Classification of Manifolds & Homotopy Theory

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Convention: Manifolds are smooth, oriented, compact, possibly with boundary.

Goal: Classify closed manifolds up to diffeomorphism, at least in the metastable range.

Want this to be <u>effective</u>.

Def: n-manifold M is <u>almost closed</u> if $\partial M \simeq S^{n-1}$.

$$\Theta_n \curvearrowright \quad M \mapsto M \# \Sigma$$

$$\{\text{closed } n\text{-manifolds}\}/\text{diff} \longrightarrow \{\text{almost closed } n\text{-manifold}\}/\text{diff} \longrightarrow \Theta_{n-1}$$

$$M \longmapsto M - \mathrm{int}\, D^n$$

$$N \longmapsto \partial N$$

Fibers = orbits of Θ_n-action.

Step ①: Classify almost closed manifolds.

focus

Step ②: $\boxed{\text{Determine boundary of a.c. man.}}$ + Θ_n-action on a lift.

Almost closed manifolds

Classic result of Wall:

(n-1)-connected 2n-manifolds

$H_0 \cong \mathbb{Z}$, H_n only nonzero homology.

Thm (Wall '62): $n \geq 3$

$$\{\text{almost closed } (n-1)\text{-conn. } 2n\text{-manifolds}\}/\text{diff}$$

$$\Updownarrow$$

$$\left\{ \begin{array}{l} H \text{ f.g. free ab grp.} \\ H \otimes H \to \mathbb{Z} \quad (-1)^n\text{-symm., non-sing.} \\ \alpha : H \to \pi_n BSO(n) \\ x^2 = HJ\alpha(x), \quad \alpha(x+y) = \alpha(x) + \alpha(y) + x \cdot y \, \tau_{S^n} \end{array} \right\} \Big/ \text{iso.}$$

$$\Updownarrow \; n \geq 4$$

$$\left\{ \begin{array}{l} H \text{ f.g. free ab. grp.} \\ \mathbb{Z} \to \underbrace{(H \otimes H)_{C_2}}_{(-1)^n \text{swap}} \to (H \otimes H)^{C_2} \to H \otimes H \text{ non-singular.} \\ H \to \pi_n BSO \text{ linear} \end{array} \right\} \Big/ \text{iso}$$

n even: quadratic form on H.

n odd: skew-symm. form H, and quadratic refinement mod 2.

Wall also classified $(n-1)$-conn. $(2n+1)$-manifolds. $n>15$

Ishimoto did some cases of $(n-2)$-conn. $2n$-manifolds.

Fang + Pan

Pre-thm (Burklund-S.): $\exists$ sublinear $\delta(n)$ s.t. if $\boxed{d \geq n/3 + \delta(n)}$, then

$$\{\text{almost closed } d\text{-conn. } n\text{-manifolds}\}/\text{diff.}$$

$$\Updownarrow$$

$$\left\{ \begin{array}{l} N \in Sp_{\geq d}. \quad \underline{\text{finite}} \\ \mathbb{S}^n \to (N \otimes N)_{hC_2} \to (N \otimes N)^{hC_2} \xrightarrow[\text{non-sing.}]{} N \otimes N \\ N \to KO \end{array} \right\} \Big/ \pi_0 hAut(N).$$

Examples: ① Recovers Wall: $2n$-manifolds $N \simeq \oplus \mathbb{S}^n$.

(1.5) Recovers Wall's odd dim. result.

② $\{$almost closed $2n$-manifolds with $N \simeq \oplus(\mathbb{S}^{n-\varepsilon} \oplus \mathbb{S}^{n+\varepsilon})$ $\}$

$\boxed{\varepsilon \neq 1 \ (4)}$

$\Updownarrow$

$$\left\{ \begin{array}{l} H \\ H \to \pi_{n+\varepsilon} BSO \\ H \to \pi_{n-\varepsilon} BSO \\ H \to \underbrace{\mathrm{im}\left(\pi^s_{n+\varepsilon} \mathbb{R}P^\infty_{n-\varepsilon} \to \pi^s_{n+\varepsilon} \mathbb{R}P^\infty_{n-\varepsilon+1}\right)}_{\text{James periodicity}} \end{array} \right\} / \mathrm{Iso}.$$

$\Rightarrow$ given ε finite comp.

Boundaries of almost closed manifolds

Q: Given N highly-conn. almost closed, what is $\partial N \in \Theta_{n-1}$?

K-M:

$$0 \to \underbrace{\boxed{bP_n}}_{\text{cyclic of known order}} \to \Theta_{n-1} \to \boxed{\mathrm{coker}(J)_{n-1}}$$

Impossibly complicated.

except $n=126$.

Hope: $[\partial N]_{\mathrm{coker} J} = 0$ most of the times.

Thm (Wall, Schultz, Stolz, Burklund-Hahn-S., Burklund-S.)

N almost closed $(n-1)$-conn. $2n$-manifold.

$\Rightarrow [\partial N]_{\mathrm{coker} J} = 0 \quad n \neq 9, 12.$

Exceptions:

$$[\partial N]_{\mathrm{coker} J} = \eta \eta_4 \in \mathrm{coker}(J)_{17}$$

$$[\partial N]_{\mathrm{coker} J} = \eta^3 \bar{\kappa} \in \mathrm{coker}(J)_{23}.$$

Thm (Burklund-Hahn-S.): $\exists$ sublinear $\varepsilon(n)$ s.t. if $d \geq n/3 + \varepsilon(n)$

$\Rightarrow N$ d-conn. almost closed n-manifold

$\Rightarrow [\partial N]_{\mathrm{coker}(J)} = 0.$

$\boxed{\text{$\varepsilon(n)$ explicit replace $n/3$ by $13n/30$.}}$

$\Rightarrow$ obstruction to closing almost closed metastably connected manifold is computable.

n odd: no obstruction.

$n \equiv 2\ (4)$: Kervaire invariant.

$n \equiv 0\ (4)$: signature + characteristic #'s.