

Classification of Manifolds & Homotopy Theory

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Convention: Manifolds are smooth, oriented, compact, possibly with boundary.

Goal: Classify closed manifolds up to diffeomorphism, at least in the metastable range.

Want this to be effective.

Def: n -manifold M is almost closed if $\partial M \cong S^{n-1}$.

$\Theta_n: M \mapsto M \# \Sigma$

$\{ \text{closed } n\text{-manifolds} \} / \text{diff} \rightarrow \{ \text{almost closed } n\text{-manifolds} \} / \text{diff}$

$M \xrightarrow{\quad} M - \text{int } D^n$

Fibers = orbits of Θ_n -action.

Θ_{n-1}

$N \downarrow \partial N$

Step (1): Classify almost closed manifolds.

focus

Step (2): Determine boundary of a.c. man.
+ Θ_n -action on a lft.

Almost closed manifolds

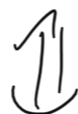
Classic result of Wall:

$(n-1)$ -connected $2n$ -manifolds

$H_0 \cong \mathbb{Z}$, H_n only nonzero homology.

Thm (Wall '62): $n \geq 3$

{ almost closed $(n-1)$ -conn. $2n$ -manifolds } / diff



$$\left\{ \begin{array}{l} H \text{ f.g. free ab. grp.} \\ H \otimes H \rightarrow \mathbb{Z} \quad (-1)^n\text{-symm., non-sing.} \\ \alpha: H \rightarrow \pi_n BSO(n) \\ x^2 = H\mathcal{J}\alpha(x), \quad \alpha(x+y) = \alpha(x) + \alpha(y) + x \cdot y \tau_n \end{array} \right\}$$



$$\left\{ \begin{array}{l} H \text{ f.g. free ab. grp.} \\ \mathbb{Z} \rightarrow (H \otimes H)_{C_2} \xrightarrow{(H \otimes H)^{C_2}} H \otimes H \text{ non-singular.} \\ H \rightarrow \pi_n BSO \end{array} \right\}$$

$\xleftarrow{\text{C-D}^n \text{ swap}}$ (between $(H \otimes H)_{C_2}$ and $(H \otimes H)^{C_2}$)
 $\xleftarrow{\text{linear}}$ (from $\pi_n BSO$ to H)

/ iso.

/ iso

n even: quadratic form on H .

n odd: skew-symm. form H , and
quadratic refinement mod 2.

Wall also classified $(n-1)$ -conn. $(2n+1)$ -manifolds.
Ishimoto did some cases of $(n-2)$ -conn. $2n$ -manifolds.

Fang + Pan
Pre-thm (Burkhard-S.): $\exists$ sublinear $d(n)$

s.t. if $d \geq n/3 + d(n)$, then

$\{ \text{almost closed } d\text{-conn. } n\text{-manifolds} \} / \text{diff.}$



$$\left\{ \begin{array}{l} N \in Sp_{\geq d} \quad \text{finite} \\ \mathbb{S}^n \rightarrow (N \otimes N)_{hC_2} \xrightarrow{\quad} (N \otimes N)_{\text{non-sing.}}^{hG} \rightarrow N \otimes N \\ N \rightarrow KO \end{array} \right\} / \pi_0 \text{Aut}(N)$$

Examples: ① Recovers Wall: $2n$ -manifolds
 $N \simeq \oplus \mathbb{S}^n$.

①.5 Recovers Wall's odd dim.
result.

② $\{ \text{almost closed } 2n\text{-manifolds} \}$
with $N \simeq \oplus (\mathbb{S}^{n-\varepsilon} \oplus \mathbb{S}^{n+\varepsilon})$

$(\epsilon \neq 1) (4)$



$$\left. \begin{array}{l} H \\ H \rightarrow \pi_{n+\epsilon} BSO \\ H \rightarrow \pi_{n-\epsilon} BSO \\ H \rightarrow \text{im}(\pi_{n+\epsilon}^S RP_{n-\epsilon}^\infty \rightarrow \pi_{n+\epsilon}^S RP_{n-\epsilon+1}^\infty) \end{array} \right\} \text{James periodicity}$$

$\Rightarrow$ given ϵ finite comp.

Boundaries of almost closed manifolds

Q: Given N highly-conn. almost closed, what is $\partial N \in \oplus_{n-1}$?

K-M:

Impossibly complicated.

$$0 \rightarrow \boxed{bP_n} \rightarrow \oplus_{n-1} \rightarrow \boxed{\text{coker}(J)_n}$$

golic of known order

except $n \leq 126$.

Hope: $[\partial N]_{\text{coker } J} = 0$ most of the times.

Thm (Wall, Schultz, Stolz, Burkland-Hohn-S, Burkland-S)

N almost closed $(n-1)$ -conn. $2n$ -manifold.

$$\Rightarrow [\partial N]_{\text{coker } J} = 0 \quad n \neq 9, 12.$$

Exceptions:

$$[\partial N]_{\text{coker } J} = \eta \eta_4 \in \text{coker}(J)_{17}$$

$$[\partial N]_{\text{coker } J} = \eta^3 \bar{\kappa} \in \text{coker}(J)_{23}.$$

Thm (Barklund-Hahn-S.): $\exists$ sublinear $\varepsilon(n)$
s.t. if $d \geq n/3 + \varepsilon(n)$

$\Rightarrow N$ d -conn. almost closed n -manifold

$$\Rightarrow [\partial N]_{\text{coker}(J)} = 0.$$

$\varepsilon(n)$ explicit
replace $n/3$
by $13n/30$.

$\Rightarrow$ obstruction to closing almost closed
metastably connected manifold is computable.

n odd: no obstruction.

$n \equiv 2(4)$: Kervaire invariant.

$n \equiv 0(4)$: signature + characteristic #5.