Self-similar Solutions to the Hypoviscous Burgers Equation at Criticality

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1. Introduction: motivations for studying self-similar solutions

2. Standard Burgers equation (review)

3. Hypoviscous Burgers equation

4. Summary and outlook
1. Introduction: motivations for studying self-similar solutions

- Role of near-identity transformation underlying source-type self-similar solutions

- Generating equation that determines the near-identity

- Exemplify with the hypoviscous Burgers equation

- Non-zero energy dissipation for it?
2. **Standard Burgers equation** (review)

\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = \nu \frac{\partial^2 u}{\partial x^2}
\]

Cole-Hopf transformation

\[
u(x, t) = -2\nu \frac{\partial}{\partial x} \log \theta(x, t) \implies \frac{\partial \theta}{\partial t} = \nu \frac{\partial^2 \theta}{\partial x^2}
\]

\[
\theta(x, t) = \frac{1}{\sqrt{4\pi \nu t}} \int \exp \left( -\frac{(x-y)^2}{4\nu t} \right) \theta_0(y) dy
\]

Hopf’s proof: The final form is given as is without any reason.

Cole’s proof: Derived it based on scale-invariance, urging us to go for the velocity potential \(\phi\)
Burgers equation \[ \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = \nu \frac{\partial^2 u}{\partial x^2} \]

scale invariance \( x \to \lambda x, t \to \lambda^2 t, u \to \lambda^{-1} u \)

If \( u(x,t) \) is a solution, so is \( u_\lambda(x,t) \equiv \lambda u(\lambda x, \lambda^2 t), \quad \forall \lambda > 0 \).

velocity potential \( u = \frac{\partial \phi}{\partial x} \implies \frac{\partial \phi}{\partial t} + \frac{1}{2} \left( \frac{\partial \phi}{\partial x} \right)^2 = \nu \frac{\partial^2 \phi}{\partial x^2} \)

scale invariance \( x \to \lambda x, t \to \lambda^2 t, \phi \to \lambda^0 \phi \)

If \( \phi(x,t) \) is a solution, so is \( \phi_\lambda(x,t) \equiv \lambda^0 \phi(\lambda x, \lambda^2 t) \).

**Type 1 critical scale-invariance**: physical dimension \([\phi] = [\nu] = \left[ \frac{L^2}{T} \right] \)

Cole suspected \( u(x,t) = F(\theta(x,t)) \) for some \( F \), and he was right.
Dynamic scaling

\( U(\xi, \tau) = \lambda(t)u(x, t), \lambda(t) = \sqrt{2a(t + t_*)}, (2at_* = 1) \)

\( \xi = \frac{x}{\lambda(t)}, \tau = \frac{1}{2a} \log \frac{t + t_*}{t_*}, \)

\[
\begin{align*}
\frac{\partial U}{\partial \tau} + U \frac{\partial U}{\partial \xi} &= \nu \frac{\partial^2 U}{\partial \xi^2} + a\xi \frac{\partial U}{\partial \xi} + aU, \\
&= a \frac{\partial}{\partial \xi}(\xi U)
\end{align*}
\]

\( \phi(x, t) = \Phi(\xi, \tau) \)

\[
\begin{align*}
\frac{\partial \Phi}{\partial \tau} + \frac{1}{2} \left( \frac{\partial \Phi}{\partial \xi} \right)^2 &= \nu \frac{\partial^2 \Phi}{\partial \xi^2} + a\xi \frac{\partial \Phi}{\partial \xi}
\end{align*}
\]
Aside: an alternative look at type 2 scale-invariance
Linearised solutions

\[ \hat{\Phi}(\xi, \tau) = \left( \frac{a}{2\pi\nu (1 - e^{-2a\tau})} \right)^{1/2} \int_{\mathbb{R}^1} \Phi_0(e^{a\tau} \eta) \exp \left( -\frac{a}{2\nu} \frac{(\xi - \eta)^2}{1 - e^{-2a\tau}} \right) d\eta \to ? \]

as \( \tau \to \infty. \)

\[ \hat{U}(\xi, \tau) = \left( \frac{a}{2\pi\nu (1 - e^{-2a\tau})} \right)^{1/2} \int_{\mathbb{R}^1} e^{a\tau} U_0(e^{a\tau} \eta) \underbrace{\exp \left( -\frac{a}{2\nu} \frac{(\xi - \eta)^2}{1 - e^{-2a\tau}} \right)}_{\to M\delta(\eta)} d\eta \]

\[ \to C \exp \left( -\frac{a\xi^2}{2\nu} \right), \]

Recall \( \lambda f(\lambda x) \to M\delta(x), \) as \( \lambda \to \infty, M = \int f \, dx. \)

Hence \( \hat{\Phi}(\xi, \tau) \to C \int_0^\xi \exp \left( -\frac{a\eta^2}{2\nu} \right) d\eta, \) retrospectively.
Repeat Cole’s argument after dynamic scaling

Assume \( \Phi(s) = Cf(s) \) for \( \exists f \), \( s \equiv \int_0^\xi \exp \left(-\frac{an\eta^2}{2\nu}\right) \, d\eta \),

then \( U = \Phi_\xi = C \exp \left(-\frac{a\xi^2}{2\nu}\right) f'(s) \), \( U_\xi + \frac{a}{\nu} \xi U = C \exp \left(-\frac{a\xi^2}{\nu}\right) f'' \).

Plug them into \( \frac{U^2}{2} = \nu U_\xi + a\xi U \)

\( \implies \frac{C}{2\nu} g^2 = g', \ g \equiv f' \): generating equation

\( \implies g(s) = \frac{1}{1 - \frac{C}{2\nu} s} \), or \( f(s) = -\frac{2\nu}{C} \log \left(1 - \frac{C}{2\nu} s\right) \)

\( f \) is a near-identity, i.e. \( f(s) \approx s \) for large \( \nu \)
Near-Gaussian form \( U(\xi) = \frac{C \exp \left( -\frac{a\xi^2}{2\nu} \right)}{1 - \frac{C}{2\nu} \int_0^\xi \exp \left( -\frac{a\eta^2}{2\nu} \right) d\eta} \)

e.g. Liu and Pierre (1984), Escobedo and Zuazua (1991)

Source-type solution \( u(x, t) = \frac{1}{\sqrt{2at}} U \left( \frac{x}{\sqrt{2at}} \right), \ (t_\ast = 0) \)

\[ \lim_{t \to 0} \frac{1}{\sqrt{2at}} U(\cdot) = M\delta(\cdot) \]

\[ \lim_{t \to \infty} t^{\frac{1}{2}} \left(1 - \frac{1}{p}\right) \left\| u(x, t) - \frac{1}{\sqrt{2at}} U \left( \frac{x}{\sqrt{2at}} \right) \right\|^p_p = 0, \text{ for } u_0 \in L^1, 1 \leq p \leq \infty \]
Small deviation from the Gaussian tells us how to handle more general solutions

\[ U = -2\nu \partial_\xi \log \left( 1 - \frac{C}{2\nu} \int_0^\xi \exp \left( -\frac{a\eta^2}{2\nu} \right) d\eta \right) \]

What can we learn if we do the same for other equations, hypoviscous Burgers, SQG & Navier-Stokes?
3. Hypoviscous Burgers equation

\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = -\nu' \Lambda u, \quad \Lambda \equiv (-\partial_{xx})^{1/2} = \partial_x H[.] \]


scale invariance \( x \rightarrow \lambda x, t \rightarrow \lambda t, u \rightarrow \lambda^0 u \)

If \( u(x, t) \) is a solution, so is \( u_\lambda(x, t) \equiv \lambda^0 u(\lambda x, \lambda t) \)

**Type 1 critical scale-invariance**: physical dimension \([u] = [\nu'] = \left[ \frac{L}{T} \right]\)

\[
\frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} = -w^2 - \nu' \Lambda w
\]

scale invariance \( x \rightarrow \lambda x, t \rightarrow \lambda t, w \rightarrow \lambda^{-1} w \)

If \( w(x, t) \) is a solution, so is \( w_\lambda(x, t) \equiv \lambda w(\lambda x, \lambda t) \)
Linearised solutions Poisson kernel (analogue of ’heat kernel’)

\[ u(x, t) = \frac{\nu t}{\pi} \int \frac{u_0(y)dy}{(x - y)^2 + (\nu' t)^2} \]

Dynamic scaling

\[ U (\xi, \tau) = u(x, t), \xi = \frac{x}{\lambda(t)}, \tau = \frac{1}{a} \log \frac{t + t_*}{t_*}, \lambda(t) = a(t + t_*) \]

\[ \frac{\partial U}{\partial \tau} + U \frac{\partial U}{\partial \xi} = -\nu' \Lambda U + a\xi \frac{\partial U}{\partial \xi} \]

In velocity gradient

\[ \frac{\partial W}{\partial \tau} + U \frac{\partial W}{\partial \xi} = -W^2 - \nu' \Lambda W + a \frac{\partial}{\partial \xi} (\xi W) \]
Linearised solutions: scaled form

$$\lim_{\tau \to \infty} \hat{W}(\xi, \tau) = \frac{M}{\pi} \frac{\mu}{\xi^2 + \mu^2},$$

where $\mu = \nu'/a$, $M \equiv \int wdx$

$$\lim_{\tau \to \infty} \hat{U}(\xi, \tau) = \frac{M}{\pi} \tan^{-1} \frac{\xi}{\mu}.$$ 

cf. After dynamic scaling

$$\hat{U}(\xi, \tau) = \frac{\nu' \left(1 - e^{-a\tau}\right)}{\pi a} \int \frac{U_0(e^{a\tau}\eta)d\eta}{(\xi - \eta)^2 + \left(\nu' \frac{1-e^{-a\tau}}{a}\right)^2},$$

$$\hat{W}(\xi, \tau) = \frac{\nu' \left(1 - e^{-a\tau}\right)}{\pi a} \int \frac{e^{a\tau}W_0(e^{a\tau}\eta)d\eta}{(\xi - \eta)^2 + \left(\nu' \frac{1-e^{-a\tau}}{a}\right)^2},$$

cf. Different scaling adopted under different BCs, Iwabuchi(2015)
Self-similar profile $W(\xi) = \lim_{\tau \to \infty} W(\xi, \tau)$

Put $W_\tau = 0$, $(UW)_\xi = -\nu' \Lambda_\xi W + a(\xi W)_\xi$.

$$\xi W - \mu H[W] = \frac{1}{a} UW \quad (*)$$

If the nonlinear term is neglected

$$\xi W - \mu H[W] = 0.$$  

To leading-order

$$W \propto \frac{\mu}{\xi^2 + \mu^2}$$

cf. $H \left[ \frac{\mu}{\xi^2 + \mu^2} \right] = \frac{\xi}{\xi^2 + \mu^2}, \quad H \left[ \frac{\xi}{\xi^2 + \mu^2} \right] = -\frac{\mu}{\xi^2 + \mu^2}, \quad (\mu > 0)$
Assume \( U(\xi) = Cf(s) \), \( W(\xi) = C\frac{\mu}{\xi^2 + \mu^2}f'(s) \), \( s \equiv \tan^{-1} \frac{\xi}{\mu} \)

\[
UW = -\nu'H[W] + a\xi W
\]

\[
\Rightarrow \frac{C}{a} \frac{ff'}{\xi^2 + \mu^2} = \frac{\xi}{\xi^2 + \mu^2}f' \left[ \frac{\mu}{\xi^2 + \mu^2}f' \right] - H \left[ \frac{\mu}{\xi^2 + \mu^2}f' \right],
\]

or \( \frac{C}{a} \frac{ff'}{\xi^2 + \mu^2} = H \left[ \frac{\mu}{\xi^2 + \mu^2} \right] f' - H \left[ \frac{\mu}{\xi^2 + \mu^2}f' \right] \), (a commutator)

\[
Rfg \cos^2 s = K[1]g - K[g], \quad g = f' : \text{generating equation for } f(s)(\approx s),
\]

where \( R = \frac{C}{\nu}, \quad K[g] = \frac{1}{\pi} \text{p.v.} \int_{-\pi/2}^{\pi/2} \frac{g(s')ds'}{\tan s - \tan s'} \), \( K[1] = \cos s \sin s \)
cf. \( \frac{R}{2} g^2 = g' \), \( g \equiv f' \) for standard Burgers
Successive approximation

\[ f_{n+1} = \frac{1}{K[1]} \left( K[f'_n] + R f_n f'_n \cos^2 s \right) \quad (n = 1, 2, \ldots) \]

\[ f_1 = s \rightarrow f'_2 = 1 + R \frac{s \cos^2 s}{\cos s \sin s} \]

\[ f'_2 = 1 + R s \cot s \]

\[ \int s \cot s ds = s - \frac{s^3}{9} - \frac{s^5}{225} - \frac{2s^7}{6615} - \ldots, \quad |s| < \pi, \notin \text{elementary functions} \]
Lifting (standard Burgers equation)

Step 1. Assume \( \Phi(\xi) = Cf(s), \ s = \hat{\Phi} = \int_0^\xi \exp \left( -\frac{an^2}{2\nu} \right) \)

\[
U(\xi) = F(\hat{\Phi}(\xi); \partial_\xi \hat{\Phi}(\xi)) \equiv \frac{\partial_\xi \hat{\Phi}}{1 - \frac{1}{2\nu} \hat{\Phi}}
\]

\( U(\xi) \) is a *near-identity* transformation of \( \partial_\xi \hat{\Phi}(\xi) \).

Reverting to the original variables, a particular solution is

\[
u(x, t) = \frac{1}{\sqrt{2at}} \left( \hat{\Phi} \left( \frac{x}{\sqrt{2at}} \right); \partial_\xi \hat{\Phi} \left( \frac{x}{\sqrt{2at}} \right) \right) \equiv \frac{1}{\sqrt{2at}} \frac{\partial_\xi \hat{\Phi}(\frac{x}{\sqrt{2at}})}{1 - \frac{1}{2\nu} \hat{\Phi}(\frac{x}{\sqrt{2at}})}.
\]
Step 2. Replacing the self-similar heat flow with the general one,

\[ u(x, t) = \frac{1}{\sqrt{2at}} F(\hat{\phi}(x, t); \sqrt{2at} \partial_x \hat{\phi}(x, t)) \]

\[ = F(\hat{\phi}(x, t); \partial_x \hat{\phi}(x, t)) \equiv \frac{\partial_x \hat{\phi}(x, t)}{1 - \frac{1}{2\nu} \hat{\phi}(x, t)}, \]

where

\[ \hat{\phi}(x, t) = \frac{1}{\sqrt{4\pi \nu t}} \int_{-\infty}^{\infty} \hat{\phi}(y, 0) \exp \left( -\frac{(x - y)^2}{4\nu t} \right) dy \]
Lifting (Hypoviscous Burgers equation)

**Step 1.** Assume that $U(\xi) = Cf(s)$, $s(= \hat{U}) \equiv \tan^{-1} \frac{\xi}{\mu}$

$$W(\xi) = Cf'(s) \frac{\mu}{\xi^2 + \mu^2} = F(\hat{U}(\xi); \partial_\xi \hat{U}(\xi))$$

$W(\xi)$ is a *near-identity* transformation of $\partial_\xi \hat{U}(\xi)$.
Reverting to the original variables, a particular solution is

$$w(x, t) = \frac{1}{at} F\left(\hat{U}\left(\frac{x}{at}\right); \partial_\xi \hat{U}\left(\frac{x}{at}\right)\right)$$

**Step 2.** Replacing the self-similar Poisson flow with the general one,

$$w(x, t) = \frac{1}{at} F(\hat{u}(x, t); at \partial_x \hat{u}(x, t))$$

$$= F(\hat{u}(x, t); \partial_x \hat{u}(x, t)) = f'(\hat{u}(x, t)) \partial_x \hat{u}(x, t)$$

where $\hat{u}(x, t) = \frac{\nu t}{\pi} \int \frac{u_0(y)dy}{(x - y)^2 + (\nu't)^2}$
4. Summary and outlook

- Self-similar solution to the hypoviscous Burgers equation

- A generating equation for the near-identity \( f(\cdot) \) is derived. (Integrable by the Poisson kernel.)

- The second-order approximation is worked out, but turns out to be poorly accurate.

- Better approximations is to be sought.
• Applications to other equations (SQG, Navier-Stokes).

• Energy dissipation rate for the hypoviscous Burgers equation