Dissipative Euler flows

$$\begin{cases} \partial_t u + \operatorname{div}(u \otimes u) + \nabla p = 0 \\ \operatorname{div} u = 0 \end{cases}$$

$$u : \mathbb{T}^3 \times I \to \mathbb{R}^3$$
$$p : \mathbb{T}^3 \times I \to \mathbb{R}$$

$$[\operatorname{div}(u \otimes u)]_i = [(u \cdot \nabla) u]_i = \sum_j u_j \partial_j u_i$$

For smooth solutions

$$\partial_t \frac{|u|^2}{2} + \operatorname{div}\left(\left(\frac{|u|^2}{2} + p\right) u\right) = 0 \qquad (LEI)$$

$$\frac{d}{dt} \frac{1}{2} \int_{\mathbb{T}^3} |u|^2 (x,t)\, dx = 0 \qquad (GEI)$$

Consider distributional solutions and continuous ones

Conservation of mass : $\int_{\partial\Omega} u \cdot \nu = 0$

Conservation of momentum,

$$\frac{d}{dt} \int_\Omega u = \int_{\partial\Omega} [u(u \cdot \nu) + p\nu]$$

HOLD $\forall \Omega \subset \mathbb{T}^3$ suff. smooth

Conjecture If $u \in C^\alpha$ then:

1) if $\alpha > \frac{1}{3}$ the GEI holds.
2) if $\alpha < \frac{1}{3}$ there are solutions which do not conserve the total kinetic energy (dissipate)

Theorem

1) Constantin - E - Titi 1994
2) Isett 2017

(dissipation : Buckmaster - D - Székelyhidi - Vicol)

There might be a "shadow" of anomalous dissipation:

$$\begin{cases} \partial_t u + \operatorname{div}(u \otimes u) + \nabla p = \varepsilon \Delta u \\ \operatorname{div} u = 0 \end{cases}$$

$$\frac{d}{dt} \frac{1}{2} \int |u|^2(x,t)\, dx = \overbrace{-\varepsilon \int |Du|^2(x,t)\, dx}^{O(1)}$$

$$\partial_t \frac{|u|^2}{2} + \operatorname{div}\left(u\left(\frac{|u|^2}{2}+p\right)\right) =$$

Duchon-Robert $\qquad \varepsilon\left(\Delta \boxed{\frac{|u|^2}{2}} - \boxed{|\nabla u|^2}\right)$

$\searrow$

$$\partial_t \frac{|u|^2}{2} + \operatorname{div}\left(u\left(\frac{|u|^2}{2}+p\right)\right) \leq 0 \qquad \text{(LE Ineq.)}$$

Conjecture (Strong form of Onsager)

If $u \in C^\alpha$ solution of Euler with $\alpha > \frac{1}{3}$
then the equality holds in (LE Ineq)

Duchon Robert $\rightarrow$ (follows from C-E-T proof!)

$\forall \alpha < \frac{1}{3}$ $\exists$ a solution $u \in C^\alpha$ for which

the inequality in (LE Ineq.) is strict.

Known in the literature:

D- Székelyhidi (2008) for bounded solutions
Isett (2019) for continuous solutions, and
Hölder, $C^{\frac{1}{15}-}$

D- Kwon (2020) $C^{\frac{1}{7}-}$

Aim of the lectures

1) Examine the "philosophy" behind C^0 iterations for the Euler equations (aim: to provide a general toolbox).

2) How to get to the Onsager conjecture from the flexibility side

3) Local energy inequality.

$\vdash\!\dashv$

Nash's scheme for C^1 isometric embeddings

(Σ, g) Riemannian manifold

$x \in \Sigma = \Omega \subset \mathbb{R}^n$

$g = g_{ij} \in \mathrm{Sym}^{n\times n}_{>}$

$u : \Omega \to \mathbb{R}^{n+2}$ ← replace it with a much larger number

$u \in C^1$ and $u^\# e = g$

$$Du^T \cdot Du = g$$

$$\boxed{\langle \partial_i u, \partial_j u \rangle = g_{ij}}$$

<u>Theorem</u> (Nash)

Assume $\Omega = \overline{B}_1 \subset \mathbb{R}^n$. Then $\exists\, u \in C^1(\overline{B}_1, \mathbb{R}^{n+2})$

s.t. $u^\# e = g$

I want to set up an iteration mechanism which starting from some app. solution produces an exact solution.

$$u_0 \to u_1 \to \dots u_k \to u_{k+1}$$

app. sol. with a certain precision

i.e. $g - u_k^\# e$ is small

app. sol. for which

metric error $\boxed{g - u_{k+1}^\# e}$ is <u>much smaller</u>

with good estimates on

$$\| u_{k+1} - u_k \|_{C^1}$$

Assume $\underbrace{g - u_k^{\#} e > 0}$ without loss of generality

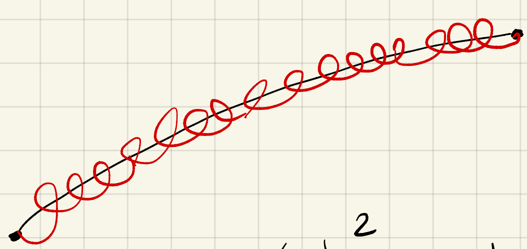
$\underbrace{-(1-\varepsilon)^2 Du_k^T Du_k + g}$ is uniformly small

↳ make it strictly positive

Toy situation : 1-d Riem. manifold

$$u_k : I \to \mathbb{R}^3$$

$$1 - |\dot{u}_k|^2 > 0$$



$$a_k(t)^2 = 1 - |\dot{u}_k|^2$$

$$u_{k+1}(t) = u_k(t) + \frac{a(t)}{\lambda}\left[n(t) \cos \lambda t + b(t) \sin \lambda t\right]$$

unit normal field

$n \perp b$

$$\dot{u}_{k+1} = \dot{u}_k(t) + \boxed{a(t)}\left[-n(t) \sin + b(t) \cos\right] + O\left(\frac{1}{\lambda}\right)$$

$$|\dot{u}_{k+1}|^2 = |\dot{u}_k|^2 + a^2 + O\left(\frac{1}{\lambda}\right)$$

$$\boxed{1 - |\dot{u}_k|^2} >> 1 - |\dot{u}_{k+1}|^2 > 0$$

$<< \delta_{k+1}$

$\leq \frac{\delta_{k+1}}{2}$

Choose a so that

$$1 - |\dot{u}_{k+1}|^2 \sim \delta_{k+1}$$

$$2\delta_k \geq 1 - |\dot{u}_k|^2 \geq a_k^2$$

$$0 < 1 - |\dot{u}_{k+1}|^2 < 2\delta_{k+1}$$

$$\|u_{k+1} - u_k\|_{C^1} \leq C\|a\|_{C^0} \leq C\sqrt{\delta_k}$$

Choose $\delta_k \downarrow 0$ so that $\sum \sqrt{\delta_k} < +\infty$:

1) C^1 convergence
2) the limit is a solution.

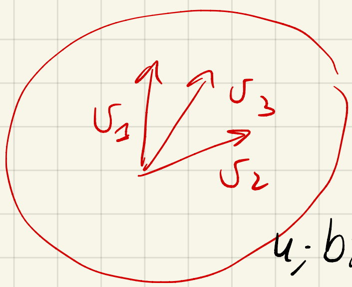
$\underbrace{g - u_k^{\#} e} > 0$ and small

$v_j \in \mathbb{R}^m$

$g - Du_k^T \, Du_k$

$y = \lambda x$

$$u_{k+1} = u_k + \sum_{i=1}^{N} \frac{1}{\lambda} \boxed{a_i(u_i(x)} \cos \boxed{(\lambda v_i \cdot x)} + b_i(x) \sin(\lambda v_i \cdot x))$$



$u_i \perp b_i \qquad |b_i| = |u_i| = 1$

$u_i \perp u_j$

$b_i \perp b_j$

$u_i, b_i \perp Du_k|_x(\mathbb{R}^m)$

$$Du_{k+1} = Du_k + \sum_{i=1}^{N} a_i(-u_i \cos + b_i \sin) \otimes v_i$$

$+ \boxed{O\left(\frac{1}{\lambda}\right)}$

indep. of λ

very close to 0 ?

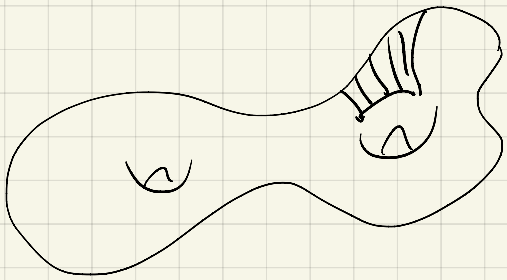
$$g - \underbrace{Du_{k+1}^T \cdot Du_{k+1} = \overbrace{\boxed{\underbrace{g - Du_k^T Du_k}}}^{n\varphi_i^2}} - \boxed{\sum_{i=1}^{N} a_i^2 \, v_i \otimes v_i} \boxed{+ O\left(\frac{1}{\lambda}\right)}$$

$$\underbrace{\sum_{i=1}^{\overset{\frac{m(m+1)}{2}}{N}} a_i^2 \, v_i \otimes v_i} = \underbrace{g - Du_k^T Du_k}$$

Easy linear algebra problem

Remark:

If Σ has nontrivial topology:



$$h = g - Du_k^T \cdot Du_k$$

$$h = \sum \varphi_i^2 h$$

for some partition of unity

"localization procedure"

Remark: Careful, you cannot choose σ_i depending on x

$$D \; \frac{a_i}{\lambda} \left(-u_i \cos\left(\lambda \sigma_i(x) \, x\right) \right)$$

$$- u_i a_i \cos \otimes \sigma_i$$

$$- u_i a_i \cos \otimes \left(D\sigma_i(x) \cdot x \right)$$

"Freezing the previous map"

for our purposes we can consider

$$Du_k \equiv \boxed{\text{constant}}$$

$$Du_k + D\,w_{k+1}$$

perturbation that I wan to add.

$$w_{k+1}(x) = W_{k+1}(x, \lambda x)$$

$$W_{k+1}(x, y)$$

$$W_{k+1}(h, A, y)$$

$$w_{k+1}(x) = W_{k+1}(h(x), A(x), \lambda x)$$

$$h = g - Du_k^T Du_k = g - A^T . A$$

$$h > 0$$

$$h - D_y W^T \cdot D_y W = 0$$

$$h - Z^T \cdot Z = 0$$

$y \in \Pi^m$

$$Z(h, A, y)$$

$$\begin{cases} \boxed{\mathrm{curl}_y Z} = 0 \\ \underbrace{Z^T \cdot Z} = h \\ \langle Z \rangle = 0 \end{cases}$$

+ error

decompose it in pieces which "I can kill"

error which I can add

$$\mathrm{curl}\left(Z(h(x), A(x), \lambda x)\right)$$

$$= \lambda\, \mathrm{curl}_y Z(\quad\quad) + \Delta_h Z + \Delta_A Z$$

y_0

$$Z(h, A, y) = \nabla_y W(h, A, y)$$

$$Z'(x, \lambda x) = \nabla\left(W(h(x), A(x), \lambda x)\right)\frac{1}{\lambda}$$

$$\mathrm{curl}\, Z' = 0$$

under control

$$Z - Z' = \frac{1}{\lambda}\left(\boxed{\Delta_h W}\, \Delta h + \boxed{\Delta_A W}\, A\right)$$

First remark:

I do not need to add all the corrections

$$a_i^2 \, v_i \otimes v_i \quad \text{all together}$$

What happens from column 2 to column 1

$$(A+Z)^T (A+Z) = g$$

$$Z^T \cdot Z + (A^T Z + Z^T A) = g - A^T A$$

$$\begin{cases} \partial_t u_k + \operatorname{div}\overbrace{(u_k \otimes u_k)}^{(u_k\cdot\nabla)u_k} + \nabla p_k = \operatorname{div}(R_k) \\ \operatorname{div} u_k = 0 \end{cases}$$

correction for divergence free

$$u_{k+1}(x,t) = u_k(x,t) + W(x,t,\lambda x,\lambda t) + w_c$$

$$p_{k+1}(x,t) = p_k(x,t) + P(x,t,\lambda x,\lambda t)$$

$\langle W \rangle_\xi$

$$\operatorname{div}_\xi W(R, v, \xi, \tau) = 0$$

$$W(R_k(x,t), u_k(x,t), \xi, \tau)$$

$$\Big(\partial_t u_k + \lambda\, \partial_\tau W(R_k, u_k, \lambda x, \lambda \tau)$$

$$+ \lambda (u_k \cdot \nabla_\xi) W(R_k, u_k, \lambda x, \lambda \tau)$$

$$- \langle W \otimes W \rangle_\xi$$

$$+ \lambda \operatorname{div}_\xi \big(W(R_k, u_k, \lambda x, \lambda \tau) \otimes W(R_k, u_k, \lambda x, \lambda t)\big)$$

$$+ \text{Other terms}\Big) = \operatorname{div} R_{k+1}$$

It is possible to invert the divergence operator

It is possible an operator div^{-1} s.t.

$$\operatorname{div}\big((\operatorname{div}^{-1} z)\big) = z \quad \text{if} \quad \langle z \rangle = 0$$

on the torus

In analogy to solving

$$\mathrm{div}\, X = f$$

$$X = \nabla(\Delta^{-1} f) \quad \text{if} \quad \int f = 0$$

$$R_{k+1} = \mathrm{div}^{-1}(\text{ Big expression })$$

$$\begin{cases} \partial_t W + (v \cdot \nabla_\xi) W + \mathrm{div}_\xi W \otimes W + \nabla_\xi P = 0 \\ \mathrm{div}_\xi W = 0 \end{cases}$$

$$\mathrm{div}^{-1}\Big(\overbrace{\partial_t u_k + \mathrm{div}(u_k \otimes u_k)}^{+\nabla p_k} + \mathrm{div}(\langle W \otimes W \rangle_\xi)$$

$$+ \underbrace{\text{Other terms}}\Big) = R_{k+1}$$

$$\mathrm{div}^{-1}(\text{Other term}) = \frac{1}{\lambda}$$

$$\mathrm{div}^{-1}\big(\mathrm{div}\,(\underbrace{R_k + \langle W \otimes W \rangle_\xi})\big)$$

$$\langle W \otimes W \rangle_\xi = -R$$

$$\langle W \rangle_\xi = 0$$