①

$$\begin{cases} \partial_t v_q + \operatorname{div}(v_q \otimes v_q) + \nabla p_q = \operatorname{div}(R_q + c_q \mathrm{Id}) \\ \operatorname{div} v_q = 0 \end{cases}$$

$$\sum_q \| v_{q+1} - v_q \|_{C^0} < +\infty$$

$$\| R_q \|_{C^0} \to 0$$

Not nec. div-free

$$v_{q+1}(x,t) = v_q(x,t) + \overbrace{W\left(v_q(x,t), R_q(x,t), \lambda x, \lambda t\right)}^{\text{Not nec. div-free}} + w_c$$

$$W(v, R, \xi, \tau) \qquad \xi \in \mathbb{T}^3$$

$$\begin{cases} \underbrace{\boxed{(\partial_\tau + v \cdot \nabla_\xi) W} + \operatorname{div}_\xi (W \otimes W) + \nabla_\xi P = 0} \\ \operatorname{div}_\xi W = 0 \\ \langle \boxed{W \otimes W} \rangle_\xi = \boxed{R} + c_q \mathrm{Id} \\ \langle W \rangle_\xi = 0 \end{cases}$$

Existence of a family: YES

$\| D_v^k W \|_{C^0}$ stay

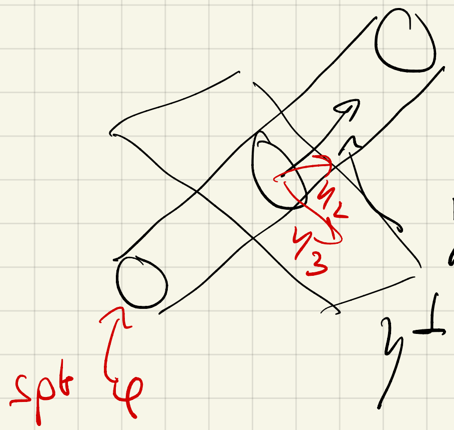
$\| D_R^k W \|_{C^0}$ bounded

$$\begin{cases} \boxed{\begin{aligned} &\langle W_s \otimes W_s \rangle_\xi = R \\ &\langle W_s \rangle_\xi = 0 \end{aligned}} \\ \operatorname{div}_\xi W_s = 0 \\ \operatorname{div}_\xi (W_s \otimes W_s) + \nabla_\xi p = 0 \end{cases}$$

<u>Beltrami fields</u> $\{W : \operatorname{curl} W = \lambda W\}$

λ constant

<u>Mikado flow</u>



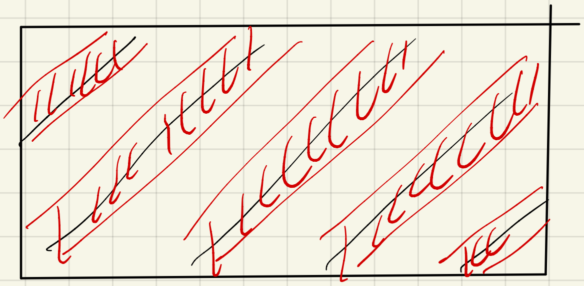
$y \in (2\pi \mathbb{Q})^3$ $\quad y^\perp$

$y_2, y_3 \perp y$

$$W = \varphi(y_1 \cdot x, y_2 \cdot x)\, y$$

pressureless solution

$$\begin{cases} \operatorname{div} W = 0 \\ \operatorname{div}(W \otimes W) = 0 \end{cases}$$



Take several diff. directions

$y^1\ y^2 \ldots\ldots,\ y^N$

and construct RoRodo's with disjoint

supports: take a linear combination

$$W(v, R, \xi, \tau) = W_s(R, \xi - v\tau)$$

$$\partial_\tau W + (v \cdot \nabla_\xi) W = 0$$

$$v_{q+1}(x,t) = v_q(x,t) + W_s(R_q(x,t), \lambda x - v_q(x,t)\lambda t)$$

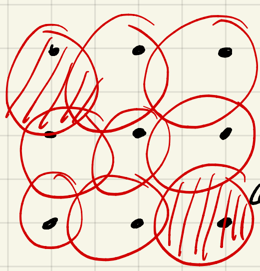
$$\begin{cases} \partial_\tau W + v \cdot \nabla_\xi W + \operatorname{div}_\xi W \otimes W + \nabla_\xi p = \dfrac{1}{\mu_q} \\ \operatorname{div}_\xi W = 0 \\ \langle W \otimes W \rangle_\xi = R \\ \langle W \rangle_\xi = 0 \end{cases}$$

$$W_s(R, \xi - \bar{v}\tau)$$ where $\bar{v}$ is a constant

$$\left(\partial_\tau + (\bar{v}\cdot\nabla)_\xi\right) W + \mathrm{div}_\xi (W \otimes W) + \nabla_\xi p = 0$$

$$\partial_\tau + (v\cdot\nabla_\xi) W + \mathrm{div}_\xi (W \otimes W) + \nabla_\xi p$$

$$= (v - \bar{v})\cdot\nabla_\xi W$$



$\bar{v}_k$

φ_k

small

small if $|v - \bar{v}| \ll 1$

$$W(R, v, \xi, \tau) = \sum \varphi_k(v)\, W_s^k(R, \xi - \bar{v}_k \tau)$$

large

$$D_v W = \sum D_v \varphi_k\, W_s^k(R, \xi - \bar{v}_k \tau)$$

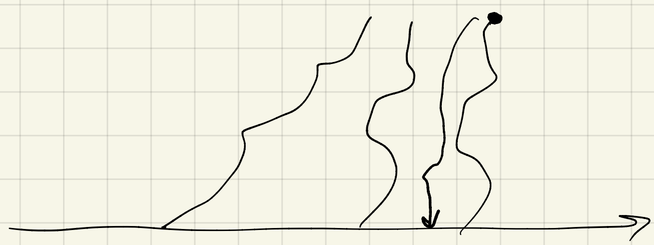
$\vdash$

How regular are the solutions that such a scheme could produce?

$$\partial_\tau W + (v_q \cdot \nabla_\xi) W + \operatorname{div}_\xi (W \otimes W) + \nabla_\xi p = 0$$

$$(\partial_t + v_q \cdot \nabla_\xi) W_s (R, \lambda x)$$

$$W_s (R, \lambda \Phi_q(t, x))$$



$$\begin{cases} \frac{d}{dt} \Psi_q(x,t) = v_q(t, \Psi_q(x,t)) \\ \Psi_q(x, 0) = x \end{cases}$$

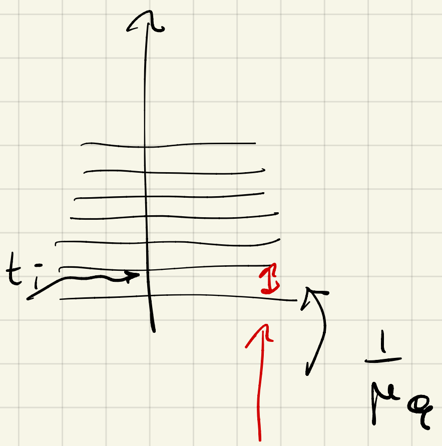
$$\Phi_q(t, \cdot) = \Psi_q^{-1}(t, \cdot)$$

$$\|D\Phi_q - Id\| \le \exp(\|Dv_q\|_0 \tau) - 1$$

$D_R W_s$ is fine

$$W_s (R, \lambda x + \lambda \underbrace{(\Phi_q(t, x) - x)})$$

Define flows Φ^i at discretized times



$$W(R, \lambda x, \lambda t)$$

$$= \sum \underbrace{\varphi_i\left(\frac{t-t_i}{\mu_q}\right)} \underbrace{W_s^i(R, \lambda \Phi^i(x,t))}$$

in this lifespan

$$\|D\Phi^i - Id\| \le \frac{C}{\mu_q}$$

$$\omega_0(x,t) = W(R_q(x,t), \lambda x, \lambda t)$$

$$(\partial_t \omega_0 + (v_q(x,t)\cdot\nabla)\omega_0 = O(\mu_q) + \boxed{D_{R_0}W\,(\partial_t R_q + (v_q\cdot\nabla) R_q)}$$

Isett : PhD thesis

$$p_{q+1} = p_q + p_o$$

$$v_{q+1} = v_q + w_o + w_c$$

plug into the equation and invert the divergence

$$\partial_t w_o + (v_q \cdot \nabla) w_o + \operatorname{div}\left(\boxed{w_o \otimes w_o + R_q}\right)$$

$$+ (w_o \cdot \nabla) v_q + \nabla p_o \qquad = \operatorname{div} R_{q+1}$$

$$\operatorname{div}(R_q)$$

$$+ \boxed{\partial_t v_q + \operatorname{div}(v_q \otimes v_q) + \nabla p_q}$$

$$R_{q+1} = \underbrace{\operatorname{div}^{-1}\left(\partial_t w_o + (v_q \cdot \nabla) w_o\right)}_{\text{Transport error}}$$

$$\operatorname{div}^{-1}\left(\operatorname{div}(w_o \otimes w_o) + R_q\right)$$

$$+ \operatorname{div}^{-1}\left((w_o \cdot \nabla) v_q\right)$$

Choose the parameters $\lambda = \lambda_{q+1}$

$\mu = \mu_q$

in such a way that $\|R_{q+1}\|_{C^0}$ is much smaller than $\|R_q\|_{C^0}$

Transport error : hope for an estimate of

type $\dfrac{\mu_q}{\lambda_{q+1}}$

Look at W_s the C^1 and C^0 size $\times\ \lambda$

$\lambda x + \lambda(\Phi(x,t) - x)$

$$w_o(x,t) = \operatorname{div}\Big(\sum \varphi^2(\mu(t-t_i))\, W_s(R_q(x,t), \lambda\Phi(x,t)) \otimes W_s(R_q(x,t), \lambda\Phi(x,t))\Big)$$

$$= \operatorname{div}^{-1} \sum \varphi^2(\mu(t-t_i)) \operatorname{div}(W_s \otimes W_s)$$

$$= \operatorname{div}^{-1} \text{ slow variable} + \sum \varphi^2(\mu(t-t_i))$$

$$= \operatorname{div}^{-1} \lambda (D\Phi - Id) \operatorname{div}_\xi W_s(R_q(x,t), \lambda\Phi(x,t)) \otimes W_s(R_q(x,t), \lambda\Phi)$$

$$+ \lambda \nabla_\xi P$$

$O(1)$

$\dfrac{1}{\mu}$ $\qquad \dfrac{\|Dv_q\|_0}{\mu}$

$$\partial_t w_o + (v_q \cdot \nabla) w_o$$

$$(\partial_t + (v_q \cdot \nabla)) w_o = \sum_i \varphi(\mu_q(t-t_i))\, W_s(R_q(x,t), \lambda\Phi_i(x,t))$$

Oscillation error $\frac{1}{\mu}$ } $\longrightarrow$ $\mu \sim \lambda^{1/2}$

Transport error $\frac{\mu^\beta}{\lambda}$ } $\longrightarrow$ $O(\lambda^{-1/2})$ Hard part

$$\boxed{\text{Nash error} \quad \mathrm{div}^{-1}\left(\left(w_o \cdot \nabla\right) v_q\right) \sim \frac{1}{\lambda}}$$

C^0 convergence

$$\underbrace{\langle W_s \otimes W_s \rangle}_{\xi} = R$$

$$|W_s(R, \xi, \tau)| \lesssim |R|^{1/2}$$

$\hookrightarrow$ $\|w_o\|_{C^0} \leq \|R_q\|_{C^0}^{1/2}$

$\hookrightarrow$ $\|v_{q+1} - v_q\|_{C^0} \leq \|R_q\|_{C^0}^{1/2}$

If $\|R_q\|_{C^0}^{1/2}$ converges exponentially to 0

then $\sum_q \|v_{q+1} - v_q\|_{C^0} < +\infty$

What about convergence in C^α

$$\|v_{q+1} - v_q\|_{C^\alpha} \le \boxed{\underbrace{\|R_q\|_{C^0}^{1/2}}_{\delta_{q+1}^{1/2}} \lambda_{q+1}^{\alpha}}$$

Converges iff $\sum \delta_{q+1}^{1/2} \lambda_{q+1}^{\alpha}$ summable

I want it to go to zero "fast"

I want it to grow reasonably slow

$$\|R_q\|_{C^0} = \delta_{q+1} \qquad \|w_o\|_{C^0} \le \|R_q\|_{C^0}^{1/2}$$

$$\operatorname{div}^{-1}\left(\boxed{(w_o \cdot \nabla) v_q}\right) = \frac{1}{\lambda_{q+1}} \delta_{q+1}^{1/2} \|Dv_q\|_{C^0} \le \frac{\delta_{q+1}^{1/2} \delta_q^{1/2} \lambda_q}{\lambda_{q+1}}$$

$$\|v_{q+1} - v_q\|_{C^1} \lesssim \delta_{q+1}^{1/2} \lambda_{q+1}$$

$$\|v_{q+1}\|_{C^1} \lesssim \delta_{q+1}^{1/2} \lambda_{q+1}$$

$$\|R_{q+1}\|_{C^0} \le \frac{\delta_{q+1}^{1/2} \delta_q^{1/2} \lambda_q}{\lambda_{q+1}}$$

should be δ_{q+2} !

$$\frac{\delta_{q+1}^{1/2} \delta_q^{1/2} \lambda_q}{\lambda_{q+1}} \le \delta_{q+2}$$

$$\boxed{\begin{aligned}\lambda_q &= (\bar{\lambda})^q \\ \delta_q &= (\bar{\lambda})^{-2\alpha_0 q}\end{aligned}}$$

$\bar{\lambda} >> 1$

would give me convergence in C^α $\forall \alpha < \alpha_0$

$$\log_{\bar{\lambda}} \frac{\delta_{q+1}^{1/2}\,\delta_q^{1/2}\,\lambda_q}{\lambda_{q+1}} \leq \log_{\bar{\lambda}} \delta_{q+2}$$

$$\cancel{-q} - 1 - \alpha_0(\cancel{q}+1) - \cancel{\alpha_0 q} + \cancel{q} \leq -2\alpha_0(\cancel{q}+2)$$

$$-\alpha_0 - 1 \qquad \leq -4\alpha_0$$

$$3\alpha_0 \leq 1$$

$$\alpha_0 = \frac{1}{3}$$

Daneri - Székelyhidi

The Mikado flows are pressureless solutions

If v is div $v = 0$

$A^{-1} v (A x)$

You can "kill" the $\frac{1}{\mu}$ error

Looks ideal:

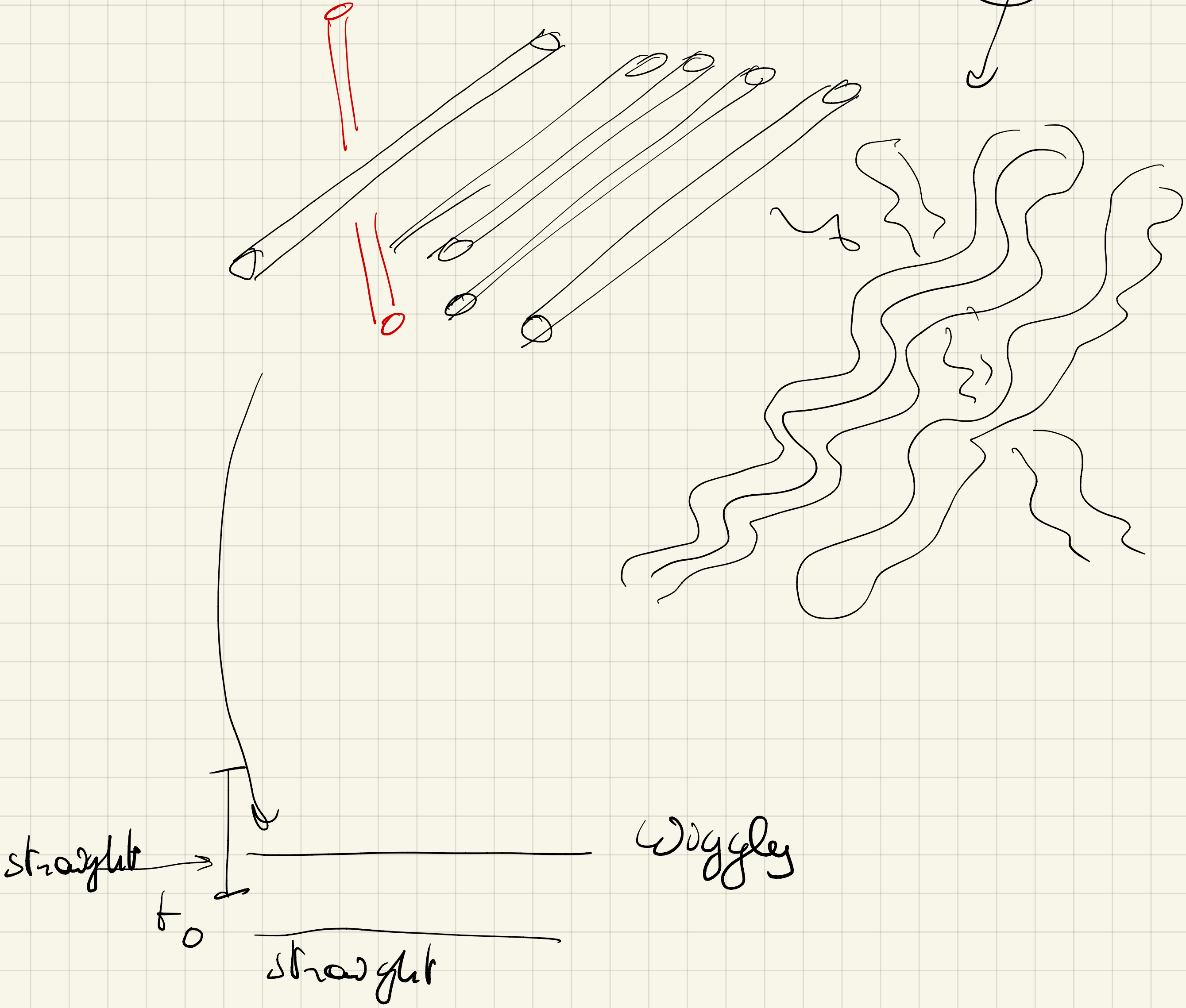
$\frac{\mu}{\lambda} \|w_o\|_{C^0}$ and $\frac{\|Dv_q\|_{C^0} \|w_o\|}{\lambda}$

Transport

$\frac{1}{\mu} \|Dv_q\|_0 \lesssim 1$ $\mu = \|Dv_q\|_0$

There is a problem

$$\sum \underline{\varphi(\mu(t-t_i))}\, W_s^i(R_q(x,t), \lambda \Phi^i(x,t))$$



Phase belt

glueing technique to achieve

this ideal case scenario