

# Cospans categories in algebraic K-theory

Quillen:  $K(R) = \Omega \mathcal{B} Q(\text{Proj}_R)$

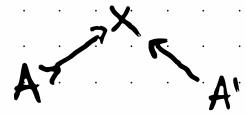
$Q(\text{Proj}_R^{\text{op}})$  has objects  $A \in \text{Proj}_R$ , morphisms  $A \xrightarrow{\quad B \quad} A'$

$\mathcal{C} \in \mathcal{C}_{\text{ob}}^{\text{st}}$   $\rightsquigarrow K(\mathcal{C}) = \Omega \mathcal{B} \mathcal{C}_{\text{sp}}(\mathcal{C})$  [Barwick-Rognes]

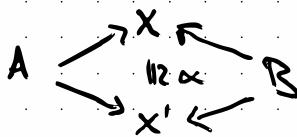
$\mathcal{C} \in \mathcal{C}_{\text{Wald}}$   $\rightsquigarrow K(\mathcal{C}) = \Omega \mathcal{B} \mathcal{C}_{\text{ob}}(\mathcal{C})$  [Raptis-Schank]

Ex  $\mathcal{C} = \text{Fin}$  finite sets,  $\mathcal{C}^{\mathcal{C}} = \text{injections}$ ,  $\mathcal{C}^{\mathcal{C}} = \text{isos}$ .

$K(\text{Fin}) = \Omega \mathcal{B} \mathcal{C}_{\text{sp}}^{\text{Lif}}$  morphisms:  
 $= QS^{\circ}$



Defn  $\mathcal{C}_{\text{sp}}$  is the 2-category with objects: finite sets,  
 morphisms: cospans  $A \rightarrow X \leftarrow B$  and  
 2morphisms: bijections



In the homotopy category  $\mathcal{C}_{\text{sp}} := h(\mathcal{C}_{\text{sp}})$  isomorphic  
 cospans are identified

# Cospans categories of manifolds

Let  $\mathcal{C}_d$  be G.M.W's bordism category.

obj: closed oriented  $(d-1)$ -dimensional submanifolds of  $\mathbb{R}^m$

mor: moduli space of embedded bordisms

$$\text{Hom}_{\mathcal{C}_d}(M, N) \simeq \coprod_{[W]: M \rightarrow N} \text{BDiff}(W \text{ rel } \partial W)$$

This is a topologically enriched category ( $\rightsquigarrow (\infty, 1)$ -category)  
 "the geometric version of  $\text{Cb}(\mathcal{D}(Z), \mathfrak{g})$ "

Much simpler:  $\text{Cob}_d := h(\mathcal{C}_d) \rightsquigarrow$  identify diffeomorphic bordisms

$$\text{Cob}_1 \quad \begin{array}{c} + \\ \text{---} \\ - \\ \text{---} \\ + \end{array} = \begin{array}{c} + \\ \text{---} \\ - \\ \text{---} \\ + \end{array}$$

$$\text{Cob}_2 \quad \begin{array}{c} \text{---} \\ \cup \\ \text{---} \end{array} = \begin{array}{c} \text{---} \\ \cup \\ \text{---} \end{array}$$

$\text{Cob}_2^{K \leq 0}$  the subcategory containing all morphisms  $W: M \rightarrow N$  that do not contain  $D^2$  or  $S^2$ .

Aside: What makes a cobordism category "geometric"?

There is a symmetric monoidal functor

$$\pi_0 : \mathcal{C}_d \longrightarrow \mathcal{C}_{\text{sp}}$$
$$M \xrightarrow{\omega} N \quad \pi_0 M \xrightarrow{\pi_0 \omega} \pi_0 N$$

and every object and morphism of  $\mathcal{C}_d$  canonically decomposes into its connected components.

Defn: A labelled cospan category is a sym. mon. (oo,1) cat,  $\mathcal{C}$  with a sym. mon. functor

$$\pi : \mathcal{C} \longrightarrow \mathcal{C}_{\text{sp}}$$
 s.t.  
$$\mathcal{C} \times \mathcal{C} \xrightarrow{\otimes} \mathcal{C}$$
$$\mathcal{C}_{\text{sp}} \times \mathcal{C}_{\text{sp}} \xrightarrow{\cong} \mathcal{C}_{\text{sp}}$$
$$\begin{array}{ccc} \pi \times \pi & \downarrow & \downarrow \pi \\ \mathcal{C}_{\text{sp}} \times \mathcal{C}_{\text{sp}} & \xrightarrow{\cong} & \mathcal{C}_{\text{sp}} \end{array}$$

Conjecture: These are exactly the monoidal envelopes of  $\infty$ -properads. [as defined by Hackney-Robertson-Yau]

# The classifying space of $\text{Csp}$

Thm (S.)  $B\text{Csp} \simeq *$  (Proof idea later)

Q: Is there an easy proof of this?

Consider the functor  $\text{Csp} \rightarrow \text{Csp}$  that identifies isomorphic cospan.  
It fits into a pushout square of sym. mon.  $(\infty, 1)$ -categories.

$$\begin{array}{ccc} \text{Csp}_\phi & \longrightarrow & \text{Csp} \\ \downarrow & \Gamma & \downarrow \\ \text{Csp}_\phi & \longrightarrow & \text{Csp} \end{array}$$

$\text{Csp}_\phi = \begin{cases} \text{obj} = \{\phi\} \\ \text{mor} = \coprod_{n \geq 0} \text{BS}_n \end{cases}$   
 $\text{Csp}_\phi = \begin{cases} \text{obj} = \{*\} \\ \text{mor} = N \end{cases}$

$$\begin{array}{ccc} \text{QS}' & \longrightarrow & B\text{Csp} \simeq * \\ \Rightarrow \quad \downarrow & \Gamma & \downarrow \\ S' & \longrightarrow & B\text{Csp} \simeq \tau_{2,2}\text{QS}^2 \end{array}$$

# The classifying space of $\text{Cob}_d$

There's a canonical functor  $\mathcal{C}_d \rightarrow \text{Cob}_d$

$$\mathbf{Diff}^0(\omega) \mapsto *$$

Note:  $\mathbf{B}\mathbf{Diff}^0(\omega) \longrightarrow S\mathcal{B}\mathcal{C}_d \longrightarrow S\mathcal{B}\text{Cob}_d$

\*

$[\text{Thm (Elkot)}] \Rightarrow H^*(S\mathcal{B}\mathcal{C}_d; \mathbb{Q}) \leftarrow H^*(S\mathcal{B}\text{Cob}_d)$  is 0 for  $d$  even  
and at most hits signature classes for  $d$  odd

Rank For  $d=3$  one can construct  $\alpha \in H^2(S\text{Cob}_3; \mathbb{Q})$  that lifts  
to a non-trivial multiple of  $\chi_p \in H^1(S\mathcal{B}\mathcal{C}_3; \mathbb{Q})$

However, this doesn't mean that  $S\mathcal{B}\text{Cob}_d$  is less complicated than  $S\mathcal{B}\mathcal{C}_d$ .

$$\begin{array}{ccc} d=1 & \mathcal{C}_{\text{closed}} & \longrightarrow \mathcal{C}_1 \\ \text{Thm (S.)} & \downarrow & \vdash \downarrow \text{ in } \mathcal{C}_{\text{top}} \\ \mathcal{C}_{\text{closed}} & \longrightarrow \mathcal{G}_1 \end{array}$$

and hence  $S\mathcal{B}\text{Cob}_1 \simeq \text{hofib}(S^{2n-2}M\text{SO}_2 \rightarrow K(2,2))$

(and more interesting things happen in  $\mathcal{C}_{\text{closed}}/\mathcal{C}_{\text{closed}}$ )

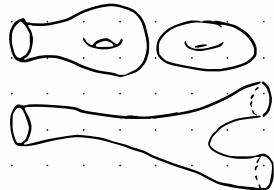
# The classifying space of $\text{Col}_2$

The map  $B\mathcal{C}_2 \rightarrow B\text{Col}_2$  factor through  $S^1$ :

$$B\mathcal{C}_2^{D_4} \xrightarrow{\sim} B\mathcal{C}_2 \simeq \Sigma^{\infty} MTSO_2$$

$$\downarrow \quad \quad \quad \downarrow$$

$$B\text{Col}_2^{D_4} \longrightarrow B\text{Col}_2$$



Then (Tillmann 96)  $B\text{Col}_2 \simeq B\text{Col}_2^{D_4} \times_{\overset{\text{S}^1}{\sim}} X$   
 $X$  is 1-connected  $\Sigma^\infty$ -space

Standard strategy:

$$BDiff(\Sigma_\infty) \xleftarrow[\text{group completion}]{} B\mathcal{C}_2^{D_4} \xleftarrow[\text{surgery}]{} B\mathcal{C}_2 \xleftarrow[\text{scanning}]{} MTSO_2$$

Reversed:  $S^1 \simeq B\text{Col}_2^{D_4} \xrightarrow{\sim} B\text{Col}_2$

Thm(S.) There's a fiber sequence:

$$B\text{Col}_2^{D_4} \longrightarrow B\text{Col}_2 \longrightarrow Q(V \sum_{j \geq 0} \Sigma^2 \mathbb{B} f_j)$$

"obstruction term"

and  $B\mathcal{C}_2 \simeq B\text{Col}_2 \times_{\overset{J^{\geq 0}}{\sim}} \pi_{\geq 3} Q\mathbb{S}^2$

The obstruction term:  $\mathbf{BF}_g$

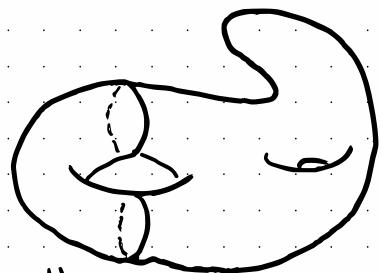
Defn The factorisation category is the full subcategory

$\widehat{\mathcal{F}}_g \subset (\mathbf{Cb}_2)^{\frac{1}{\phi}}$  on those  $(\phi \xrightarrow{W_0} M \xrightarrow{W_1} \phi)$   
satisfying:  $M \neq \phi$  and  $W_0 \cup_M W_1 \cong \Sigma_g$

Morphisms  $(M, W_0, W_1) \rightarrow (N, V_0, V_1)$

are morphisms  $X: M \rightarrow N$  st.

$$\begin{array}{ccc} W_0 & \xrightarrow{\phi} & V_0 \\ \downarrow & & \downarrow \\ M & \xrightarrow{X} & N \\ \downarrow & & \downarrow \\ W_1 & \xrightarrow{\phi} & V_1 \end{array}$$



$$[\phi \xrightarrow{W_0} M \xrightarrow{W_1} \phi]$$

$$\begin{array}{c} \text{Diagram showing a map } X: M \rightarrow N \\ M \xrightarrow{X} N \end{array}$$

$$\begin{array}{ccc} & & \text{Diagram showing a surface with three handles and a hole, representing a complex manifold} \\ & & \xrightarrow{\quad V_0 \quad} \\ [\phi & \xrightarrow{V_0} & N & \xrightarrow{V_1} & \phi] \end{array}$$

Lemma  $\mathbf{BF}_g \simeq *$

Proof idea: The full subcategory on diagrams  
is  $(\mathbf{Fin}^{(i)})^{\mathbf{op}}$  and is basically initial.



# $\text{Cob}_2^{X \leq 0}$ and tropical moduli spaces

Recall  $\text{Cob}_2^{X \leq 0} \subset \text{Cob}_2$  contains all bordisms without disks (or spheres)

Then There's a fiber sequence:

$$BC\text{ob}_2^{X \leq 0, 0_+} \xrightarrow{\quad \pi \quad} BC\text{ob}_2^{X \leq 0} \longrightarrow Q(V \sum_{j \geq 1} \mathbb{R} F_j^{X \leq 0})$$

splits

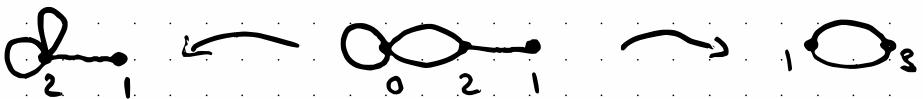
Moreover:

- $BC\text{ob}_2^{X \leq 0, 0_+} \cong S'$

$$\bullet \mathbb{R} F_g \cong \begin{cases} SO(2) & (g=1) \\ \mathbb{R} J_g \cong \Delta_g & (g \geq 2) \end{cases}$$

Here:

$$J_g = \begin{cases} \text{objects: } N\text{-weighted stable graphs of genus } g \\ \text{morphisms: edge collapses} \end{cases}$$

Ex: 

Then (Chen-Gabai-Payne)  $H_*(\Delta_g; \mathbb{Q}) \cong \sum_{j=1}^{g-1} H_*(G_{g,j})$

commutative  
 graph complex  
 ↓  
 grt

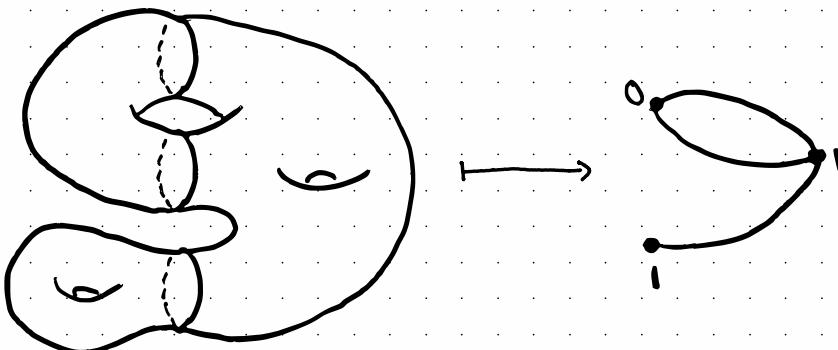
Cor  $\pi_*^{\mathbb{Q}}(BC\text{ob}_2^{X \leq 0}) = H_*(S') \oplus H_*(SO(2)) \oplus \bigoplus_{g \geq 2} \sum_{j=1}^{g-1} H_*(G_{g,j})$

## Why tropical curves?

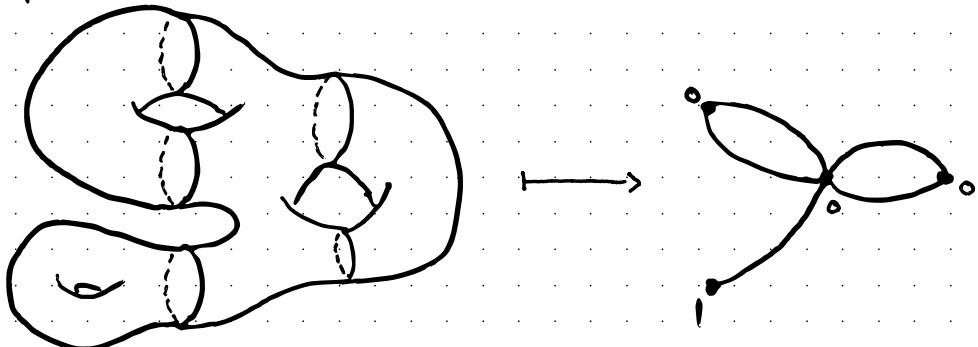
$B\mathbb{F}_g^{\text{tor}}$   $\simeq$   $B\mathcal{I}_g$  is given by a zig-zag, but the rational equivalence to  $\Delta_g$  can be given explicitly:

$$B\mathbb{F}_g^{\text{tor}} \longrightarrow \Delta_g$$

0-simplices



1-simplices

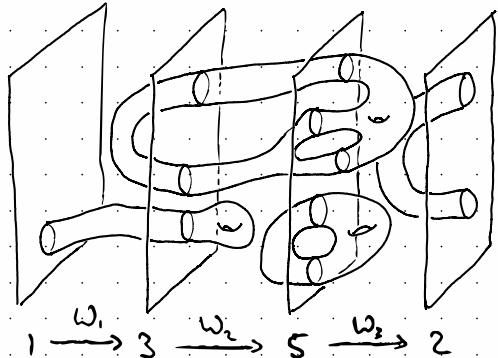


$$\emptyset \xrightarrow{\omega_0} M \xrightarrow{X} N \xrightarrow{V_1} \emptyset$$

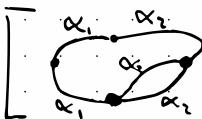
There is a continuous map:

$$B\mathcal{C}ob_2^{\alpha \leq 0}$$

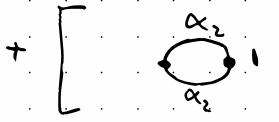
$$\longrightarrow \text{FM}(\Sigma^2(\Delta_2 \sqcup \Delta_3 \sqcup \dots))$$



→



$$a = \alpha_0, b = \alpha_3$$



$$a = \alpha_0 + \alpha_1, b = \alpha_3$$

$$1 \xrightarrow{w_1} 3 \xrightarrow{w_2} 5 \xrightarrow{w_3} 2$$

$$(\alpha_0, \alpha_1, \alpha_2, \alpha_3) \in \Delta^3$$