Self-similar Solutions to the Hypoviscous Burgers Equation at Criticality

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1. Introduction: motivations for studying self-similar solutions

- Role of near-identity transformation underlying source-type self-similar solutions

- Generating equation that determines the near-identity

- Exemplify with the hypoviscous Burgers equation

- Non-zero energy dissipation for it ?

2. **Standard Burgers equation** (review)

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \nu\frac{\partial^2 u}{\partial x^2}$$

Cole-Hopf transformation

$$u(x,t) = -2\nu\frac{\partial}{\partial x}\log\theta(x,t) \Longrightarrow \frac{\partial \theta}{\partial t} = \nu\frac{\partial^2 \theta}{\partial x^2}$$

$$\theta(x,t) = \frac{1}{\sqrt{4\pi\nu t}}\int \exp\left(-\frac{(x-y)^2}{4\nu t}\right)\theta_0(y)dy$$

Hopf's proof: The final form is given as is without any reason.
Cole's proof: Derived it based on scale-invariance,
urging us to go for the velocity potential ϕ

Burgers equation $\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \nu \frac{\partial^2 u}{\partial x^2}$

scale invariance $x \to \lambda x, t \to \lambda^2 t, u \to \lambda^{-1} u$

If $u(x,t)$ is a solution, so is $u_\lambda(x,t) \equiv \lambda u(\lambda x, \lambda^2 t), \quad \forall \lambda > 0.$

velocity potential $u = \frac{\partial \phi}{\partial x} \Longrightarrow \frac{\partial \phi}{\partial t} + \frac{1}{2}\left(\frac{\partial \phi}{\partial x}\right)^2 = \nu \frac{\partial^2 \phi}{\partial x^2}$

scale invariance $x \to \lambda x, t \to \lambda^2 t, \phi \to \lambda^{0} \phi$

If $\phi(x,t)$ is a solution, so is $\phi_\lambda(x,t) \equiv \lambda^{0} \phi(\lambda x, \lambda^2 t).$

Type 1 critical scale-invariance: physical dimension $[\phi] = [\nu] = \left[\frac{L^2}{T}\right]$

Cole suspected $u(x,t) = F(\theta(x,t))$ for some F, *and he was right.*

Dynamic scaling

$$U(\xi,\tau) = \lambda(t)u(x,t),\ \lambda(t) = \sqrt{2a(t+t_*)},\ (2at_* = 1)$$

$$\xi = \frac{x}{\lambda(t)},\ \tau = \frac{1}{2a}\log\frac{t+t_*}{t_*},$$

$$\frac{\partial U}{\partial \tau} + U\frac{\partial U}{\partial \xi} = \nu\frac{\partial^2 U}{\partial \xi^2} + \underbrace{a\xi\frac{\partial U}{\partial \xi} + aU}_{=a\frac{\partial}{\partial \xi}(\xi U)},$$

: type 2 critical scale-invariance

$$\phi(x,t) = \Phi(\xi,\tau)$$

$$\frac{\partial \Phi}{\partial \tau} + \frac{1}{2}\left(\frac{\partial \Phi}{\partial \xi}\right)^2 = \nu\frac{\partial^2 \Phi}{\partial \xi^2} + a\xi\frac{\partial \Phi}{\partial \xi}$$

Aside: an alternative look at type 2 scale-invariance

Linearised solutions

$$\widehat{\Phi}(\xi,\tau) = \left(\frac{a}{2\pi\nu(1-e^{-2a\tau})}\right)^{1/2} \int_{\mathbb{R}^1} \Phi_0(e^{a\tau}\eta) \exp\left(-\frac{a}{2\nu}\frac{(\xi-\eta)^2}{1-e^{-2a\tau}}\right) d\eta \to ?$$

as $\tau \to \infty$.

$$\widehat{U}(\xi,\tau) = \left(\frac{a}{2\pi\nu(1-e^{-2a\tau})}\right)^{1/2} \int_{\mathbb{R}^1} \underbrace{e^{a\tau}U_0(e^{a\tau}\eta)}_{\to M\delta(\eta)} \exp\left(-\frac{a}{2\nu}\frac{(\xi-\eta)^2}{1-e^{-2a\tau}}\right) d\eta$$

$$\to C \exp\left(-\tfrac{a\xi^2}{2\nu}\right),$$

Recall $\lambda f(\lambda x) \to M\delta(x)$, as $\lambda \to \infty, M = \int f dx$.

$$\text{Hence } \widehat{\Phi}(\xi,\tau) \to C \int_0^{\xi} \exp\left(-\frac{a\eta^2}{2\nu}\right) d\eta, \text{ retrospectively}$$

Repeat Cole's argument *after dynamic scaling*

$$\text{Assume } \Phi(s) = Cf(s) \text{ for } \exists f,\ s \equiv \int_0^{\xi} \exp\left(-\frac{a\eta^2}{2\nu}\right) d\eta,$$

$$\text{then } U = \Phi_\xi = C\exp\left(-\frac{a\xi^2}{2\nu}\right) f'(s),\ U_\xi + \frac{a}{\nu}\xi U = C\exp\left(-\frac{a\xi^2}{\nu}\right) f''.$$

Plug them into $\boxed{\dfrac{U^2}{2} = \nu U_\xi + a\xi U}$

$\Longrightarrow \dfrac{C}{2\nu}g^2 = g',\ g \equiv f'$: generating equation

$$\Longrightarrow g(s) = \frac{1}{1 - \frac{C}{2\nu}s}, \text{ or } f(s) = -\frac{2\nu}{C}\log\left(1 - \frac{C}{2\nu}s\right)$$

f is a near-identity, i.e. $f(s) \approx s$ for large ν

Near-Gaussian form $U(\xi) = \dfrac{C \exp\left(-\dfrac{a\xi^2}{2\nu}\right)}{1 - \dfrac{C}{2\nu}\int_0^\xi \exp\left(-\dfrac{a\eta^2}{2\nu}\right) d\eta}$

e.g. Liu and Pierre (1984), Escobedo and Zuazua (1991)

Source-type solution $u(x,t) = \dfrac{1}{\sqrt{2at}} U\left(\dfrac{x}{\sqrt{2at}}\right), \quad (t_* = 0)$

$\lim_{t\to 0} \frac{1}{\sqrt{2at}} U(\cdot) = M\delta(\cdot)$

$\lim_{t\to\infty} t^{\frac{1}{2}\left(1-\frac{1}{p}\right)} \left\| u(x,t) - \frac{1}{\sqrt{2at}} U\left(\frac{x}{\sqrt{2at}}\right)\right\|_p = 0$, for $u_0 \in L^1, 1 \le p \le \infty$

Small deviation from the Gaussian tells us how to handle more general solutions

$$U = -2\nu \partial_\xi \log \left(1 - \frac{C}{2\nu} \int_0^\xi \exp\left(-\frac{a\eta^2}{2\nu} \right) d\eta \right)$$

What can we learn if we do the same for other equations, hypoviscous Burgers, SQG & Navier-Stokes ?

3. Hypoviscous Burgers equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = -\nu' \Lambda u, \quad \Lambda \equiv (-\partial_{xx})^{1/2} = \partial_x H[\cdot]$$

Well-posedness: Kiselev, Nazarov & Shterenberg (2008)

scale invariance $x \to \lambda x, t \to \lambda t, u \to \lambda^{0} u$

If $u(x,t)$ is a solution, so is $u_\lambda(x,t) \equiv \lambda^{0} u(\lambda x, \lambda t)$

Type 1 critical scale-invariance: physical dimension $[u] = [\nu'] = \left[\frac{L}{T}\right]$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} = -w^2 - \nu' \Lambda w$$

scale invariance $x \to \lambda x, t \to \lambda t, w \to \lambda^{-1} w$

If $w(x,t)$ is a solution, so is $w_\lambda(x,t) \equiv \lambda w(\lambda x, \lambda t)$

Linearised solutions Poisson kernel (analogue of 'heat kernel')

$$u(x,t) = \frac{\nu t}{\pi} \int \frac{u_0(y)dy}{(x-y)^2 + (\nu' t)^2}$$

Dynamic scaling

$$U(\xi, \tau) = u(x,t), \xi = \frac{x}{\lambda(t)}, \tau = \frac{1}{a} \log \frac{t+t_*}{t_*}, \ \ \lambda(t) = a(t+t_*)$$

$$\frac{\partial U}{\partial \tau} + U \frac{\partial U}{\partial \xi} = -\nu' \Lambda U + a\xi \frac{\partial U}{\partial \xi}$$

In velocity gradient

$$\frac{\partial W}{\partial \tau} + U \frac{\partial W}{\partial \xi} = -W^2 - \nu' \Lambda W + a \frac{\partial}{\partial \xi}(\xi W)$$

Linearised solutions: scaled form

$$\lim_{\tau\to\infty}\widehat{W}(\xi,\tau)=\frac{M}{\pi}\frac{\mu}{\xi^2+\mu^2},$$

where $\mu=\nu'/a,\ M\equiv\int w dx$

$$\lim_{\tau\to\infty}\widehat{U}(\xi,\tau)=\frac{M}{\pi}\tan^{-1}\frac{\xi}{\mu}.$$

cf. After dynamic scaling

$$\widehat{U}(\xi,\tau)=\frac{\nu'}{\pi}\frac{1-e^{-a\tau}}{a}\int\frac{U_0(e^{a\tau}\eta)d\eta}{(\xi-\eta)^2+\left(\nu'\frac{1-e^{-a\tau}}{a}\right)^2},$$

$$\widehat{W}(\xi,\tau)=\frac{\nu'}{\pi}\frac{1-e^{-a\tau}}{a}\int\frac{e^{a\tau}W_0(e^{a\tau}\eta)d\eta}{(\xi-\eta)^2+\left(\nu'\frac{1-e^{-a\tau}}{a}\right)^2},$$

cf. Different scaling adopted under different BCs, Iwabuchi(2015)

Self-similar profile $W(\xi) = \lim_{\tau\to\infty} W(\xi,\tau)$

$$\text{Put } W_\tau = 0, \;\; (UW)_\xi = -\nu' \Lambda_\xi W + a(\xi W)_\xi.$$

$$\boxed{\xi W - \mu H[W] = \frac{1}{a} UW \qquad (*)}$$

If the nonlinear term is neglected

$$\xi W - \mu H[W] = 0.$$

To leading-order

$$W \propto \frac{\mu}{\xi^2 + \mu^2}$$

$$\text{cf. } H\left[\frac{\mu}{\xi^2+\mu^2}\right] = \frac{\xi}{\xi^2+\mu^2}, \; H\left[\frac{\xi}{\xi^2+\mu^2}\right] = -\frac{\mu}{\xi^2+\mu^2}, \; (\mu > 0)$$

Assume $U(\xi) = Cf(s), W(\xi) = C\frac{\mu}{\xi^2+\mu^2}f'(s),\ s \equiv \tan^{-1}\frac{\xi}{\mu}$

$$UW = -\nu' H[W] + a\xi W$$

$$\Longrightarrow \frac{C}{a}\frac{ff'}{\xi^2+\mu^2} = \frac{\xi}{\xi^2+\mu^2}f' - H\left[\frac{\mu}{\xi^2+\mu^2}f'\right],$$

or $\frac{C}{a}\frac{ff'}{\xi^2+\mu^2} = H\left[\frac{\mu}{\xi^2+\mu^2}\right]f' - H\left[\frac{\mu}{\xi^2+\mu^2}f'\right]$, (a commutator)

$\boxed{Rfg\cos^2 s = K[1]g - K[g]}, g = f'$: generating equation for $f(s)(\approx s)$,

where $R = \frac{C}{\nu},\ K[g] = \frac{1}{\pi}\text{p.v.}\int_{-\pi/2}^{\pi/2}\frac{g(s')ds'}{\tan s - \tan s'},\ K[1] = \cos s \sin s$

cf. $\boxed{\frac{R}{2}g^2 = g',}$ $g \equiv f'$ for standard Burgers

Successive approximation

$$f'_{n+1} = \frac{1}{K[1]}\Big(K[f'_n] + R\, f_n f'_n \cos^2 s\Big)\ \ (n = 1, 2, \ldots)$$

$$f_1 = s \to f'_2 = 1 + R\frac{s\cos^2 s}{\cos s \sin s}$$

$$f'_2 = 1 + R\, s \cot s$$

$\int s \cot s ds = s - \frac{s^3}{9} - \frac{s^5}{225} - \frac{2s^7}{6615} - \ldots,\ |s| < \pi,\ \notin$ elementary functions

Lifting (standard Burgers equation)

Step 1. Assume $\Phi(\xi) = Cf(s), s = \widehat{\Phi} = \int_0^\xi \exp\left(-\frac{a\eta^2}{2\nu}\right)$

$$U(\xi) = F(\widehat{\Phi}(\xi); \partial_\xi \widehat{\Phi}(\xi)) \equiv \frac{\partial_\xi \widehat{\Phi}}{1 - \frac{1}{2\nu}\widehat{\Phi}}$$

$U(\xi)$ is a *near-identity* transformation of $\partial_\xi \widehat{\Phi}(\xi)$.

Reverting to the original variables, a particular solution is

$$u(x,t) = \frac{1}{\sqrt{2at}}\left(\widehat{\Phi}\left(\frac{x}{\sqrt{2at}}\right); \partial_\xi \widehat{\Phi}\left(\frac{x}{\sqrt{2at}}\right)\right) \equiv \frac{1}{\sqrt{2at}} \frac{\partial_\xi \widehat{\Phi}(\frac{x}{\sqrt{2at}})}{1 - \frac{1}{2\nu}\widehat{\Phi}(\frac{x}{\sqrt{2at}})}.$$

Step 2. Replacing the self-similar heat flow with the general one,

$$\begin{aligned} u(x,t) &= \frac{1}{\sqrt{2at}} F\left(\widehat{\phi}\,(x,t)\,;\sqrt{2at}\,\partial_x \widehat{\phi}\,(x,t)\right) \\ &= F\left(\widehat{\phi}\,(x,t)\,;\partial_x \widehat{\phi}\,(x,t)\right) \equiv \frac{\partial_x \widehat{\phi}(x,t)}{1 - \frac{1}{2\nu}\widehat{\phi}(x,t)}, \end{aligned}$$

where

$$\widehat{\phi}(x,t) = \frac{1}{\sqrt{4\pi\nu t}} \int_{-\infty}^{\infty} \widehat{\phi}(y,0) \exp\left(-\frac{(x-y)^2}{4\nu t}\right) dy$$

Lifting (Hypoviscous Burgers equation)
Step 1. Assume that $U(\xi) = Cf(s)$, $s(= \widehat{U}) \equiv \tan^{-1}\frac{\xi}{\mu}$

$$W(\xi) = Cf'(s)\frac{\mu}{\xi^2+\mu^2} = F(\widehat{U}(\xi); \partial_\xi \widehat{U}(\xi))$$

$W(\xi)$ is a *near-identity* transformation of $\partial_\xi \widehat{U}(\xi)$.
Reverting to the original variables, a particular solution is

$$w(x,t) = \frac{1}{at}F\left(\widehat{U}\left(\frac{x}{at}\right); \partial_\xi \widehat{U}\left(\frac{x}{at}\right)\right)$$

Step 2. Replacing the self-similar Poisson flow with the general one,

$$\begin{aligned} w(x,t) &= \frac{1}{at}F\left(\widehat{u}\left(x,t\right); at\partial_x \widehat{u}\left(x,t\right)\right) \\ &= F\left(\widehat{u}\left(x,t\right); \partial_x \widehat{u}\left(x,t\right)\right) = f'(\widehat{u}\left(x,t\right))\,\partial_x \widehat{u}\left(x,t\right) \end{aligned}$$

$$\text{where } \widehat{u}(x,t) = \frac{\nu t}{\pi}\int \frac{u_0(y)dy}{(x-y)^2 + (\nu' t)^2}$$

4. Summary and outlook

- Self-similar solution to the hypoviscous Burgers equation

- A generating equation for the near-identity $f(\cdot)$ is derived. (Integrable by the Poisson kernel.)

- The second-order approximation is worked out, but turns out to be poorly accurate.

- Better approximations is to be sought.

- Applications to other equations (SQG, Navier-Stokes).

- Energy dissipation rate for the hypoviscous Burgers equation