, identify the curves where the vector field diverges: no solution curve from a regular initial value will reach here, so remove these curves.

We then have a bundle over the independent variable space, whose fibres are (complex projective) rational surfaces with a number of curves removed, foliated by solution curves transverse to the fibres.

Example: P_{IV}

Standard Hamiltonian form of P_{IV}

$$q' = 2pq - q^2 - tq - a_1, \quad p' = p^2 - tp - 2pq + a_2.$$



- For each of the standard Hamiltonian forms of the Painlevé equations, after compactifying to $\mathbb{CP}^2$ we require *exactly nine blowups*.
- For P_{II}-P_{VI} we can use $\mathbb{CP}^1 \times \mathbb{CP}^1$ instead of $\mathbb{CP}^2$, then require *exactly eight blowups*.

Surface types for Painlevé differential equations

For each of $\mathrm{P}_{\mathrm{I-VI}}$, Okamoto noticed that in each fibre, the inaccessible divisors had intersection configuration given by an *affine Dynkin diagram*.

	P_{I}	P_{II}		$\mathrm{P}_{\mathrm{III}}$		P_{IV}	P_{V}	P_{VI}
Surface	$E_8^{(1)}$	$E_7^{(1)}$	$D_8^{(1)}$	$D_7^{(1)}$	$D_6^{(1)}$	$E_6^{(1)}$	$D_5^{(1)}$	$D_4^{(1)}$

Associating a Dynkin diagram to the surface is done formally in terms of root system structures in its *Picard lattice* [Sakai, 2001, Saito and Takebe, 2002, Saito et al., 2002].

The Picard lattice

For a nonsingular complex projective rational surface $\mathcal{X}$, we have:

- **Picard group / divisor class group**: $\mathrm{Pic}(\mathcal{X}) \cong \mathrm{Cl}(\mathcal{X}) = \mathrm{Div}(\mathcal{X})/\sim$ (abelian group of equivalence classes of formal integer sums of irreducible curves on $\mathcal{X}$, modulo linear equivalence)
- **Intersection pairing**: Symmetric bilinear form on $\mathrm{Pic}(\mathcal{X})$ via intersections of curves

Example: when $\mathcal{X}$ is an eight-point blowup of $\mathbb{CP}^1 \times \mathbb{CP}^1$:

- $\mathrm{Pic}(\mathcal{X}) = \mathbb{Z}\mathcal{H}_1 + \mathbb{Z}\mathcal{H}_2 + \sum_{i=1}^{8} \mathbb{Z}\mathcal{E}_i$, where
  - $\mathcal{H}_1, \mathcal{H}_2$ are classes of hyperplanes in each $\mathbb{CP}^1$ factor,
  - $\mathcal{E}_1, \ldots, \mathcal{E}_8$ are exceptional divisor classes from the eight blowups.
- Intersection pairing:

$$\mathcal{H}_1 \cdot \mathcal{H}_1 = \mathcal{H}_2 \cdot \mathcal{H}_2 = \mathcal{H}_1 \cdot \mathcal{E}_i = \mathcal{H}_2 \cdot \mathcal{E}_j = 0, \quad \mathcal{H}_1 \cdot \mathcal{H}_2 = 1, \quad \mathcal{E}_i \cdot \mathcal{E}_j = -\delta_{ij},$$

- For the surfaces associated with Painlevé equations, the inaccessible divisors give a basis for a root lattice in $\mathrm{Pic}(\mathcal{X})$ associated with an affine root system [Sakai, 2001]. The type of this root system is the **surface type**.

The identification procedure: differential case

The identification procedure for differential Painlevé equations

(Step 1) **Construct a space of initial conditions.**

(Step 2) **Determine the surface type.**

(Step 3) **Find an identification with the standard model on the level of $\mathrm{Pic}(\mathcal{X})$.**

(Step 4) **Find the birational map between surfaces realising this identification, which gives the change of variables to the standard form.**

Example: Semi-classical Laguerre weight

Discrete and differential systems derived and studied in [Han and Chen, 2017] for the weight

$$w(x,c) = w(x,\alpha,c) := x^{\alpha} e^{-N(x+c(x^2-x))}, \qquad x \in (0,\infty),\ \alpha > -1,\ c \in [0,1],\ N > 0.$$

Discrete system for recurrence coefficients

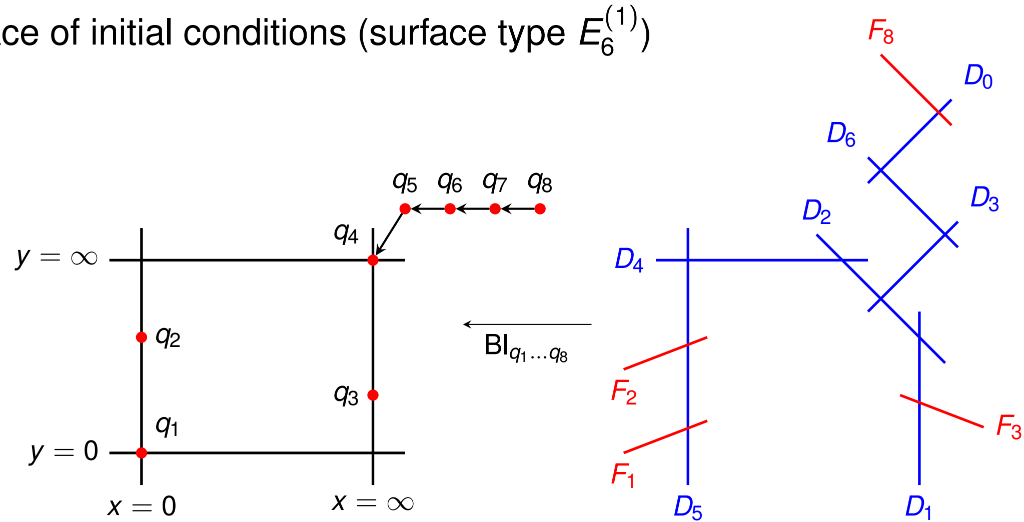
Letting $b_n = (c-1+x_n)/(2c)$, $a_n^2 = (n+Ny_n)/(2Nc)$,

$$x_n x_{n-1} = \frac{2Ncy_n(y_n - \alpha/N)}{n+Ny_n}, \qquad 2(y_{n+1}+y_n) = \frac{2\alpha}{N} - \frac{x_n^2+(c-1)x_n}{c}.$$

Differential system for recurrence coefficients with respect to c

$$\frac{x_n'}{c+1} = -N\frac{x_n^2}{4c^2} + \left(\frac{N(c-1)}{4c^2} + \frac{1}{2c(c+1)}\right) x_n - N\frac{y_n}{c} + \frac{\alpha}{2c},$$
$$\frac{y_n'}{c+1} = -N\frac{y_n^2}{2cx} + \left(\frac{\alpha}{2cx} + N\frac{x}{4c^2}\right) y_n + n\frac{x_n}{4c^2}.$$

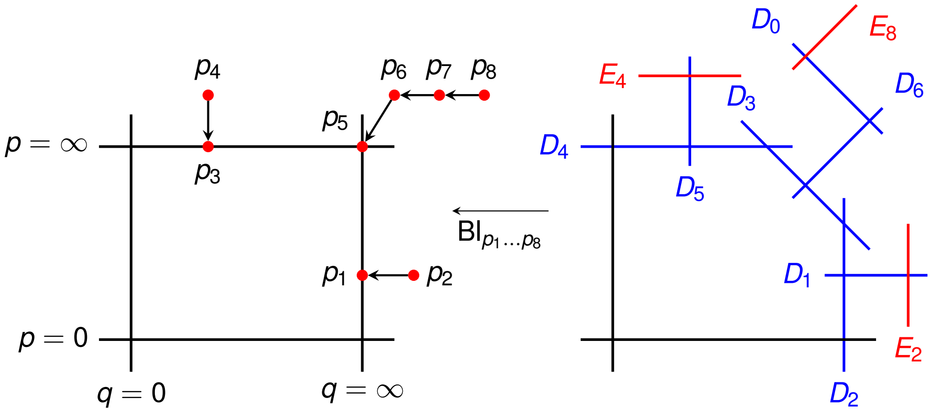
Space of initial conditions (surface type $E_6^{(1)}$)



Inaccessible divisors for the semi-classical Laguerre weight

$$D_0 = F_7 - F_8, \qquad D_1 = H_x - F_3 - F_4, \qquad D_2 = F_4 - F_5,$$
$$D_3 = F_5 - F_6, \qquad D_4 = H_y - F_4 - F_5, \qquad D_5 = H_x - F_1 - F_2, \qquad D_6 = F_6 - F_7.$$

Standard model of $E_6^{(1)}$ surfaces



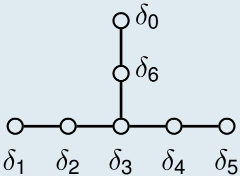
Inaccessible divisors for the standard form of P_{IV} (surface type $E_6^{(1)}$)

$$D_0 = E_7 - E_8, \qquad D_1 = E_1 - E_2, \qquad D_2 = H_q - E_1 - E_5,$$

$$D_3 = E_5 - E_6, \qquad D_4 = H_p - E_3 - E_4, \qquad D_5 = E_3 - E_4, \qquad D_6 = E_6 - E_7.$$

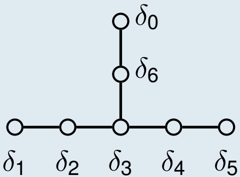
The identification procedure: differential case

Surface root basis for the semi-classical Laguerre weight



$$\delta_0 = \mathcal{F}_7 - \mathcal{F}_8, \qquad \delta_4 = \mathcal{H}_y - \mathcal{F}_4 - \mathcal{F}_5,$$
$$\delta_1 = \mathcal{H}_x - \mathcal{F}_3 - \mathcal{F}_4, \qquad \delta_5 = \mathcal{H}_x - \mathcal{F}_1 - \mathcal{F}_2,$$
$$\delta_2 = \mathcal{F}_4 - \mathcal{F}_5, \qquad \delta_6 = \mathcal{F}_6 - \mathcal{F}_7,$$
$$\delta_3 = \mathcal{F}_5 - \mathcal{F}_6.$$

Surface root basis for the standard model of surfaces for $\mathrm{P_{IV}}$



$$\delta_0 = \mathcal{E}_7 - \mathcal{E}_8, \qquad \delta_4 = \mathcal{H}_p - \mathcal{E}_3 - \mathcal{E}_5,$$
$$\delta_1 = \mathcal{E}_1 - \mathcal{E}_2, \qquad \delta_5 = \mathcal{E}_3 - \mathcal{E}_4,$$
$$\delta_2 = \mathcal{H}_q - \mathcal{E}_1 - \mathcal{E}_5, \qquad \delta_6 = \mathcal{E}_6 - \mathcal{E}_7,$$
$$\delta_3 = \mathcal{E}_5 - \mathcal{E}_6.$$

Identification on the level of Picard lattices

$$\mathcal{H}_x = \mathcal{H}_q, \quad \mathcal{H}_y = \mathcal{H}_q + \mathcal{H}_p - \mathcal{E}_1 - \mathcal{E}_3, \quad \mathcal{F}_1 = \mathcal{H}_q - \mathcal{E}_3, \quad \mathcal{F}_2 = \mathcal{E}_4,$$
$$\mathcal{F}_3 = \mathcal{E}_2, \quad \mathcal{F}_4 = \mathcal{H}_q - \mathcal{E}_1, \quad \mathcal{F}_5 = \mathcal{E}_5, \quad \mathcal{F}_6 = \mathcal{E}_6, \quad \mathcal{F}_7 = \mathcal{E}_7, \quad \mathcal{F}_8 = \mathcal{E}_8,$$

Change of variables

Standard Hamiltonian form of P_{IV}

$$q' = 2pq - q^2 - qt - a_1, \quad p' = p^2 - tp - 2pq + a_2,$$

where $q = q(t), p = p(t)$.

Theorem (Dzhamay, Filipuk, AS)

The change of variables

$$\frac{x_n(c)}{c-1} = \frac{q(t)}{t}, \qquad \frac{2cy_n(c)}{(c-1)x_n(c)} = -\frac{p(t)}{t}, \qquad 2ct^2 = (c-1)^2 N,$$

transforms the differential system from the semi-classical Laguerre weight to the standard Hamiltonian form of P_{IV}, with parameters

$$a_0 = 1 + n + \alpha, \quad a_1 = -\alpha, \quad a_2 = -n.$$

Example: Hypergeometric weight

The discrete orthogonal polynomials $p_n(x)$ with the hypergeometric weight are defined as follows [Filipuk and Van Assche, 2018]: they are orthonormal polynomials on the set $\mathbb{N} = \{0, 1, 2, \ldots\}$ of non-negative integers with respect to the hypergeometric weight w_k, so

$$\sum_{k=0}^{\infty} p_n(k)p_m(k)w_k = \delta_{m,n}, \qquad w_k = \frac{(\alpha)_k(\beta)_k}{(\gamma)_k k!}c^k, \quad \alpha, \beta, \gamma > 0,\ 0 < c < 1,$$

where $(\cdot)_k$ is the usual Pochhammer symbol and $\delta_{m,n}$ is the Kronecker delta.

Introduce variables

From the recurrence coefficients a_n, b_n, introduce variables x_n, y_n according to

$$a_n^2 = \frac{n\alpha\beta c(n+\alpha+\beta-\gamma-1) - c[n^2 + n(\alpha+\beta-\gamma-1) - \alpha\beta+\gamma]y_n - cy_n^2}{(c-1)^2(\alpha\beta - x_{n-1}x_n - y_n)},$$

$$b_n = x_n + \frac{n+(n+\alpha+\beta)c-\gamma}{1-c}.$$

The identification procedure: differential case

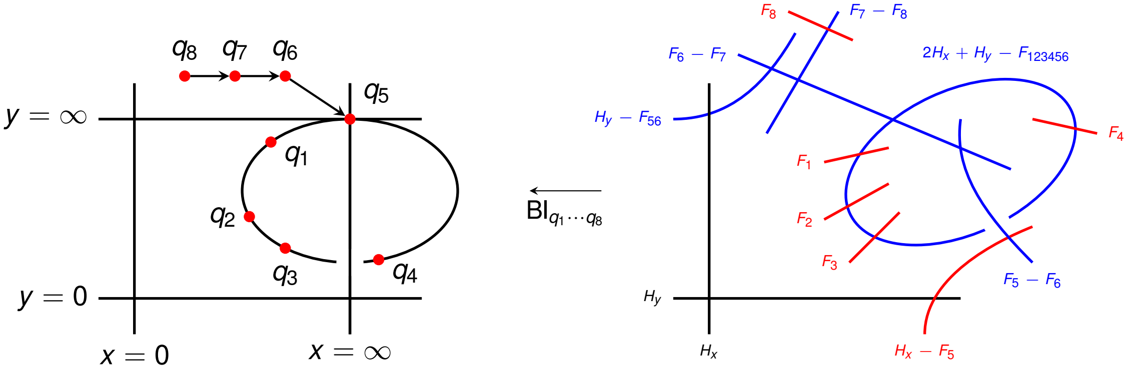
Differential system from the hypergeometric weight

$$x_n'(c) = \frac{P_1(x_n(c), y_n(c), c)}{c(c-1)\left(\alpha\beta - (n+\alpha+\beta)x_n + x_n^2 - y_n\right)},$$
$$y_n'(c) = \frac{P_2(x_n(c), y_n(c), c)}{c(c-1)\left(\alpha\beta - (n+\alpha+\beta)x_n + x_n^2 - y_n\right)},$$

where P_1, P_2 can be written explicitly, with $x = x_n(c)$, $y = y_n(c)$, as

$$\begin{aligned}
P_1(x,y,c) = {} & (1-c)x^4 + (-\alpha - \beta + 2c(\alpha+\beta+n) - \gamma - 1)x^3 \\
& + \left(\alpha(\beta+\gamma+1) + \beta\gamma + \beta - c\left(\beta^2 + 2\beta(2\alpha+n) + (\alpha+n)^2\right) + \gamma\right) x^2 \\
& + \left(\alpha\beta(2c(\alpha+\beta+n) - 1) - \gamma(\alpha\beta+\alpha+\beta)\right) x + \alpha\beta(\gamma - \alpha\beta c) \\
& + 2cy\left[x^2 - (\alpha+\beta+n)x + \alpha\beta\right] - cy^2, \\
P_2(x,y,c) = {} & n\left(\alpha^2 + \alpha\beta - \alpha + \beta^2 - \beta + \gamma + n^2 - \gamma(\alpha+\beta+n) + 2\alpha n + 2\beta n - n\right) x^2 \\
& - 2\alpha\beta n\left(\alpha + \beta - \gamma + n - 1\right) x + n\alpha\beta(\alpha\beta - \gamma) \\
& + y\Big[\left(\alpha+\beta-\gamma+2n-1\right) x^2 + 2\left(-\alpha\beta + \gamma + n^2 + n(\alpha+\beta-\gamma-1)\right) x \\
& - \alpha\gamma - \beta\gamma + \alpha\beta(\gamma - 2n + 1)\Big] + y^2\left[2x - \gamma + n - 1\right].
\end{aligned}$$

Space of initial conditions (surface type $D_4^{(1)}$)

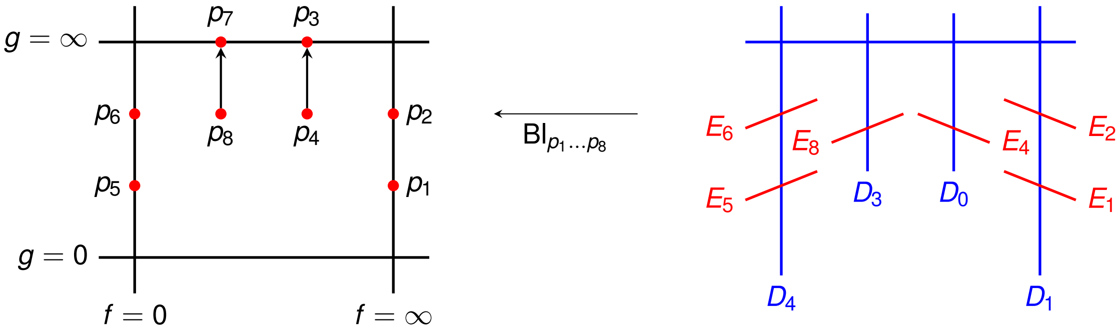


Inaccessible divisors for the differential system from hypergeometric weight

$$D_0 = F_5 - F_6, \qquad D_1 = 2H_x + H_y - F_{123456}, \qquad D_2 = F_6 - F_7,$$
$$D_3 = F_7 - F_8, \qquad D_4 = H_y - F_{56}.$$

The identification procedure: differential case

Standard model of $D_4^{(1)}$-surfaces

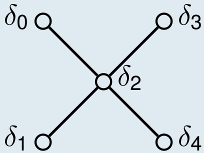


Inaccessible divisors for the standard form of P_{VI} (surface type $D_4^{(1)}$)

$$D_0 = E_3 - E_4, \qquad D_1 = H_f - E_1 - E_2, \qquad D_2 = H_g - E_3 - E_7,$$
$$D_3 = E_7 - E_8, \qquad D_4 = H_f - E_5 - E_6.$$

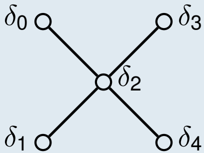
The identification procedure: differential case

Surface root basis for the hypergeometric weight



$$\delta_0 = \mathcal{F}_5 - \mathcal{F}_6, \qquad \delta_3 = \mathcal{F}_7 - \mathcal{F}_8,$$
$$\delta_1 = 2\mathcal{H}_x + \mathcal{H}_y - \mathcal{F}_{123456}, \qquad \delta_4 = \mathcal{H}_y - \mathcal{F}_{56}.$$
$$\delta_2 = \mathcal{F}_6 - \mathcal{F}_7,$$

Surface root basis for the standard model of surfaces for $\mathrm{P_{VI}}$



$$\delta_0 = \mathcal{E}_3 - \mathcal{E}_4, \qquad \delta_3 = \mathcal{E}_7 - \mathcal{E}_8,$$
$$\delta_1 = \mathcal{H}_f - \mathcal{E}_1 - \mathcal{E}_2, \qquad \delta_4 = \mathcal{H}_f - \mathcal{E}_5 - \mathcal{E}_6.$$
$$\delta_2 = \mathcal{H}_g - \mathcal{E}_3 - \mathcal{E}_7,$$

Identification on the level of Picard lattices

$$\mathcal{H}_x = \mathcal{H}_g, \quad \mathcal{H}_y = \mathcal{H}_f + 2\mathcal{H}_g - \mathcal{E}_{3456}, \quad \mathcal{F}_1 = \mathcal{E}_1, \quad \mathcal{F}_2 = \mathcal{E}_2, \quad \mathcal{F}_3 = \mathcal{H}_g - \mathcal{E}_6,$$
$$\mathcal{F}_4 = \mathcal{H}_g - \mathcal{E}_5, \quad \mathcal{F}_5 = \mathcal{H}_g - \mathcal{E}_3, \quad \mathcal{F}_6 = \mathcal{H}_g - \mathcal{E}_3, \quad \mathcal{F}_7 = \mathcal{E}_7, \quad \mathcal{F}_8 = \mathcal{E}_8,$$

Change of variables

Standard Hamiltonian form of P_{VI}

Letting $(f, g) = (q, qp)$, the standard Hamiltonian form becomes

$$\frac{f'}{f} = \frac{\partial K}{\partial g}, \quad \frac{g'}{f} = -\frac{\partial K}{\partial f},$$

$$K(f, g, t) = a_2(a_1 + a_2)(f - t) + (a_0 - 1)(f - 1)g - a_3(f - t)g + \frac{1}{f}g(f - t)(f - 1)(g - a_4),$$

Theorem (Dzhamay, Filipuk, AS)

The differential system from the hypergeometric weight is transformed to the standard form of P_{VI} via the following change of variables, where $f = f(t)$, $g = g(t)$, $x = x_n(c)$, $y = y_n(c)$:

$$f = \frac{t(x - \beta)(x - \gamma)}{\alpha\beta - (n + \alpha + \beta)x + x^2 - y}, \quad g = \gamma - x, \quad ct = 1$$

where the root variable parameters a_i from the standard form are related to α, β, γ, n by

$$a_0 = \alpha - \gamma + n, \quad a_1 = \alpha - 1, \quad a_2 = 1 - \alpha - \beta + \gamma - n, \quad a_3 = \beta + n, \quad a_4 = \beta - \gamma.$$

Geometric theory of discrete Painlevé equations: Sakai surfaces

Sakai's classification [Sakai, 2001]

- Sakai defined *generalised Halphen surfaces*, which are generalisations of those that form Okamoto's space. Among these are the surfaces associated to discrete Painlevé equations, which we call **Sakai surfaces**.
- A Sakai surface admits affine root system structures in the Picard lattice - one associated to a configuration of curves and another to the affine Weyl group of symmetries of the set of isomorphism classes of surfaces of the same type.
- **Translation symmetries define discrete Painlevé equations**.

$$
\begin{tikzcd}
 & & & & & & & A_7^{(1)'} \arrow[ddr] & \\
A_0^{(1)} \arrow[r] & A_1^{(1)} \arrow[r] & A_2^{(1)} \arrow[r] & A_3^{(1)} \arrow[r] \arrow[dr] & A_4^{(1)} \arrow[r] \arrow[dr] & A_5^{(1)} \arrow[r] \arrow[dr] \arrow[ddr] & A_6^{(1)} \arrow[r] \arrow[ur] \arrow[dr] & A_7^{(1)} \arrow[r] \arrow[dr] \arrow[ddr] & A_8^{(1)} \\
 & & & & D_4^{(1)} \arrow[r] & D_5^{(1)} \arrow[r] \arrow[dr] & D_6^{(1)} \arrow[r] \arrow[dr] & D_7^{(1)} \arrow[r] \arrow[dr] & D_8^{(1)} \\
 & & & & & & E_6^{(1)} \arrow[r] & E_7^{(1)} \arrow[r] & E_8^{(1)}
\end{tikzcd}
$$

Figure: Surface types for Sakai surfaces

The identification procedure: discrete case

The identification procedure for discrete Painlevé equations

(Step 1) **Construct a space of initial conditions (lift the mapping under blowups to a family of isomorphisms).**

(Step 2) **Find the induced mapping on the Picard lattice.**

(Step 3) **Determine the surface type.**

(Step 4) **Find a preliminary identification with the standard model on the level of Picard lattices.**

(Step 5) **Find the translation symmetry and compare it with that of the standard discrete Painlevé equation.**

(Step 6) **Adjust the identification so that the translation matches that of the standard discrete Painlevé equation.**

(Step 7) **Find the birational map between surfaces realising this identification, which gives the change of variables to the standard form.**

Example: Hypergeometric weight

Discrete system from the hypergeometric weight

$$
\begin{aligned}
&(y_n-\alpha\beta+(\alpha+\beta+n)x_n-x_n^2)(y_{n+1}-\alpha\beta+(\alpha+\beta+n+1)x_n-x_n^2)\\
&\qquad=\frac{1}{c}(x_n-1)(x_n-\alpha)(x_n-\beta)(x_n-\gamma),\\
&(x_n+\mathfrak{Y}_n)(x_{n-1}+\mathfrak{Y}_n)\\
&\qquad=\frac{(y_n+n\alpha)(y_n+n\beta)(y_n+n\gamma-(\gamma-\alpha)(\gamma-\beta))(y_n+n-(1-\alpha)(1-\beta))}{(y_n(2n+\alpha+\beta-\gamma-1)+n((n+\alpha+\beta)(n+\alpha+\beta-\gamma-1)-\alpha\beta+\gamma)^2},
\end{aligned}
$$

where

$$
\mathfrak{Y}_n=\frac{y_n^2+y_n(n(n+\alpha+\beta-\gamma-1)-\alpha\beta+\gamma)-\alpha\beta n(n+\alpha+\beta-\gamma-1)}{y_n(2n+\alpha+\beta-\gamma-1)+n((n+\alpha+\beta)(n+\alpha+\beta-\gamma-1)-\alpha\beta+\gamma)}.
$$

- The space of initial conditions is given by the same surfaces we constructed for the differential system.

A standard discrete Painlevé equation of surface type $D_4^{(1)}$

The standard d-P_{V} equation

$$\bar{f}f = \frac{tg(g-a_4)}{(g+a_2)(g+a_1+a_2)}, \qquad g + \underline{g} = a_0 + a_3 + a_4 + \frac{a_3}{f-1} + \frac{ta_0}{f-t},$$

$$\bar{a}_0 = a_0 - 1, \quad \bar{a}_1 = a_1, \quad \bar{a}_2 = a_2 + 1, \quad \bar{a}_3 = a_3 - 1, \quad \bar{a}_4 = a_4.$$

- The identification used for the change of variables to P_{VI} *does not* match the translations.
- Thus we need to make an adjustment:

Final identification on the level of Picard lattices

$$\mathcal{H}_x = \mathcal{H}_f + \mathcal{H}_g - \mathcal{E}_{78}, \quad \mathcal{H}_y = 3\mathcal{H}_f + 2\mathcal{H}_g - \mathcal{E}_{3456} - 2\mathcal{E}_{78}, \quad \mathcal{F}_1 = \mathcal{E}_1, \quad \mathcal{F}_2 = \mathcal{E}_2,$$
$$\mathcal{F}_3 = \mathcal{H}_f + \mathcal{H}_g - \mathcal{E}_{678}, \quad \mathcal{F}_4 = \mathcal{H}_f + \mathcal{H}_g - \mathcal{E}_{578}, \quad \mathcal{F}_5 = \mathcal{H}_f + \mathcal{H}_g - \mathcal{E}_{478},$$
$$\mathcal{F}_6 = \mathcal{H}_f + \mathcal{H}_g - \mathcal{E}_{378}, \quad \mathcal{F}_7 = \mathcal{H}_f - \mathcal{E}_8, \quad \mathcal{F}_8 = \mathcal{H}_f - \mathcal{E}_7,$$

The identification procedure: discrete case

Theorem (Dzhamay, Filipuk, AS)

The discrete system from the hypergeometric weight is transformed to the standard d-P_{V} equation via the following change of variables, which realises the final identification:

$$f = \frac{t(x_n-\beta)(x_n-\gamma)}{((x_n-\alpha)(x_n-\beta)-nx_n-y_n)},$$

$$g = -\frac{(x_n-\gamma)(((x_n-\alpha)(x_n-\beta)-nx_n-y_n)-t(x_n-\beta)(x_n-\gamma+\beta+n))}{((x_n-\alpha)(x_n-\beta)-nx_n-y_n)-t(x_n-\beta)(x_n-\gamma)},$$

with parameters related according to

$$a_0=\gamma-n-\alpha, \qquad a_1=\alpha-1, \quad a_2=1+n+\beta-\gamma,$$
$$a_3=-n-\beta, \qquad a_4=\gamma-\beta, \quad ct=1$$

- This change of variables also transforms the differential system to the standard form of P_{VI}, with the same parameter correspondence.

Insights from the geometric approach beyond the identification problem

Hamiltonian forms of differential equations for recurrence coefficients

Modified Laguerre weight [Chen and Its, 2010]

$$w(x) = w(x, s) = x^{\alpha} e^{-x} e^{-s/x}, \quad x \in (0, \infty), \quad \alpha,\ s > 0.$$

Introducing variables $c_n = c_n(s)$, $b_n = b_n(s)$ defined in terms of the recurrence coefficients, we have

$$\begin{aligned} s\frac{dc_n}{ds} &= 2b_n + (2n + 1 + \alpha + c_n)c_n - s, \\ s\frac{db_n}{ds} &= \frac{2}{c_n}(b_n^2 - sb_n) + (2n + \alpha + 1)b_n - ns, \end{aligned} \qquad n \in \mathbb{Z}_{\geq 0}.$$

Theorem (Dzhamay, Filipuk, Ligęza, AS)

The system above can be written in the Hamiltonian form

$$\frac{1}{c_n^2}\frac{dc_n}{ds} = \frac{\partial K}{\partial b_n}, \qquad \frac{1}{c_n^2}\frac{db_n}{ds} = -\frac{\partial K}{\partial c_n},$$

$$K = \frac{b_n(c_n^2 + (1 + \alpha + 2n)c_n + b_n - s)}{sc_n^2} - \frac{n}{c_n}.$$

Determinantal expressions for recurrence coefficients

- Suppose we can find a transformation that maps the differential and discrete systems for the recurrence coefficients to a pair consisting of
  - A Painlevé differential equation
  - A standard example of a discrete Painlevé equation of the same surface type
- The initial condition for the discrete system will likely correspond to a classical seed solution of the Painlevé differential equation, and the discrete system will generate a hierarchy of classical special solutions.

Example: Hypergeometric weight

$$x_0(c) = \frac{\alpha\beta c}{\gamma}\frac{{}_2F_1(\alpha+1,\beta+1;\gamma+1;c)}{{}_2F_1(\alpha,\beta;\gamma;c)} + \frac{(\alpha+\beta)c-\gamma}{c-1}, \quad y_0(c) = 0.$$

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