Multiple orthogonal polynomials and branched continued fractions

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Overview

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- Along any increasing nearest-neighbor path $\mathbf{n}_0 = \mathbf{0}, \mathbf{n}_1, \mathbf{n}_2, \ldots$ in $\mathbb{N}^r$, the sequence $\widehat{P}_k(x) = P_{\mathbf{n}_k}(x)$ satisfies an $(r+2)$-term recurrence

$$x\widehat{P}_n(x) \;=\; \sum_{k=n-r}^{n+1} \pi_{nk} \, \widehat{P}_k(x)$$

with $\pi_{n,n+1} = 1$: $(r,1)$-banded unit-lower-Hessenberg matrix Π

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- Write $A = \mathcal{O}(\Pi)$

Classical continued fractions

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$$\sum_{n=0}^{\infty} \underbrace{S_n(\boldsymbol{\alpha})}_{\text{Stieltjes–Rogers polynomial}} \; t^n \;=\; \cfrac{1}{1 - \cfrac{\alpha_1 t}{1 - \cfrac{\alpha_2 t}{1 - \cdots}}}$$

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- This is combinatorialists' notation. Analysts take $t^n \to \dfrac{1}{z^{n+1}}$

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Theorem (Flajolet 1980)

- The Jacobi–Rogers polynomial $J_n(\boldsymbol{\beta}, \boldsymbol{\gamma})$ is the generating polynomial for Motzkin paths of length n, in which each rise gets weight 1, each level step at height i gets weight γ_i, and each fall from height i gets weight β_i.
- The Stieltjes–Rogers polynomial $S_n(\boldsymbol{\alpha})$ is the generating polynomial for Dyck paths of length $2n$, in which each rise gets weight 1 and each fall from height i gets weight α_i.

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- J is output matrix for tridiagonal production matrix

$$\Pi = \begin{bmatrix} \gamma_0 & 1 & & & \\ \beta_1 & \gamma_1 & 1 & & \\ & \beta_2 & \gamma_2 & 1 & \\ & & \ddots & \ddots & \ddots \end{bmatrix}$$

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- Analogously: Partial Dyck paths from $(0,0) \to (2n, 2k)$
  $\longrightarrow$ Generalized Stieltjes–Rogers polynomials $S_{n,k}(\boldsymbol{\alpha})$

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- Define *m*-Stieltjes–Rogers polynomial $S_n^{(m)}(\boldsymbol{\alpha})$ to be generating polynomial for *m*-Dyck paths of length $(m+1)n$ in which each rise gets weight 1 and each *m*-fall from height i gets weight α_i.

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- Generalizing what Flajolet did for $m = 1$: Their generating function can be written as a branched continued fraction ...

Branched continued fractions

Pétréolle-A.S.-Zhu 2018

$$\sum_{n=0}^{\infty} S_n^{(m)}(\boldsymbol{\alpha})\, t^n$$

$$= \cfrac{1}{1 - \alpha_m t \displaystyle\prod_{i_1=1}^{m} \cfrac{1}{1 - \alpha_{m+i_1} t \displaystyle\prod_{i_2=1}^{m} \cfrac{1}{1 - \alpha_{m+i_1+i_2} t \displaystyle\prod_{i_3=1}^{m} \cfrac{1}{1 - \cdots}}}}$$

$$= \cfrac{1}{1 - \cfrac{\alpha_m t}{\left(1 - \cfrac{\alpha_{m+1} t}{\left(1 - \cfrac{\alpha_{m+2} t}{(\cdots)\cdots(\cdots)}\right)\cdots\left(1 - \cfrac{\alpha_{2m+1} t}{(\cdots)\cdots(\cdots)}\right)}\right)\cdots\left(1 - \cfrac{\alpha_{2m} t}{\left(1 - \cfrac{\alpha_{2m+1} t}{(\cdots)\cdots(\cdots)}\right)\cdots\left(1 - \cfrac{\alpha_{3m} t}{(\cdots)\cdots(\cdots)}\right)}\right)}}$$

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- E.g. for $m = 2$:

$$\Pi = \begin{bmatrix} \alpha_2 & 1 & & & \\ \alpha_2\alpha_3 + \alpha_2\alpha_4 & \alpha_3 + \alpha_4 + \alpha_5 & 1 & & \\ \alpha_2\alpha_4\alpha_6 & \alpha_4\alpha_6 + \alpha_5\alpha_6 + \alpha_5\alpha_7 & \alpha_6 + \alpha_7 + \alpha_8 & 1 & \\ & \ddots & \ddots & \ddots & \ddots \end{bmatrix}$$

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- Sequence $\mathbf{L} = (\mathcal{L}_k)_{k \ge 0}$ of linear functionals and sequence $\mathbf{P} = (P_n(x))_{n \ge 0}$ of monic polynomials are dual to each other in case $\mathcal{L}_k(P_n(x)) = \delta_{kn}$

Duality between linear functionals and monic polynomials

Proposition (very easy)

Given any sequence $(P_n(x))_{n \geq 0}$ of monic polynomials, there exists a unique sequence $(\mathcal{L}_k)_{k \geq 0}$ of linear functionals satisfying $\mathcal{L}_k(P_n(x)) = \delta_{kn}$, and it is normalized.

Conversely, given any normalized sequence $(\mathcal{L}_k)_{k \geq 0}$ of linear functionals, there exists a unique sequence $(P_n(x))_{n \geq 0}$ of monic polynomials that satisfies $\mathcal{L}_k(P_n(x)) = \delta_{kn}$.

The relation between these sequences is:

> *The representing matrix A of the sequence $(\mathcal{L}_k)_{k \geq 0}$ and the representing matrix B of the sequence $(P_n(x))_{n \geq 0}$ are inverses of each other:* $B = A^{-1}$

Linear recurrence $\longleftrightarrow$ production matrix

Proposition (quite easy)

Given any sequence $(P_n(x))_{n \ge 0}$ of monic polynomials, there exists a unique unit-lower-Hessenberg matrix $\Pi = (\pi_{nk})_{n,k \ge 0}$ such that

$$x P_n(x) \;=\; \sum_{k=0}^{n+1} \pi_{nk} \, P_k(x)$$

Conversely, given any unit-lower-Hessenberg matrix $\Pi = (\pi_{nk})_{n,k \ge 0}$, there exists a unique sequence $(P_n(x))_{n \ge 0}$ of polynomials satisfying this recurrence with the initial condition $P_0(x) = 1$, and it is monic.

The relation between these objects is: The representing matrix B of the sequence $(P_n(x))_{n \ge 0}$ satisfies $B = \mathcal{O}(\Pi)^{-1}$ or equivalently $\Pi = B \Delta B^{-1}$.

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and these correspondences are given by

$$A = \mathcal{O}(\Pi) = B^{-1}$$

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- Now

$$\Gamma_k(P_n(x)) \;=\; \sum_{j=0}^{n} b_{nj}\,\gamma_{jk} \;=\; (B\Gamma)_{nk}$$

So $\mathbf{P}$ is orthogonal to $\mathbf{\Gamma} \iff B\Gamma$ vanishes below the diagonal $\iff$ $B\Gamma$ is an upper-triangular matrix $U \iff \Gamma = B^{-1}U$

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  - But $\mathbf{P}$ is also orthogonal with respect to *any* sequence $\mathbf{\Gamma} = (\Gamma_k)_{k \ge 0}$ where $\Gamma = AU$ with U upper-triangular, i.e. Γ_k is any linear combination of $\mathcal{L}_0, \ldots, \mathcal{L}_k$.

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- Such a sequence $\mathbf{P}$ exists (and is unique) whenever R is a field and all the leading principal minors $\Delta_1, \Delta_2, \ldots$ of Γ are nonzero.

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- Continue to assume that R is a field and all the leading principal minors $\Delta_1, \Delta_2, \ldots$ of the Hankel matrix $\Gamma = H_\infty(\boldsymbol{\ell})$ are nonzero.
- Then $\boldsymbol{\ell}$ is given by a classical J-fraction

$$\sum_{n=0}^{\infty} \ell_n \, t^n \;=\; \cfrac{1}{1 - \gamma_0 t - \cfrac{\beta_1 t^2}{1 - \gamma_1 t - \cfrac{\beta_2 t^2}{1 - \gamma_2 t - \cdots}}}$$

with tridiagonal production matrix

$$\Pi \;=\; \begin{bmatrix} \gamma_0 & 1 & & & \\ \beta_1 & \gamma_1 & 1 & & \\ & \beta_2 & \gamma_2 & 1 & \\ & & \ddots & \ddots & \ddots \end{bmatrix}$$

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- **Fact:** The Hankel matrix $\Gamma = H_\infty(\ell)$ has the *LDL*$^{\mathrm{T}}$ factorization

$$\Gamma \;=\; \mathsf{J} D \mathsf{J}^{\mathrm{T}}$$

where $\mathsf{J} = \mathcal{O}(\Pi)$ is the unit-lower-triangular matrix of generalized Jacobi–Rogers polynomials, and $D = \mathrm{diag}(1, \beta_1, \beta_1\beta_2, \ldots)$.

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- All this is of course well-known! (but it's nice to recover)

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- Now set $\Gamma_k = \mathcal{L}^{\star, k+1}$: then the sequence $\widehat{\mathbf{P}} = (\widehat{P}_k(x))_{k \geq 0}$ is orthogonal to the sequence $\mathbf{\Gamma} = (\Gamma_k)_{k \geq 0}$.

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- The sequence $\widehat{P}_k(x) = P_{\mathbf{n}_k}(x)$ satisfies an $(r+2)$-term recurrence

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- **Special case:** Find recurrence for MOPs along the stepline.
  Matrix Π is a production matrix for the moments of $\mathcal{L}^{(1)}$.

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- Now fix $a_1, a_2 > 0$ and consider $(\mu_1, \mu_2) = (\mu_{a_1,a_2}, \mu_{a_1+1,a_2})$

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$$\int\limits_0^\infty x^n\, d\mu_{a_1,a_2}(x) \;=\; \frac{\Gamma(a_1+n)\,\Gamma(a_2+n)}{\Gamma(a_1)\,\Gamma(a_2)} \;=\; a_1^{\overline{n}}\, a_2^{\overline{n}}$$

- Now fix $a_1, a_2 > 0$ and consider $(\mu_1, \mu_2) = (\mu_{a_1,a_2}, \mu_{a_1+1,a_2})$
- Let $P_{\mathbf{n}}(x)$ be the (monic) MOPs polynomials of type II, and let $\widetilde{P}_n(x)$ be those polynomials on the stepline:

$$\widetilde{P}_{2k}(x) \;=\; P_{k,k}(x)\,, \qquad \widetilde{P}_{2k+1}(x) \;=\; P_{k+1,k}(x)$$

Example: Bessel K_ν weights, continued

- Van Assche + Yakubovich computed 4-term recurrence:

$$x\,\widetilde{P}_n(x) \;=\; \widetilde{P}_{n+1}(x) \,+\, \pi_{n,n}\widetilde{P}_n(x) \,+\, \pi_{n,n-1}\widetilde{P}_{n-1}(x) \,+\, \pi_{n,n-2}\widetilde{P}_{n-2}(x)$$

where

$$\begin{aligned}
\pi_{n,n} &= a_1 a_2 \,+\, (2a_1+2a_2-1)n \,+\, 3n^2 \\
\pi_{n,n-1} &= n(a_1+n-1)(a_2+n-1)(a_1+a_2+3n-2) \\
\pi_{n,n-2} &= n(n-1)(a_1+n-1)(a_1+n-2)(a_2+n-1)(a_2+n-2)
\end{aligned}$$

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- Then $P_n^{(m)}(a_1, \ldots, a_m; a_{m+1}) = S_n^{(m)}(\boldsymbol{\alpha})$ where $S_n^{(m)}$ is the m-Stieltjes–Rogers polynomial and the coefficients $\boldsymbol{\alpha} = (\alpha_i)_{i \geq m}$ are

$$\boldsymbol{\alpha} \;=\; a_1 \cdots a_m,\; a_2 \cdots a_{m+1},\; a_3 \cdots a_{m+1}(a_1+1),\; a_4 \cdots a_{m+1}(a_1+1)(a_2+1),\; \ldots$$

(m-fold products of shifted a_i's)

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- Plug in: Agrees with Van Assche–Yakubovich!

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A big thank you to Walter for helping to discover this!