Uniform convergent expansions of special functions in terms of elementary functions

Ester Pérez Sinusía

Department of Applied Mathematics & IUMA, University of Zaragoza, Spain

OPSTOFA Seminar
June 28th, 2021

Research group

Group members	Center
Blanca Bujanda	Public University of Navarra (UPNA), INAMAT, Spain
José Luis López	
Pedro Pagola	
Pablo Palacios	
Chelo Ferreira	University of Zaragoza (UZ), IUMA, Spain
Ester Pérez Sinusía	

UPNA

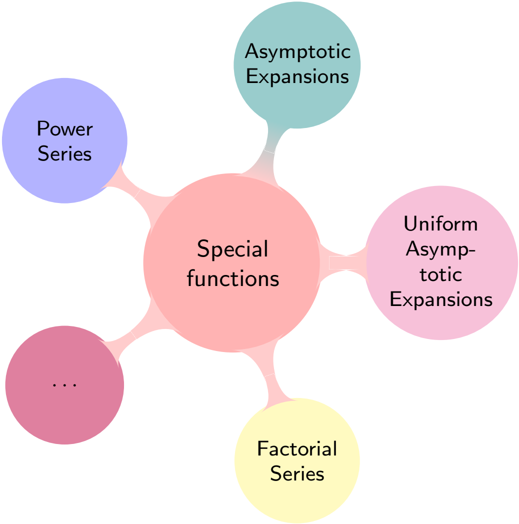
Fac. Veterinaria (UZ)

EINA (UZ)

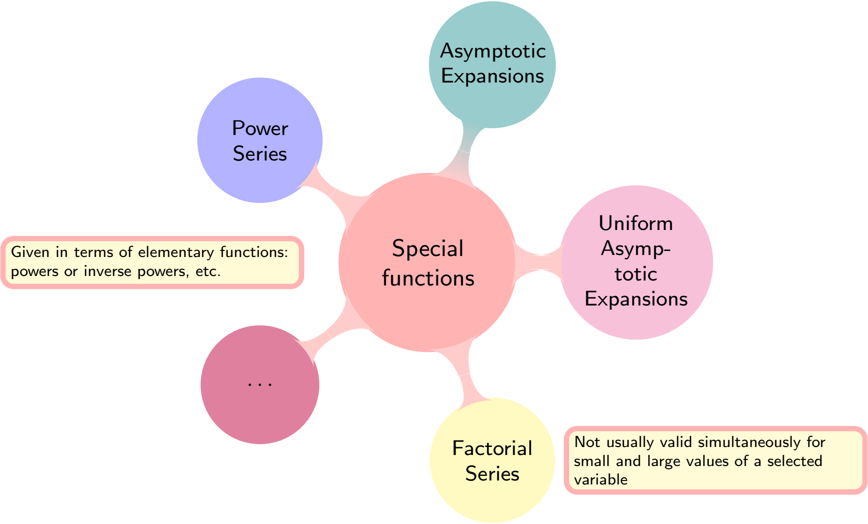
Content

Content

Expansions of special functions



Expansions of special functions



Expansions of special functions

Goal

- Derive convergent expansions in terms of elementary functions that hold uniformly in z in a large region that includes small and large values of $|z|$.
- Provide error bounds for these expansions.

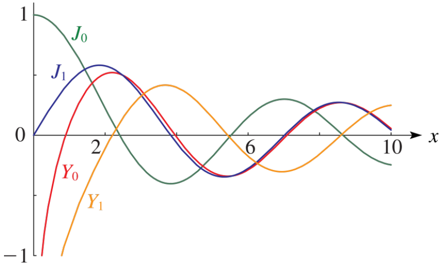
Content

Bessel functions

Definition

$$z^2 \frac{\mathrm{d}^2 w}{\mathrm{d}z^2} + z \frac{\mathrm{d}w}{\mathrm{d}z} + (z^2 - \nu^2) w = 0$$

$J_\nu(z)$ and $Y_\nu(z)$ Bessel functions of the first and second kind



https://dlmf.nist.gov/10.3.F1.mag

Bessel functions

Chapter 10

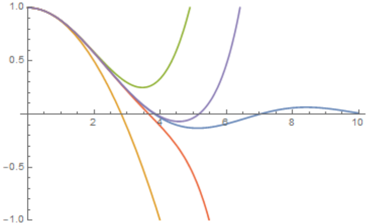
Bessel Functions

F. W. J. Olver[1] and L. C. Maximon[2]

Bessel function

Known Expansions: Power Series

$$J_\nu(z) = \left(\tfrac{1}{2}z\right)^\nu \sum_{k=0}^{n-1} (-1)^k \frac{\left(\frac{1}{4}z^2\right)^k}{k!\Gamma(\nu+k+1)} + R_n^{J,0}(\nu, z)$$



NIST Digital Library of Mathematical Functions http://dlmf.nist.gov/10.8

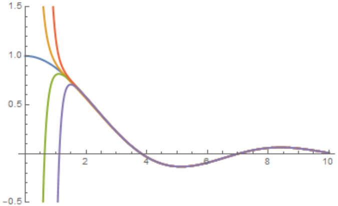
Bessel function

Known Expansions: Asymptotic expansions for large argument

$$J_\nu(z) = \left(\frac{2}{\pi z}\right)^{\frac{1}{2}} \left(\cos\omega \sum_{k=0}^{n-1} \frac{(-1)^k a_{2k}(\nu)}{z^{2k}} - \sin\omega \sum_{k=0}^{\infty} \frac{(-1)^k a_{2k+1}(\nu)}{z^{2k+1}}\right) + R_n^{(J,\infty)}$$

$$a_k(\nu) = \frac{(4\nu^2-1^2)(4\nu^2-3^2)\cdots(4\nu^2-(2k-1)^2)}{k!8^k}, \quad k \geq 1,$$

$$a_0(\nu) = 1, \quad \omega = z - \tfrac{1}{2}\nu\pi - \tfrac{1}{4}\pi$$



NIST Digital Library of Mathematical Functions `http://dlmf.nist.gov/10.17`

Integral representation

Integral representation

$$J_\nu(z) = \frac{2(\frac{1}{2}z)^\nu}{\pi^{\frac{1}{2}}\Gamma\left(\nu+\frac{1}{2}\right)}\int_0^1 (1-t^2)^{\nu-\frac{1}{2}}\cos(zt)\,\mathrm{d}t$$

- Valid for $\Re\nu > -1/2$.
- Analytic continuation formula:

$$J_\nu(e^{im\pi}z) = e^{im\nu\pi}J_\nu(z), \quad m = 0, \pm 1, \pm 2, \ldots$$

Just the approximaton for $\Re z \geq 0$.

Integral representation

Integral representation and expansions of the Bessel function

$$J_\nu(z) = \frac{2(\frac{1}{2}z)^\nu}{\pi^{\frac{1}{2}}\Gamma\left(\nu+\frac{1}{2}\right)} \int_0^1 (1-t^2)^{\nu-\frac{1}{2}} \cos(zt)\, \mathrm{d}t$$

	Method	**Properties**
PS	Expand $\cos(zt)$ at the origin	Convergent
AE	Cauchy's theorem + Watson's lemma	Asymptotic

Not uniform

Integral representation

Integral representation and expansions of the Bessel function

$$J_\nu(z) = \frac{2(\frac{1}{2}z)^\nu}{\pi^{\frac{1}{2}}\Gamma\left(\nu+\frac{1}{2}\right)} \int_0^1 (1-t^2)^{\nu-\frac{1}{2}} \cos(zt)\,\mathrm{d}t$$

	Method	**Properties**
PS	Expand $\cos(zt)$ at the origin	Convergent
AE	Cauchy's theorem + Watson's lemma	Asymptotic

Not uniform

Uniform convergent expansions

Uniform convergent expansions

$$J_\nu(z) = \frac{2\left(\frac{1}{2}z\right)^\nu}{\pi^{\frac{1}{2}}\Gamma\left(\nu+\frac{1}{2}\right)} \bar{J}_\nu(z),$$

$$\boxed{\bar{J}_\nu(z) = \int_0^1 (1-t^2)^{\nu-\frac{1}{2}} \cos(zt)\, dt}$$

1. Taylor expansion at the origin of $(1-t^2)^{\nu-\frac{1}{2}}$.
2. Interchange series and integral.
3. Bound remainder term independently of $\Re z$.

Uniform convergent expansions

1. Taylor expansion at the origin of $(1-t^2)^{\nu-\frac{1}{2}}$

$$\bar{J}_\nu(z) = \int_0^1 (1-t^2)^{\nu-\frac{1}{2}} \cos(zt)\,\mathrm{d}t$$

$$(1-t^2)^{\nu-1/2} = \sum_{k=0}^{n-1} \frac{(1/2-\nu)_k}{k!} t^{2k} + r_n(t,\nu), \quad t\in[0,1),$$

where

$$r_n(t,\nu) := \frac{(1/2-\nu)_n t^{2n}}{n!}\, {}_2F_1\left(\begin{matrix} n+1/2-\nu, \;\; 1 \\ n+1 \end{matrix} \,\middle|\, t^2 \right).$$

Uniform convergent expansions

2 Replace Taylor expansion and interchange series and integral

$$\bar{J}_\nu(z) = \sum_{k=0}^{n-1} \frac{(1/2-\nu)_k}{k!} A_k(z) + R_n(z,\nu)$$

Explicit formula for the coefficients $A_k(z)$

$$A_k(z) := \int_0^1 t^{2k} \cos(zt)dt = (-1)^k \frac{d^{2k}}{dz^{2k}} \left(\frac{\sin z}{z}\right)$$

$$= (-1)^k \frac{(2k)!}{z^{2k+1}} \left[\sin z \sum_{j=0}^{k} \frac{(-z^2)^j}{(2j)!} - z \cos z \sum_{j=0}^{k-1} \frac{(-z^2)^j}{(2j+1)!} \right]$$

Uniform convergent expansions

2 Replace Taylor expansion and interchange series and integral

$$\bar{J}_\nu(z) = \sum_{k=0}^{n-1} \frac{(1/2-\nu)_k}{k!} A_k(z) + R_n(z,\nu)$$

Recurrence relation for the coefficients $A_k(z)$

$$A_{n+1}(z) = \frac{1}{z}\left[\sin z + 2(n+1)\frac{\cos z}{z}\right] - \frac{2(n+1)(2n+1)}{z^2} A_n(z),$$

$$A_0(z) = \frac{\sin z}{z}$$

Uniform convergent expansions

Rearranging terms, for $n = 1, 2, 3, \ldots$, the Bessel function $J_\nu(z)\ldots$

$$\boxed{\frac{\sqrt{\pi}\Gamma(\nu+1/2)}{2(z/2)^\nu} J_\nu(z) = P_{n-1}(z,\nu)\frac{\sin z}{z} - Q_{n-1}(z,\nu)\cos z + R_n(z,\nu)}$$

$$P_n(z,\nu) := \sum_{m=0}^{n} \frac{a_{n,m}(\nu)}{(-z^2)^m}, \quad a_{n,m}(\nu) := \sum_{k=m}^{n} \frac{(1/2-\nu)_k (2k)!}{k!(2(k-m))!},$$

$$Q_n(z,\nu) := \sum_{m=1}^{n} \frac{b_{n,m}(\nu)}{(-z^2)^m}, \quad b_{n,m}(\nu) := \sum_{k=m}^{n} \frac{(1/2-\nu)_k (2k)!}{k!(2(k-m)+1)!}$$

Elementary functions

Uniform convergent expansions

3 Bounding the remainder term

$$\boxed{R_n(z,\nu) := \int_0^1 r_n(t,\nu)\cos(zt)dt}$$

$$r_n(t,\nu) = \frac{(1/2-\nu)_n t^{2n}}{n!}\,{}_2F_1\left(\begin{array}{c} n+1/2-\nu, \quad 1 \\ n+1 \end{array}\middle| t^2\right)$$

$$|R_n(z,\nu)| \le e^{|\Im z|}\int_0^1 |r_n(t,\nu)|dt$$

Bounds for the remainder term

Bounds and properties

For $n > \Re\nu - 1/2$

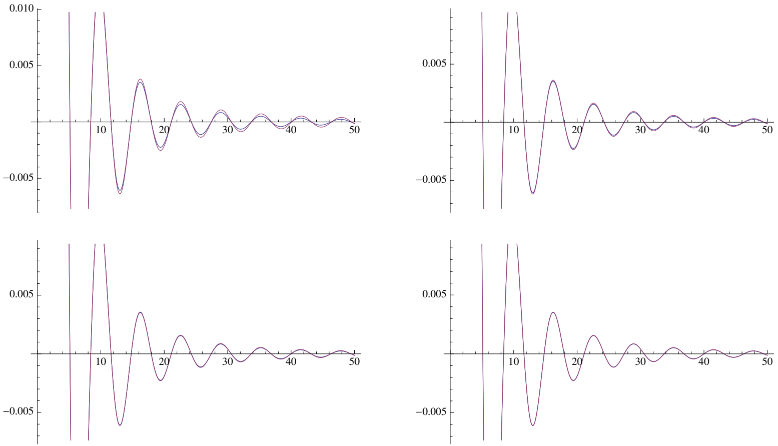
$$\boxed{|R_n(z,\nu)| \leq \frac{2|(1/2-\nu)_n|}{(n-1)!(2n-1)(2\Re\nu+1)} e^{|\Im z|}}$$

1. It behaves as $n^{-\Re\nu-1/2}$ as $n \to +\infty \to$ convergent.
2. Uniform in z in any fixed horizontal strip.

For real $\nu > 1/2$ and $n \geq \nu - 1/2$

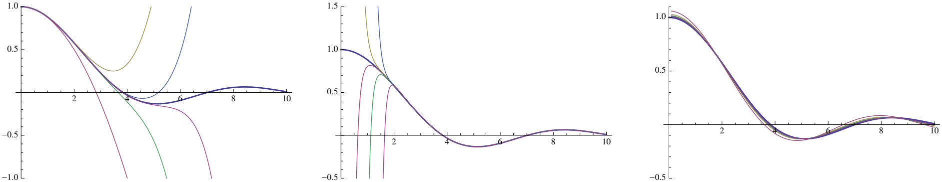
$$|R_n(z,\nu)| \leq \frac{4|(1/2-\nu)_n|}{(n-1)!(2\nu-1)|z|} e^{|\Im z|}$$

Numerical experiments



Plot of the function $\bar{J}_2(x)$ and the approximation for $n = 10$, $n = 15$ (top) and $n = 20$, $n = 25$ (bottom) in the real interval $[0, 50]$.

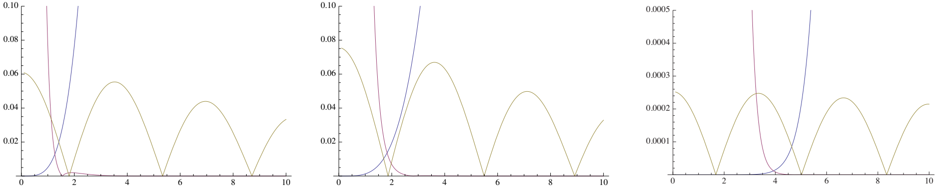
Numerical experiments



(a) Power series (b) Asymptotic expansion (c) Uniform convergent expansion

Comparison between the three approximations of $(2/z)^{\nu} J_{\nu}(z)$ for $\nu = 1$, $z \in [0, 10]$ and $n = 1, 2, \ldots, 5$.

Numerical experiments



Absolute error in the approximation of $(2/z)^{\nu} J_{\nu}(z)$ in the interval $z \in [0, 10]$ given by the three expansions for $n = \nu = 1$ (left), $n = 1$ and $\nu = 2$ (middle) and $n = \nu = 3$ (right).

Remarks

Remarks

1. The formulas derived may be extended to $\Re\nu \leq -1/2$ using

$$J_{\nu-1}(z) + J_{\nu+1}(z) = \frac{2\nu}{z} J_\nu(z)$$

Remarks

Remarks

2. For $x > 0$:

$$\frac{15\pi}{2x^3} J_3(x) = \left[\frac{3x^4 - 140x^2 + 360}{8x^6} + \theta_1(x)\right] x \sin x + \left[\frac{5(x^2 - 18)}{2x^4} + \theta_2(x)\right] \cos x,$$

with $|\theta_1(x)| < 0.0062$ and $|\theta_2(x)| < 0.051$.

Content

General theory of uniform approximations

Is it possible to design a general theory of uniform approximations of special functions based on integral transforms?

$$F(z) = \int_a^b h(t,z)g(t)dt$$

- (a,b) is a bounded or unbounded interval
- $h(\cdot,z)g(\cdot)$ is integrable on (a,b)
- $g(t)$ analytic in $\Omega \subset \mathbb{C}$ and includes $(a,b) \subset \Omega$
- Often, $F(z)$ is a special function

General theory of uniform approximations

$$F(z) = \int_a^b h(t,z)g(t)dt$$

$$\Downarrow$$

Bounded interval

$[a,b]$ bounded $\rightarrow [0,1]$

$$F(z) = \int_0^1 h(t,z)g(t)dt$$

Unbounded interval

(a,b) unbounded $\rightarrow [0,\infty)$

$$F(z) = \int_0^\infty \tilde{h}(u,z)\tilde{g}(u)du$$

$$[u = -\log t]$$

$$F(z) = \int_0^1 h(t,z)g(t)dt$$

General theory of uniform approximations

$$F(z)=\int_a^b h(t,z)g(t)dt$$

$$\Downarrow$$

Bounded interval

$[a,b]$ bounded $\to [0,1]$

$$F(z)=\int_0^1 h(t,z)g(t)dt$$

Unbounded interval

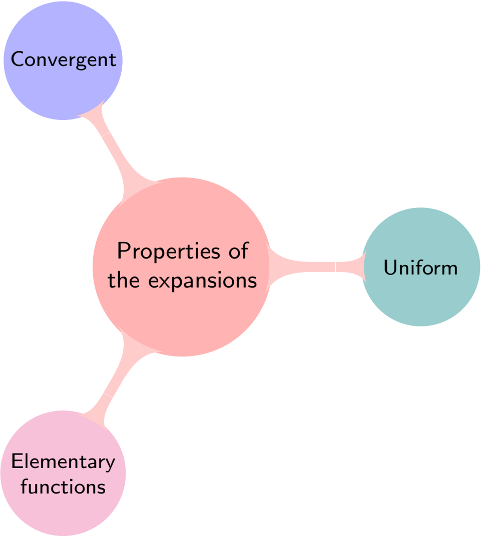
(a,b) unbounded $\to [0,\infty)$

$$F(z)=\int_0^\infty \tilde{h}(u,z)\tilde{g}(u)du$$

$$[u=-\log t]$$

$$F(z)=\int_0^1 h(t,z)g(t)dt$$

General theory of uniform approximations

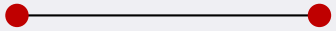


General theory of uniform approximations

Cases

We consider four different cases concerning the position of the end points $t = 0, 1$ of the integration interval with respect to Ω

- Case (i) $[0, 1] \subset \Omega$.



- Case (ii) $(0, 1] \subset \Omega$, $[0, 1] \not\subset \Omega$.

- Case (iii) $[0, 1) \subset \Omega$, $[0, 1] \not\subset \Omega$.

- Case (iv) $(0, 1) \subset \Omega, [0, 1] \not\subset \Omega$.

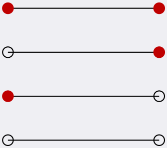
General theory of uniform approximations

Hypotheses

$$F(z) = \int_0^1 h(t,z)g(t)dt, \;\; z \in D$$

1. $g(t)$ analytic in an open region Ω that contains $(0,1)$ and $f(t) := t^{1-\sigma}(1-t)^{1-\gamma}g(t)$, with $0 < \sigma, \gamma \leq 1$ bounded in Ω

$$\begin{cases} \sigma = \gamma = 1 \\ \sigma < 1, \gamma = 1 \\ \sigma = 1, \gamma < 1 \\ \sigma, \gamma < 1 \end{cases}$$



To include the possibility of an integrable singularity at $t = 0$ and/or at $t = 1$.

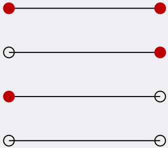
General theory of uniform approximations

Hypotheses

$$F(z) = \int_0^1 h(t,z) g(t) dt, \ z \in D$$

1. $g(t)$ analytic in an open region Ω that contains $(0,1)$ and $f(t) := t^{1-\sigma}(1-t)^{1-\gamma} g(t)$, with $0 < \sigma, \gamma \leq 1$ bounded in Ω

$$\begin{cases} \sigma = \gamma = 1 \\ \sigma < 1, \gamma = 1 \\ \sigma = 1, \gamma < 1 \\ \sigma, \gamma < 1 \end{cases}$$



To include the possibility of an integrable singularity at $t = 0$ and/or at $t = 1$.

General theory of uniform approximations

Hypotheses

$$F(z) = \int_0^1 h(t,z)g(t)dt, \;\; z \in D$$

2. We can choose a point t_0 such that the disk $D_r(t_0)$ for $g(t)$ satisfies $(0,1) \subset D_r(t_0) \subset \Omega$.

To impose that $(0,1) \subset D_r(t_0) \subset \Omega$.

General theory of uniform approximations

Hypotheses

$$F(z) = \int_0^1 h(t,z)g(t)dt, \ z \in D$$

2. We can choose a point t_0 such that the disk $D_r(t_0)$ for $g(t)$ satisfies $(0,1) \subset D_r(t_0) \subset \Omega$.

To impose that $(0,1) \subset D_r(t_0) \subset \Omega$ (not always possible!).

General theory of uniform approximations

Hypotheses

$$F(z) = \int_0^1 h(t,z)g(t)dt, \ \ z \in D$$

3. We assume that $|h(t,z)| \leq Ht^{\alpha}(1-t)^{\beta}$ for $(t,z) \in [0,1] \times \mathcal{D}$, with $H > 0$ independent of z and t and $\alpha + \sigma > 0$, $\beta + \gamma > 0$.

It is natural to assume this form for the bound of the function $h(t,z)$, as the function $h(\cdot,z)g(\cdot)$ must be integrable in $[0,1]$.

General theory of uniform approximations

Hypotheses

$$F(z) = \int_0^1 h(t,z) g(t) dt, \;\; z \in D$$

3. We assume that $|h(t,z)| \leq H t^{\alpha}(1-t)^{\beta}$ for $(t,z) \in [0,1] \times \mathcal{D}$, with $H > 0$ independent of z and t and $\alpha + \sigma > 0$, $\beta + \gamma > 0$.

It is natural to assume this form for the bound of the function $h(t,z)$, as the function $h(\cdot, z) g(\cdot)$ must be integrable in $[0,1]$.

General theory of uniform approximations

Hypotheses

$$F(z) = \int_0^1 h(t,z)g(t)dt, \ \ z \in D$$

4. The moments of h, $M[h(\cdot,z);k] := \int_0^1 h(t,z)(t-t_0)^k dt$ are elementary functions of z.

'Elementary' means that the moments $M[h(\cdot,z);k]$ are functions of fewer variables than $F(z)$ (this means that at least one of the 'extra' variables of $F(z)$ is in $g(t)$).

General theory of uniform approximations

Hypotheses

$$F(z) = \int_0^1 h(t,z) g(t) dt, \;\; z \in D$$

4. The moments of h, $M[h(\cdot,z);k] := \int_0^1 h(t,z)(t-t_0)^k dt$ are elementary functions of z.

'Elementary' means that the moments $M[h(\cdot,z);k]$ are functions of fewer variables than $F(z)$ (this means that at least one of the 'extra' variables of $F(z)$ is in $g(t)$).

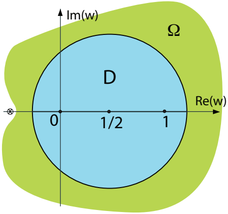
How to obtain the expansion?

STEP 1 → STEP 2 → STEP 3

$\int_0^1 h(z,t)g(t)dt$

Consider the Taylor expansion of $g(t)$ at t_0, such that $(0,1) \subset D_r(t_0) \subset \Omega$

$$g(t) = \sum_{k=0}^{n-1} A_k (t-t_0)^k + g_n(t)$$



where

$$g_n(t) := \frac{(t-t_0)^n}{2\pi i} \oint_{C_r} \frac{g(w)dw}{(w-t)(w-t_0)^n}, \quad t \in (0,1)$$

How to obtain the expansion?

STEP 1 → STEP 2 → STEP 3 $\int_0^1 h(z,t)g(t)dt$

Introduce the expansion into the integral

$$F(z) = \sum_{k=0}^{n-1} A_k M[h(\cdot,z),k] + R_n(z)$$

where the moments of h are

$$M[h(\cdot,z),k] = \int_0^1 h(t,z)(t-t_0)^k dt,$$

and the remainder term

$$R_n(z) = \int_0^1 h(t,z) g_n(t) dt.$$

How to obtain the expansion?

STEP 1 → STEP 2 → STEP 3 $\int_0^1 h(z,t)g(t)dt$

Introduce the expansion into the integral

$$F(z) = \sum_{k=0}^{n-1} A_k M[h(\cdot, z), k] + R_n(z)$$

where the moments of h are

$$M[h(\cdot, z), k] = \int_0^1 h(t,z)(t-t_0)^k dt,$$

and the remainder term

$$R_n(z) = \int_0^1 h(t,z) g_n(t) dt.$$

How to obtain the expansion?

STEP 1 → STEP 2 → STEP 3

$\int_0^1 h(z,t)g(t)dt$

Introduce the expansion into the integral

$$F(z) = \sum_{k=0}^{n-1} A_k M[h(\cdot,z),k] + R_n(z)$$

where the moments of h are

$$M[h(\cdot,z),k] = \int_0^1 h(t,z)(t-t_0)^k dt,$$

and the remainder term

$$R_n(z) = \int_0^1 h(t,z)g_n(t)dt.$$

How to obtain the expansion? Coefficients

STEP 1 → STEP 2 → STEP 3

$\sum_{k=0}^{n-1} A_k M[h(\cdot,z),k]$

In the case that the initial interval of integration is unbounded

Coefficients A_k given in terms of

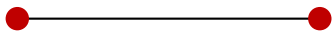
- Nörlund polynomials
- Partial ordinary Bell numbers

How to obtain the expansion? Bounds

STEP 1 → STEP 2 → STEP 3

$$R_n(z) = \int_0^1 h(t,z) g_n(t)\,dt$$

Case (i) $[0,1] \subset \Omega$.



$$|g_n(t)| \leq \frac{1}{2\pi a^n} \oint_{C_r} \frac{|g(w)dw|}{|w-t|} = \frac{M}{a^n}, \quad t \in [0,1], \quad a > 1$$

$$|h(t,z)| \leq H t^{\alpha-1}$$

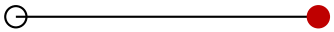
$$|R_n(z)| \leq \frac{MH}{a^n} = \mathcal{O}(a^{-n}),\ n \to \infty$$

How to obtain the expansion? Bounds

STEP 1 ⟶ STEP 2 ⟶ **STEP 3**

$$R_n(z) = \int_0^1 h(t,z) g_n(t) dt$$

Case (ii) $(0,1] \subset \Omega$.



$t^{1-\sigma} g(t)$ bounded in Ω

$$R_n(z) = \int_0^{t_0} h(t,z) g_n(t) dt + \int_{t_0}^1 h(t,z) g_n(t) dt, \quad |h(t,z)| \leq H t^{\alpha-1},$$

$$|g_n(t)| \leq \begin{cases} \frac{M(t_0-t)^n t^{\sigma-1}}{t_0^n} & \text{if } t \in [0,t_0] \\ \frac{M}{a^n} & \text{if } t \in [t_0,1] \end{cases}$$

$$|R_n(z)| \leq \frac{M H t_0^{\alpha+\sigma} \Gamma(\alpha+\sigma) n!}{\Gamma(n+\alpha+\sigma+1)} = \mathcal{O}(n^{-\sigma-\alpha}), \; n \to \infty$$

How to obtain the expansion? Bounds

STEP 1 ⟶ STEP 2 ⟶ STEP 3

$R_n(z) = \int_0^1 h(t,z) g_n(t) dt$

$$R_n(z) = \mathcal{O}(a^{-n} + A\, n^{-\sigma-\alpha} + B\, n^{-\gamma-\beta}), \quad n \to \infty$$

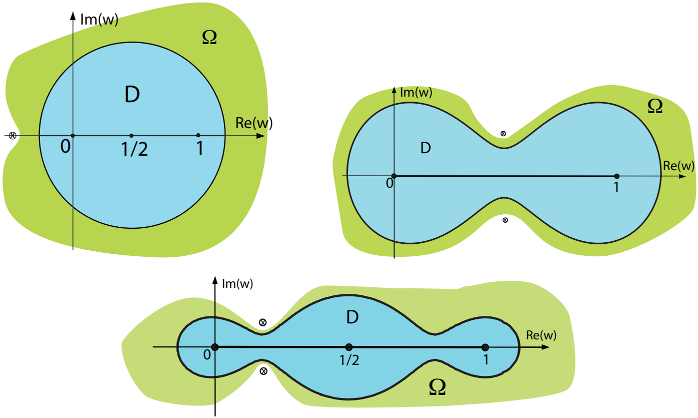
$$(A,B) := \begin{cases} (0,0) & \text{in case (i)}, \\ (1,0) & \text{in case (ii)}, \\ (0,1) & \text{in case (iii)}, \\ (1,1) & \text{in case (iv)}, \end{cases}$$

- Expansion uniformly convergent for $z \in \mathcal{D}$ in the four cases.
- Convergence exponential in (i) and of power type in (ii)-(iv).

Some remarks

Function $g(t)$ may posses singularities located near the integration interval $(0,1)$ such that $D_r(t_0) \not\subset \Omega$ for any $t_0 \in \Omega$.

Solution: Multipoint Taylor expansions



Example: Multipoint Taylor expansion

Consider the hypergeometric function

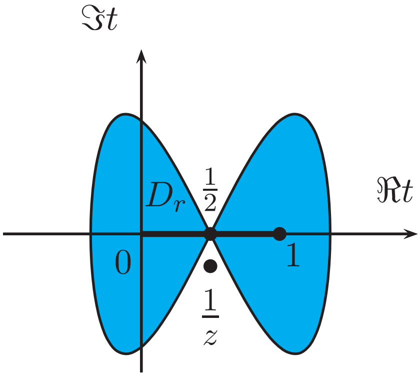
$$\frac{\Gamma(z_1)\Gamma(z_2)}{\Gamma(z_1+z_2)}\,{}_2F_1(d, z_1, z_1+z_2; u) = \int_0^1 t^{z_1-1}(1-t)^{z_2-1}(1-zt)^{-d}\,dt$$

We can apply the method with

- $g(t) = (1-zt)^{-d}$
- $h(t, z_1, z_2) = t^{z_1-1}(1-t)^{z_2-1}$, considering z_1 and z_2 as uniform variables
- $\alpha = \beta = \delta - 1$, $\sigma = \gamma = 1$ (case (i))

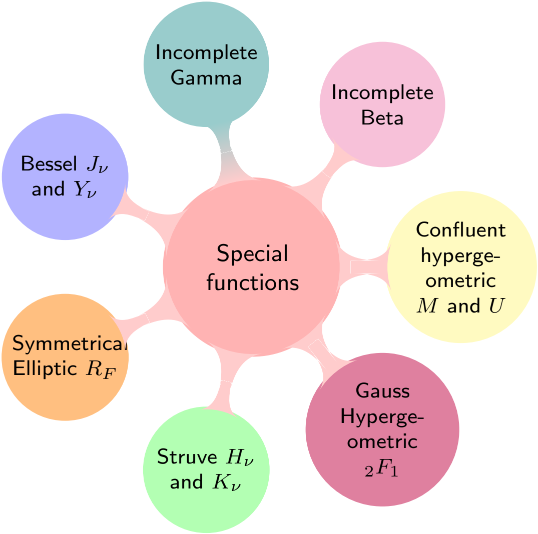
Example: Multipoint Taylor expansion

We take the points $t_1 = 0$ and $t_2 = 1$ as base points $(m = 2)$ in order to better avoid the singularity at $t = 1/z$



Content

Uniform convergent expansions of special functions



Uniform convergent expansions of special functions

Special function	Integral
Incomplete Gamma	$\gamma(a,z) = z^a \int_0^1 t^{a-1} e^{-zt} dt$
Incomplete Beta	$\beta_z(a,b) = z^a \int_0^1 t^{a-1} (1-zt)^{b-1} dt$
Confluent M	$M(a,b;z) = \frac{\Gamma(b)}{\Gamma(a)\Gamma(b-a)} \int_0^1 t^{a-1}(1-t)^{b-a-1} e^{zt} dt$
Hypergeometric	${}_2F_1(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 \frac{t^{b-1}(1-t)^{c-b-1}}{(1-zt)^a} dt$
Confluent U	$U(c,b,z) = \frac{1}{\Gamma(c)} \int_0^\infty e^{-zu} u^{c-1} (1+u)^{b-c-1} du$

Uniform convergent expansions of special functions

Special function	Integral
Incomplete Gamma	$\gamma(a,z) = z^a \int_0^1 t^{a-1} e^{-zt} dt$
Incomplete Beta	$\beta_z(a,b) = z^a \int_0^1 t^{a-1}(1-zt)^{b-1} dt$
Confluent M	$M(a,b;z) = \frac{\Gamma(b)}{\Gamma(a)\Gamma(b-a)} \int_0^1 t^{a-1}(1-t)^{b-a-1} e^{zt} dt$
Hypergeometric	${}_2F_1(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 \frac{t^{b-1}(1-t)^{c-b-1}}{(1-zt)^a} dt$
Confluent U	$U(c,b,z) = \frac{1}{\Gamma(c)} \int_0^\infty e^{-zu} u^{c-1}(1+u)^{b-c-1} du$

The incomplete Gamma function

The incomplete Gamma function

For $\Re a > 0$, $z \in \mathbb{C}$ and $n = 1, 2, 3, \ldots$,

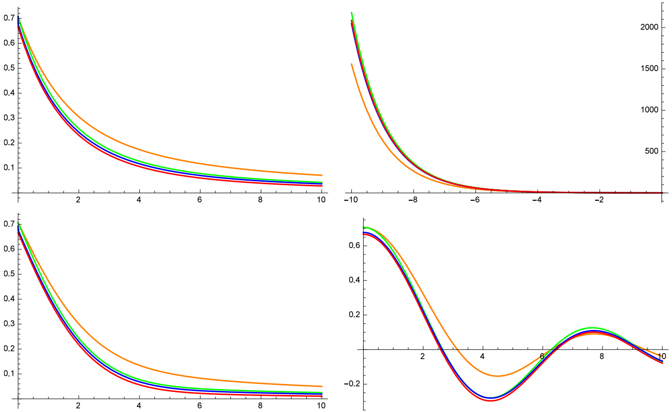
$$\boxed{z^{-a}\gamma(a,z) = 2^{1-a}\sum_{k=0}^{n-1}\frac{(1-a)_k}{k!}\gamma_k(z) + R_n(a,z)}$$

$$\gamma_k(z) := \frac{(-2)^k\,k!}{z^{k+1}}\left[e_k\left(-\frac{z}{2}\right) - e^{-z}e_k\left(\frac{z}{2}\right)\right],\quad e_k(z) := \sum_{j=0}^{k}\frac{z^j}{j!},$$

$$\gamma_{n+1}(z) = \frac{1+(-1)^n e^{-z}}{z} - 2\frac{n+1}{z}\gamma_n(z), \quad \gamma_0(z) = \frac{1-e^{-z}}{z}.$$

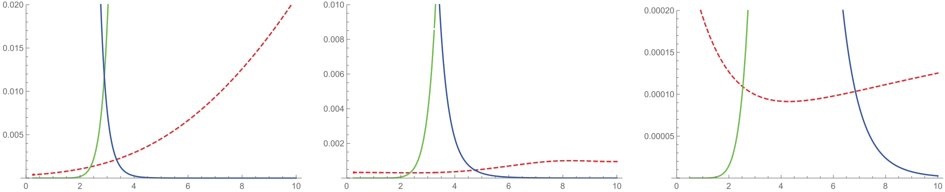
$R_n(a,z) \sim n^{-\Re a}$ as $n \to \infty$ uniformly in z with $\Re z \geq \Lambda$, for any fixed $\Lambda \in \mathbb{R}$.

The incomplete Gamma function



Graphics of $\gamma_{3/2}(z)$ (red) and the approximations for $n = 1$ (orange), $n = 2$ (green), $n = 3$ (blue) in several intervals: $[0, 10]$ (top left), $[-10, 0]$ (top right), $[0, 10e^{i\pi/4}]$ (bottom left) and $[0, 10i]$ (bottom right)

The incomplete Gamma function



Relative errors in the approximation of $\gamma_{5/2}(z)$ by using the uniform expansion (red and dashed), the power series expansion (green) and the asymptotic expansion (blue) in the intervals $z \in [0, 10]$ (left), $z \in [0, 10e^{i\pi/4}]$ (middle) and $z \in [0, 10e^{i\pi}]$ (right) with $n = 10$.

The incomplete Beta function

The incomplete Beta function

For $\Re a > 0$, $\Re b \le 1$, $z \in \mathbb{C} \setminus [1, \infty)$ and $n = 1, 2, 3, \ldots$,

$$\boxed{z^{-a} B_z(a,b) = 2^{1-a} \sum_{k=0}^{n-1} \frac{(1-a)_k}{k!} \beta_k(z,b) + R_n(z,a,b)}$$

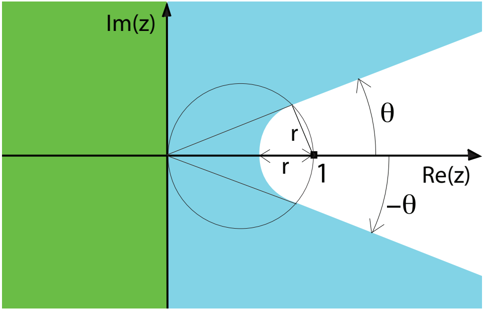
$$\beta_k(z,b) := \frac{k!}{z(b)_{k+1}} \sum_{j=0}^{k} \frac{(-b-k)_j}{j!} \left(-\frac{2}{z}\right)^{k-j} \left[(-1)^j - (1-z)^{b+k-j}\right],$$

$$\beta_k(z,b) = \frac{1}{zb}\left[1 - (-1)^k (1-z)^b\right] - \frac{2k}{zb} \beta_{k-1}(z, b+1),$$

$$\beta_0(z,b) = \frac{1}{zb}\left[1 - (1-z)^b\right]$$

Elementary functions

The incomplete Beta function

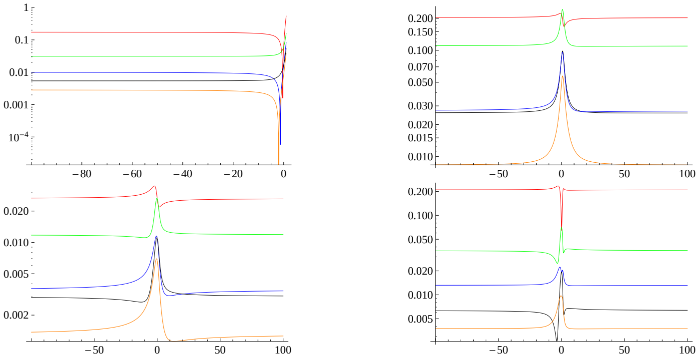


$$S_\theta := \{\theta \le |\arg(z)| \le \pi\} \cup \{z \in \mathbb{C}; |z - 1/2| < 1/2 \text{ and } |z-1| > \sin\theta\}$$

$$|R_n(z,a,b)| \le [\sin(\theta)]^{\Re b - 1} \frac{e^{\pi|\Im b|}|(1-a)_n|}{n!\, 2^{\Re a - 1}\, \Re a} \max\{2^{\Re a - n - 1}, 1\} \text{ Green and Blue}$$

$$|R_n(z,a,b)| \le \frac{e^{\pi|\Im b|}|(1-a)_n|}{n!\, 2^{\Re a - 1}\, \Re a} \max\{2^{\Re a - n - 1}, 1\} \text{ Green}$$

The incomplete Beta function



Relative errors on a logarithmic scale for $n = 1$ (red), $n = 2$ (green), $n = 3$ (blue), $n = 4$ (black) and $n = 5$ (orange) for $z = \rho e^{i\theta}$ with $\theta = 0$, $a = 1.5$, $b = 3$ (top left), $\theta = \pi/4$, $a = 1.3 + 0.75i$, $b = 2$ (top right), $\theta = \pi/2$, $a = 1.1$, $b = 2.25 + 0.25i$ (bottom left), $\theta = -\pi/3$, $a = 1.5 - 0.2i$, $b = 3.0 - i$ (bottom right) and $\rho \in [-100, 1)$ or $\rho \in [-100, 100]$.

The confluent hypergeometric function

The confluent hypergeometric function

For $\Re b > \Re a > 0$ and $n = 1, 2, 3, \ldots$,

$$\boxed{M(a,b;z) = \frac{\Gamma(b)}{\Gamma(a)\Gamma(b-a)} \sum_{k=0}^{n-1} A_k(a,b) F_k(z) + R_n(a,b,z)}$$

$$A_n(a,b) := 2^{n+2-b} \sum_{k=0}^{n} (-1)^k \frac{(1-a)_k (a+1-b)_{n-k}}{k!(n-k)!}$$

$$F_n(z) := \frac{n!}{(-z)^{n+1}} \left[e_n\left(\frac{z}{2}\right) - e^z e_n\left(-\frac{z}{2}\right)\right], \quad e_n(z) := \sum_{k=0}^{n} \frac{z^k}{k!},$$

$$F_{n+1}(z) = \frac{e^z + (-1)^n}{z 2^{n+1}} - \frac{n+1}{n} F_n(z), \qquad F_0(z) = \frac{e^z - 1}{z}.$$

Elementary functions

The confluent hypergeometric function

The confluent hypergeometric function

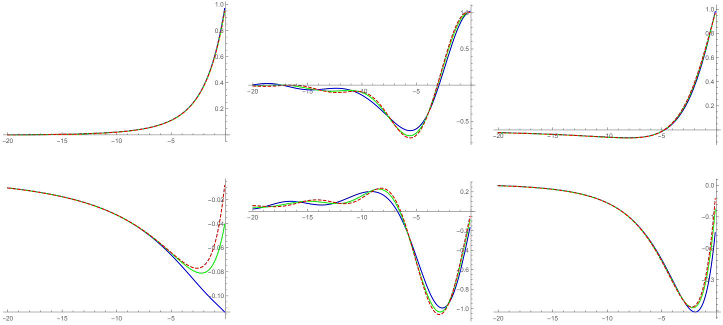
For $1-\Re b+n>0$,

$$\boxed{|R_n(a,b,z)|\le H(z)\frac{2|\Gamma(b)|\,\Gamma(1-\Re b+n)}{\pi\,|\Gamma(a)||\Gamma(b-a)|}\left(\frac{|\sin[(b-a)\pi]|}{\Gamma(1-\Re a+n)}+\frac{|\sin(a\pi)|}{\Gamma(1+\Re a-\Re b+n)}\right)}$$

$$H(z):=\begin{cases} e^{\Re z} & \text{if } \Re z>0,\\ 1 & \text{if } \Re z\le 0.\end{cases}$$

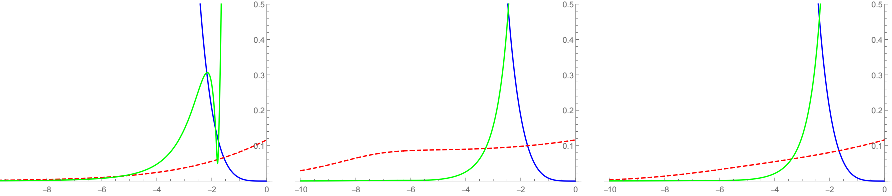
The remainder behaves as $R_n(a,b,z)\sim n^{-\min\{\Re a,\Re b-\Re a\}}$ as $n\to\infty$ uniformly in z with $\Re z\ge\Re z_0$, for any fixed $z_0\in\mathbb{C}$.

The confluent hypergeometric function



Graphics of $M(2.1+i, 4.2+1.2i; z)$ (red dashed) and the approximations for $n=3$ (blue), $n=5$ (green) in several intervals: $[-20, 0]$ (left), $[-20i, 0]$ (middle) and $[-20e^{i\pi/4}, 0]$ (right). Top graphics → real part; bottom graphics → imaginary part.

The confluent hypergeometric function



Relative errors in the third order approximation ($n = 3$) of $M(2.1 + i, 4.2 + 1.2i, z)$ by using the power series expansion (blue), the asymptotic expansion (green) and the uniform convergent expansion (red dashed) in the intervals $z \in [-10, 0]$ (left), $z \in [-10e^{-i\pi/3}, 0]$ (middle) and $z \in [-10e^{-i\pi/4}, 0]$ (right).

The Gauss hypergeometric function

The Gauss hypergeometric function

For $\Re a \geq 0$, $\Re c > \Re b > 0$, $z \in S_\theta$, with $0 < \theta \leq \pi/2$, and $n = 1, 2, 3, \ldots$,

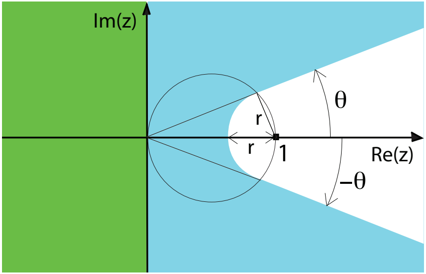
$$\boxed{{}_2F_1(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \sum_{k=0}^{n-1} A_k(b,c) H_k(z,a) + R_n(a,b,c;z)}$$

$$A_k(b,c) := 2^{k+2-c} \sum_{j=0}^{k} (-1)^j \frac{(1-b)_j (1+b-c)_{k-j}}{j!(k-j)!},$$

$$H_k(z,a) := \frac{(-1)^k}{2^k z^{k+1}} \sum_{j=0}^{k} \binom{k}{j} 2^j (z-2)^{k-j} \times \left[\frac{1-(1-z)^{j+1-a}}{j+1-a} (1-\delta_{j,a-1}) - \delta_{j,a-1} \log(1-z) \right]$$

Elementary functions

The Gauss hypergeometric function

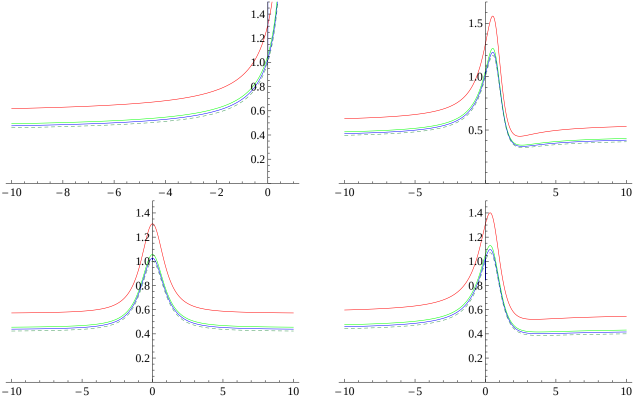


$$S_\theta := \{\theta \le |\arg(z)| \le \pi\} \cup \{z \in \mathbb{C}; |z-1/2| \le 1/2 \text{ and } |z-1| \ge \sin\theta\},$$

$$\boxed{|R_n(a,b,c;z)| \le \frac{2e^{\pi|\Im a|}|\Gamma(c)|\,\Gamma(1-\Re c+n)}{\pi\,|\Gamma(b)||\Gamma(c-b)|[\sin(\theta)]^{\Re a}}\left(\frac{|\sin[(c-b)\pi]|}{\Gamma(1-\Re b+n)}+\frac{|\sin(b\pi)|}{\Gamma(1+\Re b-\Re c+n)}\right)}$$

The remainder term behaves as $R_n(z,a,b,c) \sim n^{-\min\{\Re b, \Re c - \Re b\}}$ as $n \to \infty$ uniformly in $|z|$ in the extended sector S_θ.

The Gauss hypergeometric function



Plots of the absolute value of ${}_2F_1(0.5, 1.3, 2.5, z)$ (dashed) and the approximations for $n = 2$ (red), $n = 4$ (green) and $n = 6$ (blue) in several intervals: $[-10, 1]$ (top left), $[-10e^{i\pi/4}, 10e^{i\pi/4}]$ (top right), $[-10e^{i\pi/2}, 10e^{i\pi/2}]$ (bottom left) and $[-10e^{-i\pi/3}, 10e^{-i\pi/3}]$ (bottom right).

The confluent hypergeometric function U

Define $\mathcal{D} = \{z \in \mathbb{C};\ \Re z \geq \delta > 0\}$ and $b, c \in \mathbb{C}$ with $\Re c > 0$

$$U(c,b,z) = \frac{1}{\Gamma(c)} \int_0^\infty e^{-zu} u^{c-1} (1+u)^{b-c-1} du$$
$$= \frac{1}{\Gamma(c)} \int_0^1 t^{z-1} (-\log t)^{c-1} (1-\log t)^{b-c-1} dt.$$

- $g(t) = (-\log t)^{c-1} (1-\log t)^{b-c-1}$
- $h(t,z) = t^{z-1}$
- $\alpha = \delta - 1$, $\beta = 0$, any $0 < \sigma < 1$ and $\gamma = 1$ if $\Re c \geq 1$ or $\gamma = \Re c$ if $0 < \Re c < 1$
- We consider $t_1 = 1/2$ as the base point.

The confluent hypergeometric function U

$$U(c,b,z)=\frac{1}{\Gamma(c)}\left[\sum_{k=0}^{n-1}A_k(c,b)G_k(z)+R_n(z)\right]$$

- Moments:

$$G_k(z):=\int_0^1 t^{z-1}\left(t-\frac{1}{2}\right)^k dt=\sum_{j=0}^{k}\binom{k}{j}\left(\frac{-1}{2}\right)^{k-j}\frac{1}{z+j}.$$

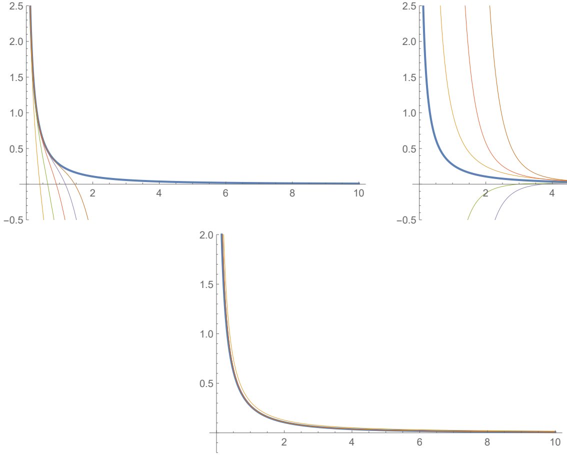
- Coefficients:

$$A_0(c,b)=(\log 2)^{c-1}(1+\log 2)^{b-c-1},$$

$$A_n(c,b)=\frac{A_0(c,b)}{n!}\sum_{k=1}^{n}\frac{(-1)^k b(n,k)}{(1+\log 2)^k}(b-c-k)_k$$

$$\times_2 F_1\left(1-c,-k;-c+b-k;1+\frac{1}{\log 2}\right),n\geq 1$$

The confluent hypergeometric function U



Approximations of $U(2, \frac{3}{2}, x)$ (thicker graphics) given by the Taylor expansion (left), the asymptotic expansion (middle) and the uniform expansion (right) for $x \in [0, 10]$ and $n = 1, 2, 3, 4, 5$. The approximations are similar for complex x and other values of c, b.

Content

The error function

Hindawi
International Journal of Mathematics and Mathematical Sciences
Volume 2018, Article ID 5146794, 12 pages
https://doi.org/10.1155/2018/5146794

Research Article

A New Special Function and Its Application in Probability

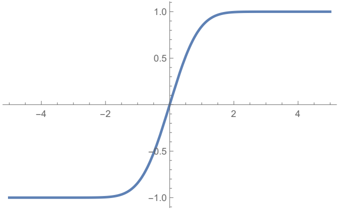
Zeraoulia Rafik,[1] Alvaro H. Salas,[2] and David L. Ocampo[2,3]

$$I(a) = \int_0^a e^{-x^2 \operatorname{erf} x} dx$$

The error function

The error function

$$\operatorname{erf} z = \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2} dt$$



Applications

- Statistics and probability theory.
- Uniform asymptotic expansions of integrals.
- Stokes phenomenon.

Uniform convergent expansions for the error function?

Integral representations

$$\operatorname{erf} z = \frac{z}{\sqrt{\pi}} \int_0^1 \frac{e^{-z^2 t}}{\sqrt{t}} dt$$

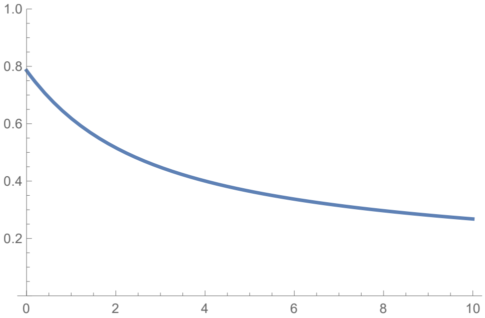
$$\operatorname{erfc} z = \frac{2}{\pi} e^{-z^2} \int_0^\infty \frac{e^{-z^2 t^2}}{t^2+1} \mathrm{d}t$$

$$\int_0^\infty \frac{e^{-at}}{\sqrt{t+z^2}} \mathrm{d}t = \sqrt{\frac{\pi}{a}} e^{az^2} \operatorname{erfc}\left(\sqrt{a} z\right)$$

No good results

The error function

$$F(a) := \frac{\pi}{4} e^{a}(1 - (\operatorname{erf}\sqrt{a})^2) = \int_0^1 \frac{e^{-at^2}}{1+t^2}\, dt, \quad \Re a > 0$$



The error function

Expansion

For $a \in \mathbb{C}$ with $\Re a > 0$, $n = 1, 2, 3, \ldots$,

$$F(a) = \sum_{k=0}^{n-1} (-1)^k \gamma_k(a) + R_n(a),$$

where the functions $\gamma_k(a)$ are the elementary functions

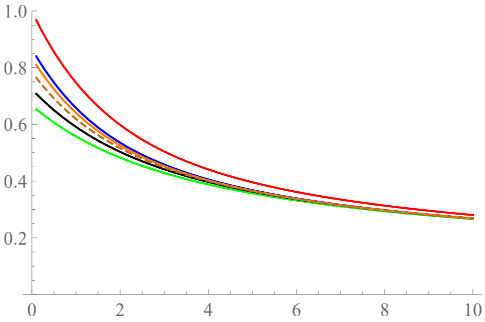
$$\gamma_k(a) := \int_0^1 e^{-at^2} t^{2k}\, dt = -\frac{e^{-a}}{2a} \sum_{j=0}^{k-1} \frac{(k-j+1/2)_j}{a^j} + \frac{\sqrt{\pi}}{2a^{k+1/2}} (1/2)_k \operatorname{erf} \sqrt{a},$$

that satisfy the recurrence relation

$$\gamma_k(a) = -\frac{e^{-a}}{2a} + \frac{2k-1}{2a} \gamma_{k-1}(a), \quad \gamma_0(z) = \frac{1}{2}\sqrt{\frac{\pi}{a}} \operatorname{erf} \sqrt{a}.$$

The error function

$$F(a) := \frac{\pi}{4} e^{a}(1 - (\operatorname{erf}\sqrt{a})^2) = \int_0^1 \frac{e^{-at^2}}{1+t^2}\, dt, \quad \Re a > 0$$



The error function

Second order equation in $x = \operatorname{erf}\sqrt{a}$

$$x^2 + \frac{4}{\sqrt{\pi}} e^{-a} A_n(a) x - 1 - \frac{4}{\pi} e^{-2a} B_n(a) + \frac{4}{\pi} e^{-a} R_n(a) = 0,$$

where

$$A_n(a) := \sum_{k=0}^{n-1} (-1)^k (1/2)_k \frac{1}{2a^{k+1/2}},$$

$$B_n(a) := \sum_{k=0}^{n-1} (-1)^k \sum_{j=0}^{k-1} \frac{(k-j+1/2)_j}{2a^{j+1}}.$$

The error function

Expansion for $\operatorname{erf}\sqrt{a}$

$$\operatorname{erf}\sqrt{a} \approx \frac{2}{\sqrt{\pi}} e^{-a} \frac{\frac{\pi}{4} e^{2a} + B_n(a)}{A_n(a) + \sqrt{A_n(a)^2 + \frac{\pi}{4} e^{2a} + B_n(a)}}$$

and

$$|r_n(a)| \leq \frac{4e^{-\Re a}}{\pi(2n+1)} \leq \frac{4}{\pi(2n+1)},$$

$$|r_n(a)| \leq \frac{4e^{-\Re a}}{\pi} \frac{\frac{1}{2}(\Re a)^{-n-1/2} n! (1-e^{-\Re a})^{n+\frac{1}{2}}}{1 + \frac{1}{\pi}(n-1)! \frac{|a|^{1/2-n} e^{-\Re a}}{n+|a|+1/2}}$$

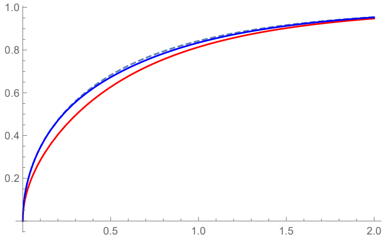
and when $\Re a \to 0^+$

$$|r_n(a)| \leq \frac{4}{\pi^2} \frac{(n+|a|+1/2)|a|^{n-1/2}}{(n-1)!}.$$

The error function

$$\operatorname{erf}(\sqrt{a}) \approx \frac{\sqrt{\pi}\sqrt{a}e^{a}}{\sqrt{1+\pi a e^{2a}}+1}$$

$$\operatorname{erf}(\sqrt{a}) \approx \frac{4\sqrt{a}e^{-a}\left(\pi e^{2a}a^{2}+3\right)}{\sqrt{\pi}\left(4a^{2}+\sqrt{16\pi e^{2a}a^{5}+16a^{4}+32a^{3}+28a^{2}-12a+9}-2a+3\right)}$$



The error function: Other expansions

- Power series

$$\operatorname{erf} z = \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{n!(2n+1)}, \quad z \in \mathbb{C}$$

- Other power series

$$\operatorname{erf} z = \frac{2}{\sqrt{\pi}} e^{-z^2} \sum_{n=0}^{\infty} \frac{2^n z^{2n+1}}{1 \cdot 3 \cdots (2n+1)}, \quad z \in \mathbb{C}$$

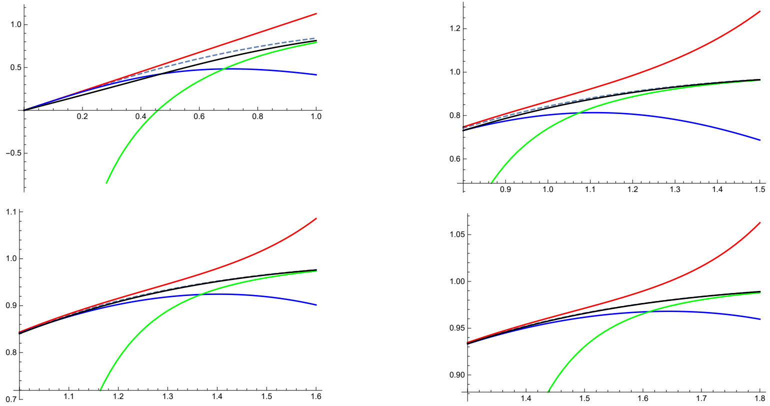
The error function: Other expansions

- Asymptotic expansion

$$\operatorname{erf} z \sim 1 - \frac{e^{-z^2}}{\sqrt{\pi}} \sum_{m=0}^{\infty} (-1)^m \frac{\left(\frac{1}{2}\right)_m}{z^{2m+1}}, \quad |\operatorname{ph} z| < \frac{3\pi}{4}$$

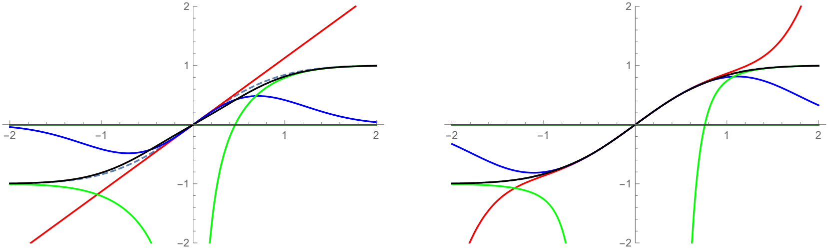
$$\operatorname{erf}(-z) \sim \frac{e^{-z^2}}{\sqrt{\pi}} \sum_{m=0}^{\infty} (-1)^m \frac{\left(\frac{1}{2}\right)_m}{z^{2m+1}} - 1$$

The error function



Graphics of $\operatorname{erf} a$ (dashed) and the power series (red), the other power series (blue), the asymptotic expansion (green) and our new expansion (black), for $n = 1, 3, 5$ and 7.

The error function



Graphics of $\operatorname{erf} a$ (dashed) and the power series (red), the other power series (blue), the asymptotic expansion (green) and our new expansion (black), for $n = 1$ and $n = 3$.

Content

Final remarks and future work

1. We have designed a general theory of uniformly convergent approximations of special functions based on their integral representations.
2. Apply the method to other special functions.
3. Investigate the stability of the recurrence relations for the coefficients if they are applied forward or look for other recurrence relations.
4. Investigate if the new expansions can be interested from a numerical point of view depending on the range of the variable: for moderate values. The empirical results point in that direction.
5. For intermediate values, compare the results with Chebyshev expansions or quadrature formulas.

Thank you for your attention!