# LOCALLY ANALYTIC REPRESENTATIONS OF $p$-ADIC GROUPS

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## Lecture 1

The $p$-adic representation theory of $p$-adic groups is the subject of much ongoing research. It is motivated by the desire to better understand the conjectural $p$-adic Langlands correspondence, but it is also an interesting branch of representation theory in its own right. In these lectures, we introduce the notion of locally analytic representations, and a particularly nice subclass of them called admissible representations.

Today: we focus on the topological/analytic background.

**Non-archimedean fields.** Throughout, $K$ will denote our base field.

**Definition 1.** A non-archimedean absolute value (NAAV) on $K$ is a function $|\cdot| : K \to \mathbb{R}$ such that for all $a, b \in K$:

(i) $|a| \geq 0$;
(ii) $|a| = 0 \iff a = 0$;
(iii) $|a \cdot b| = |a| \cdot |b|$; and
(iv) $|a + b| \leq \max\{|a|, |b|\}$.

This gives a metric on $K$ via $d(a, b) := |a - b|$, making $K$ into a topological field. The unit ball $\mathcal{O}_K := \{a \in K : |a| \leq 1\}$ is a subring.

**From now on, we assume that $K$ is equipped with a NAAV and that it is complete, i.e. Cauchy sequences converge.**

*Remark.* We can more generally topologise $K^n$ for any $n$ by equipping it with the norm $||(a_1, \ldots, a_n)|| := \max\{|a_1|, \ldots, |a_n|\}$.

**Canonical example: the $p$-adics.** Let $p$ be a prime number and let $a \in \mathbb{Q}$. Define $|a|_p := p^{-r}$ if $a = p^r \cdot \frac{m}{n}$ where $(m, p) = (n, p) = 1$. This is a NAAV on $\mathbb{Q}$ (exercise). The completion of $\mathbb{Q}$ with respect to $|\cdot|_p$ is denoted by $\mathbb{Q}_p$, the *field of $p$-adic numbers*, and the unit ball of $\mathbb{Q}_p$ is denoted by $\mathbb{Z}_p$, the ring of *$p$-adic integers*. Moreover, the NAAV on $\mathbb{Q}_p$ extends uniquely to a NAAV an $K$ for any finite field extension $K/\mathbb{Q}_p$.

Concretely, elements of $\mathbb{Z}_p$ are 'infinite base $p$ expansions', i.e. can be represented uniquely as a series

$$a_0 + a_1 p + a_2 p^2 + \ldots + a_n p^n + \ldots,$$

where $a_i \in \{0, 1, \ldots, p-1\}$ for all $i$. We then have $\mathbb{Q}_p = \mathbb{Z}_p[1/p]$.

**Back to general $K$.** Convergence of series will be important to us. The following will be crucial:

**Fact/exercise.** If $(a_n)$ is a sequence in $K$, then

$$\sum_{n \geq 0} a_n \text{ converges} \iff a_n \to 0 \text{ as } n \to \infty.$$

This is a consequence of (iv) in the definition of a NAAV.

In particular, as a function, a power series $f(x) = \sum_{n \geq 0} a_n x^n$ makes sense (i.e. converges) on a ball $B_\varepsilon(0) := \{a \in K : |a| \leq \varepsilon\}$ if and only if $\varepsilon^n |a_n| \to 0$ as $n \to \infty$.

**$p$-adic Lie groups.** Given $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$, we adopt the notation $|\alpha| := \alpha_1 + \dots + \alpha_n$ and $t^\alpha := t_1^{\alpha_1} \cdots t_n^{\alpha_n}$.

**Definition 2.** If $U \subseteq K^n$ is open, then a function $f : U \to K^m$ is *locally analytic* if for all $x_0 \in U$, there exists $\varepsilon > 0$ and power series $F_i(t) = \sum_{\alpha \in \mathbb{N}^n} a_{\alpha,i} t^\alpha$ $(1 \leq i \leq m)$, where $a_{\alpha,i} \in K$ and $\varepsilon^{|\alpha|} \cdot |a_{\alpha,i}| \to 0$ as $|\varepsilon| \to \infty$, such that for all $x \in U$ with $||x - x_0|| \leq \varepsilon$ we have $f(x) = (F_1(x - x_0), \dots, F_m(x - x_0))$.

*Remark.* This notion can be generalised to functions $f : U \to V$ where $V$ is a suitable (locally convex, Hausdorff) topological vector space. These are the functions which can be locally described by converging power series with coefficients in $V$.

Next we introduce manifolds:

**Definition 3.** Let $M$ be a Hausdorff topological space. An *atlas of dimension* $n$ on $M$ is a set $\mathcal{A} = \{(U_i, \varphi_i)\}_{i \in I}$ such that

- $U_i \subset M$ is open for all $i \in I$ and $M = \bigcup_{i \in I} U_i$;
- $\varphi_i : U_i \to K^n$ is a homeomorphism onto an open subset of $K^n$ for all $i \in I$; and
- for all $i, j \in I$, the maps

$$\varphi_i(U_i \cap U_j) \underset{\varphi_i \circ \varphi_j^{-1}}{\overset{\varphi_j \circ \varphi_i^{-1}}{\rightleftarrows}} \varphi_j(U_i \cap U_j)$$

  are locally analytic

We say $M$ is a (locally $K$-analytic) *manifold of dimension* $n$ if it is equipped with such an atlas, and the pairs $(U_i, \varphi_i)$ are called *charts*. We say a map $f : M \to K^m$ is *locally analytic* if $f \circ \varphi^{-1} : \varphi(U) \to K^m$ is locally analytic for each chart $(U, \varphi)$ of $M$.

Finally we can talk about groups:

**Definition 4.** A manifold $G$ is a *Lie group* if it is a group such that the multiplication $m : G \times G \to G$ is locally analytic.

**Examples.**

(i) $(K^n, +)$ or $(\mathcal{O}_K^n, +)$.

(ii) $(K^\times, \cdot)$ or $(\mathcal{O}_K^\times, \cdot)$.

(iii) $(1 + p\mathbb{Z}_p, \cdot) \leq (\mathbb{Q}_p^\times, \cdot)$, i.e. elements of the form $1 + a_1 p + a_2 p^2 + \dots$.

(iv) $\mathsf{GL}_n(K)$, $\mathsf{GL}_n(\mathcal{O}_K)$, $\mathsf{SL}_n(K)$, $\mathsf{SL}_n(\mathcal{O}_K)$.

(v) Closed subgroups of $\mathsf{GL}_n(K)$ are Lie groups, such as the Borel subgroup

$$B = \left\{ \begin{pmatrix} * & \cdots & * \\ & \ddots & \vdots \\ & & * \end{pmatrix} \in \mathsf{GL}_n(K) \right\}$$

and the maximal torus

$$T = \left\{ \begin{pmatrix} * & & \\ & \ddots & \\ & & * \end{pmatrix} \in \mathsf{GL}_n(K) \right\}.$$

(vi) We also have the *Iwahori subgroup* of $\mathsf{GL}_2(\mathbb{Z}_p)$

$$I = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathsf{GL}_2(\mathbb{Z}_p) : c \in p\mathbb{Z}_p \right\}.$$

(vii) The $K$-valued points of any connected algebraic group over $K$.

All of these examples are algebraic in nature, but the point of the analytic setup is that we may study a class of representations larger than the algebraic ones.

**Locally analytic representations.** From now on, we fix complete non-archimedean fields $L \subseteq K$ such that the NAAV on $K$ extends the one on $L$, and we fix $G$ a locally $L$-analytic Lie group. We will study representations of $G$ on $K$-vector spaces. We assume that $V$ is a suitable topological $K$-vector space (locally convex, Hausdorff) so that we can talk about locally analytic functions $f : G \to V$. We write

$$C^{\mathsf{an}}(G, V) := \{f : G \to V | f \text{ locally analytic}\}.$$

**Definition 5.** A representation $\rho : G \to \mathsf{GL}(V)$ is *locally analytic* if for each $v \in V$, the map $g \mapsto \rho(g)v$ belongs to $C^{\mathsf{an}}(G, V)$.

*Remark.* This only depends on each vector $v \in V$, so given *any* representation on $V$, it makes sense to consider the *locally analytic vectors*

$$V^{\mathsf{an}} := \{v \in V : (g \mapsto \rho(g)v) \in C^{\mathsf{an}}(G, V)\},$$

a locally analytic subrepresentation.

We finish with some examples.

**Examples.**
(i) If $G$ is algebraic (e.g. $\mathsf{GL}_n(K)$) then any algebraic representation of $G$ is locally analytic.
(ii) If $G = (\mathbb{Z}_p, +)$, we can define a character $\chi : G \to K^\times$ as follows. Pick $z \in K^\times$ such that $|z - 1| < 1$. Then, for $a \in \mathbb{Z}_p$, set

$$\chi(a) = z^a := \sum_{n=0}^{\infty} (z-1)^n \binom{a}{n}.$$

Here the binomial coefficient is defined as $\binom{a}{n} = \frac{a(a-1)\dots(a-n+1)}{n!} \in \mathbb{Q}_p$. It follows from a theorem of Amice that $\chi$ is locally analytic.
(iii) Let $G = \mathsf{GL}_2(\mathbb{Q}_p)$, $B$ the Borel subgroup, $T$ the maximal torus. Let $\chi : T \to K^\times$ be a locally analytic character. As $T$ is a quotient of $B$, we may lift $\chi$ to $B$. Then we have the locally analytic induction

$$\mathsf{Ind}_B^G(\chi) := \{f \in C^{\mathsf{an}}(G, K) : f(gb) = \chi(b^{-1})f(g)\ \forall g \in G, b \in B\}.$$

This is a locally analytic representation of $G$ when $G$ acts by left translation, called a *principal series* representation.
(iv) When $\chi = \mathbf{1}$ in (iii), we have a natural injection $\mathbf{1}_G \to \mathsf{Ind}_B^G(\mathbf{1})$ with image the constant functions $G \to K$. The quotient $\mathsf{St} := \mathsf{Ind}_B^G(\mathbf{1})/\mathbf{1}_G$ is called the *Steinberg representation*.

*Remark.* Even if $G = (\mathbb{Z}_p, +)$, we can construct infinitely many irreducible, infinite dimensional, locally analytic representations. If $z \in K^\times$ as in example (ii) and $z$ is transcendental over $\mathbb{Q}_p$, and assuming that $K$ is the smallest complete field containing $z$, then Diarra showed that $K$ is an irreducible $\mathbb{Q}_p$-representation of $G$ via

$$\rho(a)v = \sum_{n=0}^{\infty} (z-1)^n \binom{a}{n} v.$$

Hence locally analytic representations are too wild to study in general. We need a nicer subclass of representations within it.

## Lecture 2

The analytic nature of both the groups and representations makes it hard to work with them directly. In order to study these representations more algebraically, we define an algebra $D(G, K)$ such that

$$\left\{ \begin{array}{l} \text{sufficiently nice} \\ \text{loc. an. representations} \end{array} \right\} \leftrightarrow \left\{ \begin{array}{l} \text{sufficiently nice} \\ D(G,K)\text{-modules} \end{array} \right\}.$$

Here, 'sufficiently nice' will have to be some topological properties. Later, we will see how to replace some of these topological properties with more algebraic ones.

We fix fields $\mathbb{Q}_p \subseteq L \subseteq K$ with $L/\mathbb{Q}_p$ finite, and assume further that $K$ is a *spherically complete* with respect to a NAAV extending the one on $L$ (don't worry about what that means, it's a technical condition to ensure duals are non-zero and it is satisfied e.g. if $[K:\mathbb{Q}_p] < \infty$). As last time, we will be studying $K$-representations of locally $L$-analytic groups.

**Quick overview of topological notions.** (sketchy)

- All our $K$-vector spaces are *locally convex*, i.e. their topology is given by a family of seminorms.
- Given $V$ locally convex, its *continuous dual* is
$$V' := \{f : V \to K \text{ linear } | f \text{ is continuous}\}.$$
This dual is itself locally convex via the strong topology (analogue of topology of uniform convergence).
- We say $V$ is a *Banach space* if its topology is given by a single norm and if it is complete.
- More generally, $V$ is called *Fréchet* if it is metrizable and complete.
- We say $V$ is *reflexive* if $V \cong (V')'$.

**The distribution algebra.** Recall that given a locally $L$-analytic manifold $M$, we have $C^{\mathsf{an}}(M,K) = \{f : M \to K | f \text{ is locally analytic}\}$.

**Definition 6.** With $M$ as above, the *space of distributions* on $M$ is the dual $D(M,K) := C^{\mathsf{an}}(M,K)'$.

**Facts.**
- (i) If $M$ is compact then $D(M,K)$ is Fréchet (i.e. is nice).
- (ii) If $M = \coprod_{i \in I} M_i$, where the $M_i$ are pairwise disjoint compact open subsets, then $D(M,K) = \bigoplus_{i\in I} D(M_i,K)$ topologically. This is useful when $M = G$ is a Lie group and the $M_i$ are left cosets of some compact open subgroup $G_0$ (e.g. $G = \mathsf{GL}_n(\mathbb{Q}_p)$ and $G_0 = \mathsf{GL}_n(\mathbb{Z}_p)$).
- (iii) When $M = G$ is a Lie group, $D(G,K)$ is in fact a $K$-algebra.

From now on, $M = G$ is a Lie group.

**Dirac distributions.** Given $g \in G$, we have an element $\delta_g \in D(G,K)$ given by $\delta_g(f) := f(g)$ for $f \in C^{\mathsf{an}}(G,K)$. This gives an injection $G \to D(G,K)$, $g \mapsto \delta_g$.

**The convolution product.** We now sketch the construction of the product on $D(G,K)$. The key fact is that there is an isomorphism
$$D(G \times G, K) \cong D(G,K)\widehat{\otimes}_K D(G,K)$$
where this denotes some completion of the usual algebraic tensor product. Also, the group multiplication $m : G \times G \to G$ induces a map $C^{\mathsf{an}}(G,K) \to C^{\mathsf{an}}(G\times G,K)$, $f \mapsto f \circ m$. Dually this gives a map $D(G\times G,K) \to D(G,K)$.

Given $u, v \in D(G,K)$, we define their *convolution* $u * v$ to be the image of $u \otimes v$ under the composite
$$D(G,K)\widehat{\otimes}_K D(G,K) \xrightarrow{\cong} D(G\times G,K) \to D(G,K).$$
When $G$ is finite, $D(G,K)$ is just the group algebra $KG$ and this is the usual multiplication.

**Theorem 7** (Féaux de Lacroix)**.** *Convolution defines a separately continuous product on $D(G,K)$ with unit $\delta_1$. When $G$ is compact, this makes $D(G,K)$ into a Fréchet algebra i.e. $* : D(G,K) \times D(G,K) \to D(G,K)$ is continuous.*

The main moral of the story to come is that we gain more control by working with $D(G,K)$-modules rather than locally analytic $G$-representations directly.

**Explicit description of the distribution algebra.** What does the distribution algebra look like concretely?
So far the only elements we have come across are the Dirac delta distributions $\delta_g$ for $g \in G$,

$$\delta_g(f) = f(g),\ f \in C^{\mathrm{an}}(G,K).$$

**Lemma 8.** *The map $g \mapsto \delta_g$ is a continuous map of monoids $G \to D(G,K)$, i.e. $\delta_{gh} = \delta_g \cdot \delta_h$ for any $g,h \in G$.*

In particular, there is a natural algebra morphism $K[G] \to D(G,K)$. If $G$ is a compact group, we can even go further: there is the notion of a completed group algebra (or Iwasawa algebra) $K[[G]]$, and by continuity, we obtain a continuous algebra morphism $\theta : K[[G]] \to D(G,K)$.

**Theorem 9.** *If $L = \mathbb{Q}_p$, then $\theta$ is a faithfully flat injection.*

In other words, we can study the ('more classical') $K[[G]]$-modules by passing to $D(G,K)$-modules, applying $D(G,K) \otimes_{K[[G]]} -$ without losing any information.

So $D(G,K)$ contains the group algebra. But it also contains distributions induced from the Lie algebra:

Let $\mathfrak{g}$ be the Lie algebra of $G$, e.g. $G = \mathrm{SL}_2(\mathbb{Z}_p)$, $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{Z}_p)$. If $x \in \mathfrak{g}$, we can form the distribution $\mathrm{dist}(x)$ by

$$\mathrm{dist}(x)(f) = \frac{\mathrm{d}}{\mathrm{d}t}(f(\exp(tx)))|_{t=0}.$$

This gives a linear map $\mathrm{dist} : \mathfrak{g} \to D(G,K)$, sending $[x,y]$ to the commutator $\mathrm{dist}(x)\mathrm{dist}(y) - \mathrm{dist}(y)\mathrm{dist}(x)$ - so we obtain an algebra morphism $U(\mathfrak{g})_K \to D(G,K)$.

**Lemma 10.** *The map is injective. The closure of $U(\mathfrak{g})_K$ in $D(G,K)$ is a Frechet algebra which we denote by $\widehat{U(\mathfrak{g})}_K$.*

At first, the object $\widehat{U(\mathfrak{g})}_K$ might seem strange, but its elements are actually very concrete. If $x_1, \ldots, x_d$ is an ordered $K$-basis of $\mathfrak{g} \otimes K$, then by the PBW theorem, $U(\mathfrak{g})_K$ admits a $K$-basis of the form

$$x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_d^{\alpha_d},$$

where $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}_0^d$. Now an arbitrary element of $\widehat{U(\mathfrak{g})}_K$ can be written uniquely as

$$\sum_{\alpha \in \mathbb{N}_0^d} \lambda_\alpha x^\alpha,\ \lambda_\alpha \in K,\ \pi^{-|\alpha| n}\lambda_\alpha \to 0 \text{ as } |\alpha| \to \infty\ \forall n.$$

**Proposition 11.** *The Dirac distributions $\delta_g$ generate a dense subspace of $D(G,K)$.*

*Proof.* Idea: $D(G,K)' \cong C^{\mathrm{an}}(G,K)$, and an element $f$ of $C(G,K)$ is zero if and only if $f(g) = \delta_g(f)$ is zero for all $g$. □

So $D(G,K)$ is like a topological group algebra that is sufficiently thickened to also incorporate the infinitesimal information, present in form of the Lie algebra.

What is the relation between locally analytic $G$-representations and $D(G,K)$-modules? Just as with the usual group algebra, a locally analytic $G$-representation carries a $D(G,K)$-module structure (this is actually a bit subtle to show). It turns out however that it is more useful to dualize this operation to get a better handle on the topology.

**Theorem 12.** *There is an anti-quivalence of categories*

$$\{\textit{loc an } G\textit{-reps of cpct type}\} \to \{\textit{sep cts } D(G,K)\textit{-mods in nuclear Frechet spaces}\}$$
$$V \mapsto V'.$$

**Remark.** We won't explain all the topological notions in detail: compact type is a property that ensures that $V$ is reflexive, i.e. $(V')' \cong V$, and nuclear Frechet spaces can be thought of as the Frechet spaces which are dual to those of compact type.

## Lecture 3

Last time we saw how we can think of locally analytic $G$-representations (of compact type) as certain topological modules over the distribution algebra $D(G,K)$. The problem persists however that these are topological modules, and doing algebra with topological objects is hard!

**Toy model: Noetherian Banach algebras and finitely generated modules.** Let $A$ be a Noetherian Banach $K$-algebra, i.e. it is a Noetherian $K$-algebra which is complete with respect to some (submultiplicative) norm. The category of normed $A$-modules (or of Banach modules if we insist on completeness) is problematic for the very same reason. But here there is an excellent remedy.

**Theorem 13.** *Any (abstract) finitely generated $A$-module can be endowed with a canonical Banach norm such that any $A$-module map between finitely generated modules is automatically continuous. These norms are compatible with the formation of submodules, quotients, and direct sums.*
*More abstractly: there is a fully faithful functor from (abstract!) finitely generated $A$-modules to the category of Banach $A$-modules, exhibiting the former as an (abelian!) subcategory of the latter.*

*Proof.* If $M$ is a finitely generated $A$-module, there exists some surjection $A^r \to M$. We can check that this endows $M$ with a Banach norm which has the property that $M \to N$ is continuous if and only if the composition $A^r \to M \to N$ is. But if $N$ is another finitely generated $A$-module endowed with such a norm, then any map $A^r \to N$ is a sum of action maps and hence continuous. This shows that any $A$-module map $M \to N$ is automatically continuous. In particular (taking $M = N$), Banach norms arising from a different generating set give rise to an equivalent norm. Now check that any submodule of a finitely generated $A$-module is a closed subspace with respect to this norm by reducing to the case of $A^r$. □

**Remark.** This works e.g. for the Tate algebra

$$K\langle x\rangle = \{\sum_{\mathbb{N}_0} a_i x^i : |a_i| \to 0\}$$

of analytic functions on the unit disk, ensuring that $p$-adic analytic geometry (and its theory of coherent modules) is well-behaved.

It turns out that $D(G,K)$ is hardly ever Noetherian Banach. But it is the next best thing.

**Definition 14.** Let $A$ be a Frechet $K$-algebra. We say that $A$ is a **Frechet-Stein algebra** if $A$ can be written as $A = \varprojlim A_n$, where each $A_n$ is a Noetherian Banach $K$-algebra such that $A_{n+1} \to A_n$ has dense image and turns $A_n$ into a flat $A_{n+1}$-module on both sides.
An $A$-module $M$ is called **coadmissible** if $M = \varprojlim M_n$, where $M_n$ is a finitely generated $A_n$-module, and the natural morphism $A_n \otimes_{A_{n+1}} M_{n+1} \to M_n$ is an isomorphism.

Example: Let

$$A_n = K\langle \pi^n x\rangle = \left\{\sum a_i x^i : \pi^{-in} a_i \to 0\right\}$$

be the ring of analytic function on a disk of radius $|\pi|^{-n}$. Then $A = \varprojlim A_n$ is the ring of analytic functions on $X = \varinjlim \mathrm{Sp} A_n$, the 'affine line'. Then $A$ is a Frechet-Stein algebra, and coadmissible $A$-modules are precisely the global sections of coherent $\mathcal{O}_X$-modules.

Just as in this example, the $M_n$ can in general be recovered from $M$, which allows to go back and forth between $M$ and its 'Noetherian levels'.

**Lemma 15.** *If $M$ is a coadmissible $A$-module, then the natural morphism $A_n \otimes_A M \to M_n$ is an isomorphism.*

**Theorem 16.** *Any coadmissible $A$-module can be endowed with a canonical Frechet topology, such that any $A$-module morphism between coadmissible $A$-modules is automatically continuous.*

*Proof.* Equip each $M_n$ with its canonical Banach norm and take the limit. Any morphism $M \to N$ then gives rise to $A_n$-module morphisms $M_n \to N_n$ by the lemma, and these are continuous by our toy model. Thus $M \to N$ is continuous by definition of the inverse limit topology. □

**Theorem 17.** *Let $G$ be compact and let $\mathfrak{g}$ be its Lie algebra. Then $D(G, K)$ and $\widehat{U(\mathfrak{g})}_K$ are Frechet-Stein algebras.*

*Proof.* (Sketch.) For $\widehat{U(\mathfrak{g})}_K$, this is quite similar to the example of analytic functions discussed before.
What are the $A_n$s for the distribution algebra? Any compact locally $\mathbb{Q}_p$-analytic group contains an open subgroup $G_0$ which is uniformly pro-$p$. This induces a certain filtration $G_0^i$ on $G_0$. It is these different filtered pieces that give rise to various Banach completions of the distribution algebra $D(G_0, K)$. One then shows that $D(G_0, K)$ is Frechet-Stein. As $G_0$ is open in a compact $G$, it has finite index, so it is straightforward to lift this to obtain that $D(G, K)$ is Frechet-Stein. □

**Definition 18.** Let $V$ be a locally analytic $G$-rep of cpct type. We say that $V$ is **admissible** if $V'$ is a coadmissible $D(H, K)$-module for some (equivalently, for any) compact subgroup $H \leq G$.

**Proposition 19.** *Let $A$ be a Frechet-Stein algebra and let $M$ be a coadmissible $A$-module. Let $N \leq M$ be a submodule. The following are equivalent:*

(i) *$N$ is coadmissible.*
(ii) *$N$ is closed with respect to the canonical topology on $M$.*
(iii) *$M/N$ is coadmissible.*

**Corollary 20.** *The category of coadmissible $A$-modules is an abelian category and contains all finitely presented $A$-modules.*

We have thus reached our goal. We have the following commutative diagram (for $G$ compact), where the horizontal arrows are (anti-)equivalences of categories.

$$\begin{array}{ccc}
\text{admissible } G\text{-reps} & \xrightarrow{\cong} & \text{coadmissible } D(G,K)\text{-modules} \\
{\scriptstyle\subseteq}\big\downarrow & & \big\downarrow{\scriptstyle\subseteq} \\
\text{loc an } G\text{-reps} & \xrightarrow{\cong} & D(G,K)\text{-mods in nuc Fr spaces}
\end{array}$$

The category of coadmissible $D(G, K)$-modules is an abelian category of abstract $D(G, K)$-modules.

**Things we have not covered.**

(i) $p$-adic local Langlands relates (conjecturally) $n$-dimensional Galois representations to unitary Banach representations of $\mathsf{GL}_n(\mathbb{Q}_p)$. Often it is helpful to restrict a unitary representation to its locally analytic vectors (analogously to the smooth vectors for complex representations). The pLLC is so far only known for $n = 2$, where both sides can be classified quite explicitly in terms of $(\phi, \Gamma)$-modules.

(ii) Geometric tools: One can establish an equivalence of categories between coadmissible $D(G, K)$-moduels and a certain class of $p$-adic $\mathcal{D}$-modules in rigid analytic geometry, analogously to Beilinson-Bernstein theory over the complex numbers.

(iii) Induction, principal series for $\mathsf{GL}_2(K)$...

(iv) Analogues of Hecke algebras

(v) Reduction mod $p$