τ-TILTING THEORY – AN INTRODUCTION

HIPOLITO TREFFINGER

1. Introduction

The term τ*-tilting theory* was coined by Adachi, Iyama and Reiten in [3] a paper having this title that first appeared in the arXiv in 2012 and that was published in 2014. In this paper, the authors combined with a fresh approach to the the study of two classical branches of representation theory of finite dimensional algebras, namely tilting theory and Auslander-Reiten theory. The combination of these two subjects is clearly reflected in the name of this novel theory, where the greek letter τ represents the Auslander-Reiten translation in the module category of an algebra while the reference to tilting theory is obvious.

In these notes we give a small introduction to τ-tilting theory from the representation theoretic perspective, making special emphasis in the close relation between τ-tilting theory and torsion theories. Clearly, this notes do not represent a complete survey on the topic since several important aspects of the theory are not covered in detail or even mentioned. For instance, in this notes we do not cover the explicit between τ-tilting theory with cluster theory, stability conditions, or exceptional sequences, just to name a few.

These notes are organised as follows. In Section 2 we give a short historical account of the developments in representation theory that lead to the introduction of τ-tilting theory, hoping that this will help placing τ-tilting theory in the more general framework of representation theory.

Then, in Section 3 we start giving the material that is covered in the lectures by giving the definition of our main objects of study: τ-rigid and τ-tilting modules and pairs.

Afterwards, in Section 4 we explain the relationship between τ-tilting theory and torsion classes in the module category of algebras.

We finish these notes in Section 5 were we show how representation theory in general, and τ-tilting theory in particular, can be encoded using integer vectors.

We warn the reader that we do not include in these notes any proofs, these can found in the references given in the each result. Given the short time that has passed since the introduction of τ-tilting theory, to our knowledge, there is no much material on the subject available other that the original research papers, with the exception of [45]. For survey materials on more classical tilting theory, the reader is encouraged to see [7, 5].

2. Towards τ-tilting theory

It can be argued that the modern study of representation theory started with the parallel developments of almost split sequences by Auslander and Reiten [14, 16, 17] (see also [52]) and the theory of quiver representations by Gabriel [39, 40]. Gabriel showed two very important results using quivers. One of these results says that the representation theory of every finite dimensional algebra over an algebraically closed field can be understood using quiver representations. The formal statement is the following.

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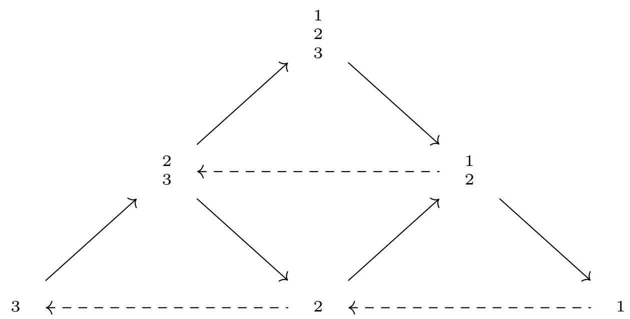


FIGURE 1. The Auslander-Reiten quiver of A

Theorem 2.1. [39] *Let A be a finite dimensional algebra over an algebraically closed field $\mathbb{K}$. Then A is Morita equivalent to the category $\mathbb{K}Q/I$, the path algebra of the quiver Q bounded by the ideal of relations I. Moreover the quiver Q is uniquely determined by A.*

In the literature, people refer to the quiver Q determined by the algebra as the *ordinary quiver* or the *Gabriel quiver* or simply the *quiver* of the algebra. In these notes we take the latter option. The reason to give it such names is that one can associate to each finite dimensional algebra another quiver known as the *Aulander-Reiten quiver* of the algebra, which encodes all the almost split sequences in mod A. For more information about the Auslander-Reiten theory of algebras, the reader is encouraged to see the course on this topic by Raquel Coelho-Simões in this same series.

The second result result of Gabriel we want to mention here is the classification of hereditary algebras of finite representation type by means of Dynkin diagrams as follows.

Theorem 2.2. [39] *Let A be a hereditary finite dimensional algebra over an algebraically closed field $\mathbb{K}$. Then A is Morita equivalent to $\mathbb{K}\Delta$, where Δ is a quiver whose underlying graph is a Dynkin diagram. Moreover there is a one-to-one correspondence between the indecomposable representations of A and the positive roots of the root systems associated to Δ.*

When this result appeared, it came with a great surprise since many fundamental properties of the path algebra of a quiver depends on the orientations of the arrows. For instance, the dimension as $\mathbb{K}$-vector spaces of a Dynkin quiver vary depending of the orientation we chose. Hence, there was no reason to believe that the number of indecomposable representations should be the same.

Example 2.3. For instance, take the algebra A and A' to be the path algebras of the quivers

$$Q_A = \ 1 \longrightarrow 2 \longrightarrow 3 \qquad Q_{A'} = \ 1 \longrightarrow 2 \longleftarrow 3$$

of type $\mathbb{A}_3$. A quick calculation shows that $\dim_{\mathbb{K}} A = 6$ while $\dim_{\mathbb{K}} A' = 5$. The Auslander-Reiten quiver of A and A' can be found in Figure 1 and Figure 2, respectively. Here the arrows correspond to the irreducible morphisms in its module category and the dashed line correspond to the Auslander-Reiten translation. In these figures we can see that the number of indecomposable representations of A and A' coincide.

As a consequence, there was a high interest to explain this phenomenon. The first explanation was given by Bernstein, Gelfand and Ponomarev in [23] by constructing the so-called reflection functors.

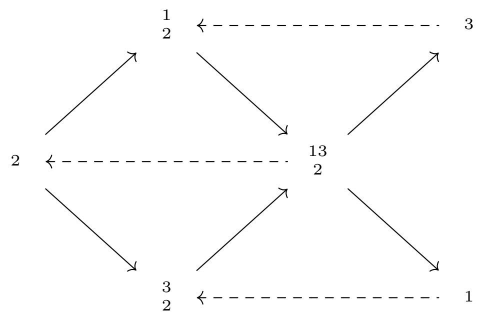


FIGURE 2. The Auslander-Reiten quiver of A'

Let Q a quiver of type Δ. Since every Dynkin diagram is a tree, there is at least one vertex $x \in Q_0$ which is a sink, that is a vertex such that all the arrows incident on that vertex are incoming arrows. Now, we construct a quiver Q' which is identical to Q, except for the fact that now the vertex x is a source, which means that every arrow incident to x is an outgoing arrow. One says that Q and Q' are reflections of each other at x. In our example, A' is the reflection of A at the vertex 3. Then, Bernstein, Gelfand and Ponomarev showed is the existence of functors, that they called reflection functors, between $\text{mod}\mathbb{K}Q$ and $\text{mod}\mathbb{K}Q'$ that induce a one-to-one correspondence between their indecomposable objects.

Some years after that, Auslander, Platzeck and Reiten [15] realised that these functors were induced by a very specific object in $\text{mod}\mathbb{K}Q$. To be more precise, note that the simple module $S(x)$ associated to the vertex $x \in Q_0$ is projective and it is not injective, so inverse of the Auslander-Reiten translation $\tau^{-1}S(x)$ of $S(x)$ is a non-zero indecomposable object of $\text{mod}\mathbb{K}Q$. Then they showed that the reflection functors described by Berstein, Gelfand and Ponomarev were induced by

$$T = \tau^{-1}S(x) \oplus \bigoplus_{x \neq y \in Q_0} P(y) \tag{1}$$

that is the sum of all of the indecomposable projectives except $S(x)$ direct sum $\tau^{-1}S(x)$. Moreover, they showed that $\mathbb{K}Q'$ was isomorphic to $\text{End}_{\mathbb{K}Q}(T)^{op}$. In particular, this approach allowed them to show the existence of reflection-like functors between the module category of any Artin algebra A having a simple projective module and $\text{End}_A(T)^{op}$, even when A is not hereditary or even when A has no quiver associated to it. Going once again to our running example, the module described by Auslander, Platzeck and Reiten in $\text{mod}A$ is $T = {\scriptstyle 2} \oplus \begin{smallmatrix}1\\2\\3\end{smallmatrix} \oplus \begin{smallmatrix}1\\2\end{smallmatrix}$

Some years later, Brenner and Butler went further and studied in [25] this phenomenon axiomatically. In this paper they introduce the notion of *tilting modules* as follows.

Definition 2.4. [25] Let A be an algebra and T be an A-module. We say that T is a tilting module if the following holds:

(i) $\text{pd}_A T \leq 1$, the projective dimension of T is at most 1.
(ii) T is rigid, that is $\text{Ext}^1_A(T, T) = 0$.
(iii) There exists a short exact sequence of the form

$$0 \to A \to T' \to T'' \to 0$$

where T', T'' are direct summands of direct sums of T.

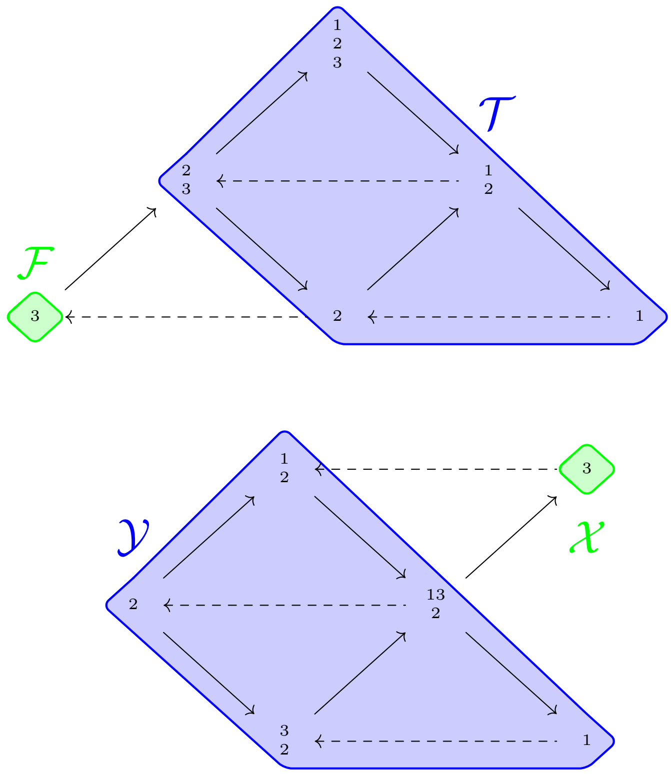


FIGURE 3. Torsion pairs and APR-tilting

In this paper they show that given any given tilting A-module T acts as a sort of translator between the representation theory of A and $B := \mathrm{End}_A(T)^{op}$, the opposite of the endomorphism algebra of T.

The first thing that they have shown is that a tilting A-module T is also a tilting B-module. Moreover they showed that T induced a torsion pair $(\mathcal{T}, \mathcal{F})$ in $\mathrm{mod}A$ and a torsion pair $(\mathcal{X}, \mathcal{Y})$ in $\mathrm{mod}B$ such that the functors $\mathrm{Hom}_A(T, -) : \mathrm{mod}A \to \mathrm{mod}B$ and $\mathrm{Ext}_A(T, -) : \mathrm{mod}A \to \mathrm{mod}B$ induce an equivalence of categories between $\mathcal{T}$ and $\mathcal{Y}$ and between $\mathcal{F}$ and $\mathcal{X}$, respectively. For the precise definition of torsion pair, please see Definition 4.1. This result of Brenner and Butler can be seen applied to our running example in Figure 3.

Since the module introduced by Auslander, Platzeck and Reiten was their motivating example, one can expect that it verifies (i)-(iii) and indeed this is the case. In fact, nowadays this module is nowadays known as the *APR-tilting module*. But, as the lector is already guessing, there are many more examples of tilting modules. Then take the module $T = {\scriptstyle 3} \oplus \begin{smallmatrix}1\\2\\3\end{smallmatrix} \oplus {\scriptstyle 1}$. One can verify that T is indeed a tilting module. Firstly, the projective dimension of T is less or equal to one since A is hereditary. Secondly, one can check that T does not admit self extensions. Finally, the short exact sequence

$$0 \to {\scriptstyle 3} \oplus \begin{smallmatrix}2\\3\end{smallmatrix} \oplus \begin{smallmatrix}1\\2\\3\end{smallmatrix} \to {\scriptstyle 3} \oplus \begin{smallmatrix}1\\2\\3\end{smallmatrix} \oplus \begin{smallmatrix}1\\2\\3\end{smallmatrix} \to {\scriptstyle 1} \to 0$$

is such that ${\scriptstyle 3} \oplus \begin{smallmatrix}1\\2\\3\end{smallmatrix} \oplus \begin{smallmatrix}1\\2\\3\end{smallmatrix}$ and ${\scriptstyle 1}$ belong to are direct summands of direct sums of T.

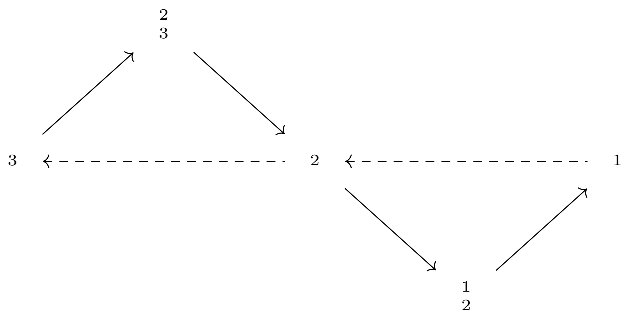


FIGURE 4. The Auslander-Reiten quiver of B

Now, the algebra $B = \mathrm{End}_A(T)^{op}$ is isomorphic to the path algebra of the quiver

$$1 \longrightarrow 2 \longrightarrow 3$$

modulo the ideal generated by the composition of the two arrows. The Auslander-Reiten quiver of B can be seen in Figure 4.

As we can see in this example, when we take an arbitrary tilting module A the number of indecomposable representations in mod$A$ and in mod$B$ are not the same. However, this is not a contradiction of the results of Brenner and Butler since their result only sees what happens inside the torsion pairs induced by T in mod$A$ and mod$B$. In this particular case, as we can see in Figure 5, the indecomposable object $\begin{smallmatrix}2\\3\end{smallmatrix}$ is does not belong to any of the two subcategories $\mathcal{T}$ and $\mathcal{F}$ induced by the tilting module T.

Although the module categories of an algebra can be to a certain extent different to the module category of the endomorphism algebra, it was shown by Happel [41], first, and then generalised by Rickard [53] that these two algebras are *derived equivalents.* Without going to the details, starting from an the module category of an algebra, one can construct a triangulated category known as the *derived category* of the algebra that encodes a wealth of homological information of the algebra. Then, the results of Happel and Rickard state that, at a derived level, the original algebra and the endomorphism algebra of the tilting module have the same derived category, which implies that they share many homological properties that we will not discuss here.

Let me do a small parenthesis here that will be important later. The algebra B is the smaller non-hereditary example of the so-called *tilted algebras.* Tilted algebras were introduced by Happel and Ringel in [42] as the endomorphism algebra of a tilting module over a hereditary algebra. The main idea behind its introduction was to use all the information available on hereditary algebras to understand a new class of algebras which have not been study systematically until that moment.

The study of tilted algebras has sparked a great deal of research which it would be impossible to describe completely here. However, I need to mention two notorious developments.

Firstly, note that the tilting theorem of Brenner and Butler does not impose any restriction on the algebra A. Then, if we are able to understand some of the representation theory of tilted algebras using the knowledge we have on hereditary algebras, we can repeat the process and understand the representation theory a new family of algebras using the knowledge we have on tilted algebras via the tilting theorem. These algebras are known as *iterated tilted algebras.* In [13], Assem and Skowroński classified all the iterated tilted of Dynkin type $\mathbb{A}$ in terms of their ordinary quiver and relations, which lead them to the definition of the so-called *gentle algebras.*

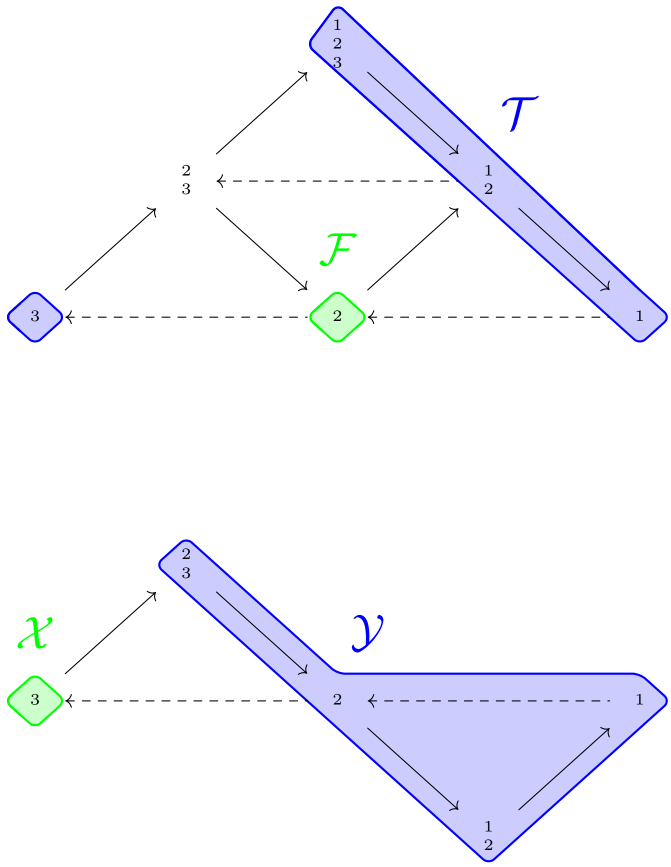


FIGURE 5. Torsion classes and a tilted algebra

Today, gentle algebras constitute a highly active area of research, deepening our understanding of representation theory of finite dimensional algebras and connecting this topic with various other branches of mathematics such as group theory and algebraic and differential geometry.

The second is the characterisation of tilted algebras found independently by Liu [48] and Skowroński [57] using the Auslander-Reiten quiver an algebra. They have shown that an algebra is tilted if and only if there is a structure with specific homological and combinatorial properties in their Auslander-Reiten quiver. Inspired by this characterisation of tilted algebras many families of algebras have been defined and determined by means of their Auslander-Reiten quiver.

Some years later, at the beginning of the twenty first century, Fomin and Zelevinsky [36, 37, 22, 38] were studying the properties of the canonical bases arising in Lie theory and this study lead to the introduction of *cluster algebras*.

This algebras are spanned by a basis of so-called *cluster variables* that are produced inductively from an *initial seed* via a process called *mutation*. Even if the process of mutation is repeated ad-infinitum, for some initial seeds there are only finitely many cluster variables that can be constructed. In this case we say that an algebra is of *finite type*. Moreover, for some of these algebras, known as *skew-symmetrizable* cluster algebras, their combinatorial construction can be expressed using quivers. One surprising result shown by Fomin and Zelevinsky in the first of the series of papers where they introduced cluster algebras is the following classification.

Theorem 2.5. [37] *Let $(Q, \{\underline{x}\})$ be the initial seed of a cluster algebra $\mathcal{A}$. Then $\mathcal{A}$ is of finite type if and only if Q is mutation equivalent to a quiver whose underlying graph is a Dynkin diagram.*

The resemblance of this result with Theorem 2.2 is striking and points towards a deep relationship between cluster theory and representations of finite dimensional algebras.

Cluster variables are always arranged in sets called *clusters* which have exactly n variables. These clusters can be arranged into a n-regular graph where there is an edge between two clusters if one can be obtained from the other performing a single mutation.

As it turns out, similar phenomena has been described in tilting theory. For instance it was shown by Skowroński in [58] that every tilting module has exactly n isomorphism classes of indecomposable direct summands. Also, Happel and Unger have shown in [43] that every partial tilting module having $n-1$ isomorphism classes of indecomposable direct summands can always be completed into a tilting module and there are at most two ways in which it can be completed to a tilting module.

Hence, one would like to categorify all the cluster phenomena using tilting theory, where the cluster variables are represented by indecomposable pretilting modules and tilting modules correspond to clusters. However, tilting theory falls short to describe the cluster phenomena for at least two reasons. The first is that, as we just mention, there are some examples of almost complete tilting modules that can be completed into a tilting module in exactly one way, which means that we can not reproduce the process of mutation in some indecomposable direct summand of this module.

The second reason, and maybe the most obvious, is that there are fewer indecomposable partial tilting modules than cluster variables. For instance, an hereditary path algebra of type $\mathbb{A}_n$ has exactly $\frac{n(n-1)}{2}$ indecomposable partial tilting modules, while the number of cluster variables in a cluster algebra of type $\mathbb{A}_n$ is $\frac{n(n+1)}{2}$, that is, there are exactly n more cluster variables than indecomposable partial tilting objects.

Then if one wants to categorify cluster algebras using tilting theory, it is necessary to extend the latter in some way. That is exactly what Buan, Marsh, Reineke, Reiten and Todorov did in [26]. In this seminal paper, instead of working with the module category of the algebra, they constructed a slightly larger triangulated category that they called the *cluster category* where everything works perfectly by the definition of the so-called *(partial) cluster-tilting objects*.

See in Figure 6 the cluster category associated to the algebra A of Example 2.3. The points that are tagged with the same object in the Auslander-Reiten quiver of $\mathcal{C}_A$ should be identified. In particular, we see that the Auslander-Reiten quiver of the cluster category of an algebra of type $\mathbb{A}_3$ is a Möbius strip. In fact the Auslander-Reiten quiver of the cluster category of any algebra of type $\mathbb{A}_n$ is a Möbius strip.

One the one hand they show that there is a one to one correspondence between cluster variables and indecomposable partial cluster-tilting objects; that there is a one to one correspondence between clusters and cluster-tilting objects; that the mutation is well-defined in all the indecomposable direct summands of any cluster-tilting object; and that the mutation of clusters and cluster-tilting objects are compatible.

On the other, they showed that there is a natural inclusion of the module category of the path algebra into the cluster category such that every (partial) tilting module in the module category becomes a (partial) cluster-tilting module. Moreover, they show that every possible mutation of tilting modules at the level of the module category becomes a mutation of cluster-tilting modules at the level of the cluster category.

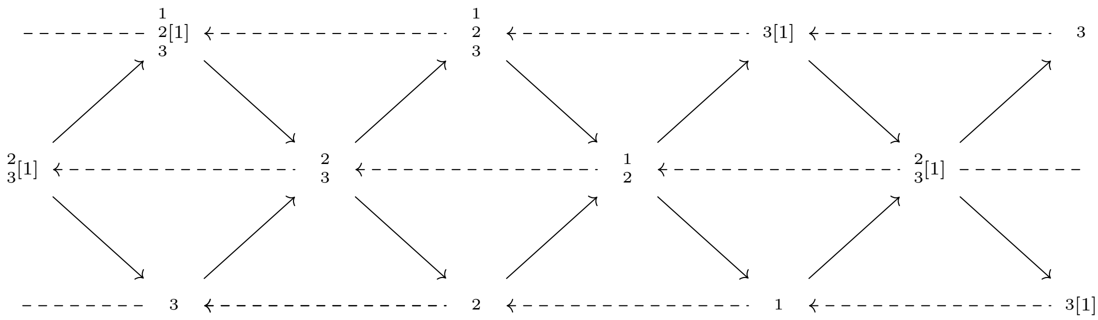


FIGURE 6. The Auslander-Reiten quiver of $\mathcal{C}_A$

We said before that Happel and Ringel showed that much of the representation theory of tilted algebras can be described from the information we have about the representation theory of the hereditary algebras. Now, the cluster categories associated to hereditary algebras have very nice properties, close to the properties of the hereditary algebras they come from. So Buan, Marsh and Reiten, emulating the construction of tilted algebras, introduced in [27] the *cluster-tilted algebras* as the endomorphism algebras of cluster-tilting objects in a cluster category. In this case, they showed that given a cluster-tilting object T in $\mathcal{C}_A$, the functor $\mathrm{Hom}_{\mathcal{C}_A}(T, -)$ induces a equivalence of categories between $\mathrm{mod}(\mathrm{End}_{\mathcal{C}_A}(T))^{op}$ and the quotient of $\mathcal{C}_A$ by the ideal $\mathcal{I}(\tau T)$ of all the morphisms that factor through τT the Auslander-Reiten translation of T .

We have mentioned already that any hereditary category A is naturally embedded in its cluster category $\mathcal{C}_A$. Moreover, if T is a tilting object in mod$A$, it turns out that T becomes a cluster-tilting object in $\mathcal{C}_A$ when we apply the natural embedding. Then starting from T we can construct a tilted algebra $\mathrm{End}_A(T)$ and a cluster-tilted algebra $\mathrm{End}_{\mathcal{C}_A}(T)$. The relation between $\mathrm{End}_A(T)$ and $\mathrm{End}_{\mathcal{C}_A}(T)$ and their module categories was studied by Assem, Brüstle and Schiffler in a series of papers [9, 8, 10, 11]. Firstly, they showed that one can recover $\mathrm{End}_{\mathcal{C}_A}(T)$ from the $\mathrm{End}_A(T)$ via a process that they called *relation extension* which bypasses the cluster category $\mathcal{C}_A$. Moreover, they have shown that every cluster-tilted algebra is the relation extension of a tilted algebra. They also have characterised all the tilted algebras that have an isomorphic Weymanrelation extension using a particular structure that can be found in the Auslander-Reiten quiver of cluster-tilted algebras which are deeply related to the structures described by Liu and Skowroński for tilted algebras.

In order to start the construction of the cluster category, Buan, Marsh, Reineke, Reiten and Todorov assumed that the quiver in the initial seed of the algebra is acyclic. However, there is no reason why one should start with an acyclic quiver. From a cluster perspective, any quiver is equally valid, so was expected for a similar cluster category to exists regardless of the quiver we choose at the start. The first problem with a more general quivers arising in cluster theory is that they have cycles, so their path algebra is infinite dimensional. Then in order to use something close to tilting theory, we need to quotient this path algebra with the correct ideal of relations. This problem was solved by Derksen, Weyman and Zelevinsky [33, 34] when they started considering certain of potentials associated to a quiver. They have shown that associated to each quiver there exists a special potential, that they called *non-degenerate*, such that one can categorify the cluster algebra associated to the quiver using their *decorated* representations. Moreover, they went further and showed that there exists a notion of mutation of non-degenerated potentials that is compatible with the cluster mutations of the quivers. We note that the notion of decorated representation gives rise a rich theory that is closely related to that of τ-tilting theory and it can be considered as a precursor of the latter.

Now that we have the correct algebras associated to the cyclic quivers in cluster theory we would like to have their corresponding cluster categories. To build these categories is not obvious. The main problem being that the construction of Buan, Marsh, Reineke, Reiten and Todorov uses heavily the structure of the derived category of the algebra and some key properties used in their construction fail when the algebra is not hereditary. Then this problem was overcome by Amiot in [4], where she used the theory of Ginzburg dg-algebras developed by Keller and Yang in [47] to construct a cluster category is compatible with the other notions of cluster categories existing to that moment.

All the phenomenon on cluster algebras and its close parallelism with tilting theory pointed to the existence of an extension of classical tilting theory were we would be always allowed to perform mutations. For hereditary algebras, the construction of this theory was performed by Ingalls and Thomas in [44], where they introduced the so-called *support tilting modules*. To explain the notion of support tilting module, let us look at the limitations of classical tilting theory.

As we did before consider A to be the path algebra of the linearly oriented $\mathbb{A}_3$ quiver. Then $T = \begin{smallmatrix}1\\2\\3\end{smallmatrix} \oplus \begin{smallmatrix}1\end{smallmatrix} \oplus \begin{smallmatrix}3\end{smallmatrix}$ is a tilting module in $\text{mod}A$.

Ideally, we would like to perform a mutation over every single direct summand of T. In other words, we would like to replace each indecomposable direct summand of T by another indecomposable in such a way that the resulting module is again tilting.

The summand $\begin{smallmatrix}1\end{smallmatrix}$ is replaceable, since we can change it by $\begin{smallmatrix}2\\3\end{smallmatrix}$ to obtain $\begin{smallmatrix}1\\2\\3\end{smallmatrix} \oplus \begin{smallmatrix}2\\3\end{smallmatrix} \oplus \begin{smallmatrix}3\end{smallmatrix}$ which is tilting.

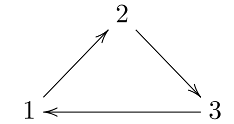
We can also mutate at the summand $\begin{smallmatrix}3\end{smallmatrix}$, because it can be replaced by $\begin{smallmatrix}1\\2\end{smallmatrix}$ to obtain the tilting module $\begin{smallmatrix}1\\2\\3\end{smallmatrix} \oplus \begin{smallmatrix}1\\2\end{smallmatrix} \oplus \begin{smallmatrix}1\end{smallmatrix}$.

However, we can not replace $\begin{smallmatrix}1\\2\\3\end{smallmatrix}$ by any other indecomposable module to obtain a new tilting module. This is a consequence to a classical result obtained independently by Assem [] and Smalø[], which implies that every indecomposable projective-injective object in $\text{mod}A$ is a direct summand of any tilting module in $\text{mod}A$. In particular, $\begin{smallmatrix}1\\2\\3\end{smallmatrix}$ can not be replaced because it is a projective-injective module in $\text{mod}A$.

The solution found by Ingalls and Thomas was to drop $\begin{smallmatrix}1\\2\\3\end{smallmatrix}$ from T altogether to obtain $T' = \begin{smallmatrix}1\end{smallmatrix} \oplus \begin{smallmatrix}3\end{smallmatrix}$ which clearly is not tilting. However, it is tilting on its *support algebra*, which is constructed by taking a quotient of A by the ideal generated by the idempotent included in the annihilator $annT'$ of T'.

More generally, they showed that for a hereditary algebra the mutation is always possible if we allow our tilting modules not to be supported over every vertex of the algebra.

Now, for more general algebras this construction fails again. For instance, if we take the algebra A to be the path algebra of the quiver



modulo the ideal generated by all paths of length 2, we have that A as a right module over itself is isomorphic to $\begin{smallmatrix}1\\2\end{smallmatrix} \oplus \begin{smallmatrix}3\\1\end{smallmatrix} \oplus \begin{smallmatrix}2\\3\end{smallmatrix}$. Note that in this case every indecomposable projective is also injective.

But at the same time we can not drop any of the direct summands since the sum of the two remaining projective modules is supported on every vertex of the algebra.

Something that we have not said before is that, by construction, in the cluster category we have that

$$\mathrm{Ext}^1_{\mathcal{C}_A}(M, N) \cong \mathrm{Hom}_{\mathcal{C}_A}(N, \tau M).$$

This isomorphism can actually be translated to the module categories of non-hereditary cluster-tiled algebras. So, we can translate the cluster-tilting objects of the cluster category to the module category of a cluster-tilted algebra to get a series of modules which categorify perfectly the corresponding cluster algebra. However, these objects are not in general partial tilting objects because they might be of infinite projective dimension.

Then, Adachi, Iyama and Reiten introduced τ-tilting theory in [3], the object of study of these lectures, by dropping the restriction on the projective dimension of the modules into consideration and changing the classical rigidity by the notion of *τ-rigidity* that we will introduce in the next section.

Before starting with the material of the lectures notes, we would like to point out that many results of τ-tilting theory were developed independently by Derksen and Fei in [32], where they studied *general presentations* using methods of a more geometric nature.

3. τ-tilting theory: Basic definitions

In this section we give the basic definitions on τ-tilting theory. We also mention some of the basic relations between τ-tilting theory and classical tilting theory.

Recall that in this notes A is always a finite dimensional algebra over an algebraically closed field, mod$A$ is the category of finitely generated right A-modules and τ denotes the Auslander-Reiten translation in mod$A$.

Given any A-module M, we denote $|M|$ the number of isomorphism classes of indecomposable direct summands of M. Throughout this document we assume that n is the number of isomorphism classes of simple A-modules. Note that in this case $|A| = n$.

Also, unless otherwise specified, every module in this notes is assumed to be basic, meaning that the set of indecomposable direct summands of M are pairwise non-isomorphic. For more background material, the reader is encouraged to see [12, 19, 54].

We start giving the definition of the most fundamental notion in this notes.

Definition 3.1. [21, 20, 3] Let A be an algebra and M be an object in mod$A$. We say that M is τ-rigid if $\mathrm{Hom}_A(M, \tau M) = 0$.

We now give without proof a series of important properties of τ-rigid modules which show in which sense τ-tilting theory is a generalisation of the classical tilting theory.

Proposition 3.2. *Let A be an algebra and T be a partial tilting module. Then T is τ-rigid. Moreover, if T is tilting then $|T| = n$.*

Proposition 3.3. [12] *Let M be a τ-rigid module. Then the following holds.*

1. *There are at most n isomorphism classes of indecomposable direct summands of M. In short, $|M| \leq n$.*
2. *M is rigid, that is, $\mathrm{Ext}^1_A(M, M) = 0$.*
3. *If the annihilator ann(M) of M is equal to the ideal $\{0\} \subset A$, then M is a partial tilting module.*
4. *If the projective dimension pd$M$ of M is at most one, then M is a partial tilting module.*
5. *If $|M| = n$ and ann$(M) = \{0\}$, then M is a tilting module.*

We now give the definition of support τ-tilting modules.

Definition 3.4. [3] Let A be an algebra. A τ-rigid A-module M is τ-tilting if $|M| = n$. We say that a τ-rigid A-module M is support τ-tilting if there exists an idempotent $e \in A$ such that M is a τ-tilting A/AeA-module, where AeA is the ideal generated by e in A.

As a direct consequence of the results in Proposition 3.2 and Proposition 3.3 we obtain the following.

Proposition 3.5. *Let A be an algebra. Then an A-module M is tilting if and only if M is τ-rigid and $pdM \leq 1$.*

This result can be presented as evidence to the statement that τ-tilting theory is a generalisation of tilting theory which is independent of the projective dimension of the objects. Following this idea, in the last decade a series of works appeared generalising classical results in tilting theory to τ-tilting theory.

However, one needs to be careful when giving such statements, since before the definition of τ-tilting theory there have been at least another generalisation of tilting theory to higher projective dimensions. I am speaking of the generalised tilting modules introduced by Miyashita in [49]. They are defined as follows.

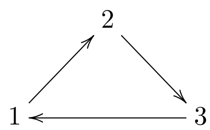
Definition 3.6. [49] Let A be an algebra and T be an A-module and r be a positive integer. We say that T is a r-tilting module if the following holds:

(i) $\mathrm{pd}_A T \leq r$, the projective dimension of T is at most r.
(ii) $\mathrm{Ext}^i_A(T, T) = 0$ for all $1 \leq i \leq r$.
(iii) There exists a short exact sequence of the form
$$0 \to A \to T^{(1)} \to T^{(2)} \to \cdots \to T^{(r)} \to 0$$
where $T^{(i)}$ is a direct summands of direct sums of T for all $1 \leq i \leq r$.

Over time, it has been shown that many important results of classical tilting theory can be generalised to generalised tilting modules. One might expect that people would try to show an explicit relation between τ-tilting modules and generalised tilting modules. However, to my knowledge, nobody has done this yet.

We now give an example of all the support τ-tilting modules in the module category of an algebra.

Example 3.7. Let A be the path algebra given by the quiver



modulo the second power of the ideal generated by all the arrows. The Auslander-Reiten quiver of A can be seen in Figure 7.

Note that every module is represented by its Loewy series and both copies of $\begin{smallmatrix}2\\3\end{smallmatrix}$ should be identified, so the Auslander-Reiten quiver of A has the shape of a cylinder. In the table 1 we give a complete list of the support τ-tilting modules in mod$A$ together with its associated idempotent.

Suppose that we are working on with the algebra of the previous example and we come across the module $M = \begin{smallmatrix}1\\2\end{smallmatrix} \oplus 1$. After a quick calculation we can see that this module is τ-rigid. But is this module support τ-tilting or is it only τ-rigid?

What I mean with this question is that M is only a support τ-tilting module only when we associate to it the idempotent e_3. However this notation does not allow us to see if M has associated the idempotent or not.

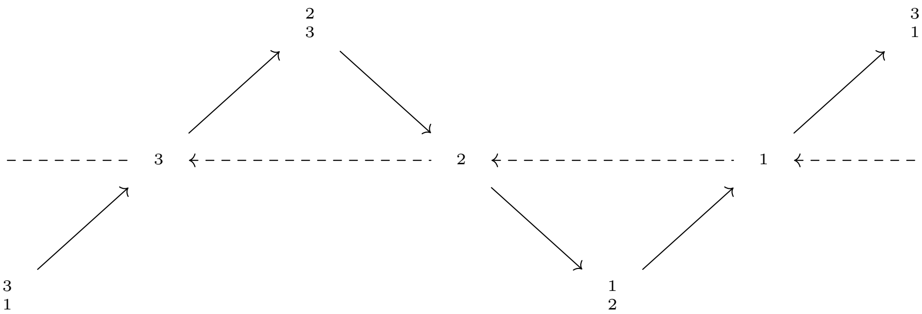


FIGURE 7. The Auslander-Reiten quiver of B

support τ-tilting module	idempotent
$\begin{smallmatrix}1\\2\end{smallmatrix}\oplus\begin{smallmatrix}2\\3\end{smallmatrix}\oplus\begin{smallmatrix}3\\1\end{smallmatrix}$	$\varnothing$
$\begin{smallmatrix}1\\2\end{smallmatrix}\oplus\begin{smallmatrix}2\\3\end{smallmatrix}\oplus 2$	$\varnothing$
$\begin{smallmatrix}1\\2\end{smallmatrix}\oplus\begin{smallmatrix}3\\1\end{smallmatrix}\oplus 1$	$\varnothing$
$\begin{smallmatrix}2\\3\end{smallmatrix}\oplus\begin{smallmatrix}3\\1\end{smallmatrix}\oplus 3$	$\varnothing$
$\begin{smallmatrix}3\\1\end{smallmatrix}\oplus 3$	e_2
$\begin{smallmatrix}3\\1\end{smallmatrix}\oplus 1$	e_2
$\begin{smallmatrix}1\\2\end{smallmatrix}\oplus 1$	e_3
$\begin{smallmatrix}1\\2\end{smallmatrix}\oplus 2$	e_3
$\begin{smallmatrix}2\\3\end{smallmatrix}\oplus 3$	e_1
$\begin{smallmatrix}2\\3\end{smallmatrix}\oplus 2$	e_1
1	e_2+e_3
2	e_1+e_3
3	e_1+e_2
0	$e_1+e_2+e_3$

TABLE 1. Support τ-tilting modules in mod$A$

As we will see, for certain problems in τ-tilting theory it is important to distinguish the τ-rigid module $\begin{smallmatrix}1\\2\end{smallmatrix}\oplus 1$ from the support τ-tilting module $\begin{smallmatrix}1\\2\end{smallmatrix}\oplus 1$ and for that we need to the notion of τ-rigid and τ-tilting pairs. Before doing so, recall that given an idempotent $e \in A$ we have that the right ideal eA is a projective module and that every projective arises this way.

Definition 3.8. Let A be an algebra, M be an A-module and P be a projective module. We say that the pair (M, P) is τ-rigid if M is a τ-rigid module and $\mathrm{Hom}_A(P, M) = 0$. A τ-rigid pair is τ-tilting if $|M| + |P| = n$.

As you would expect, these two notations are equivalent. Indeed, given a support τ-tiling module M with associated idempotent e we have that (M, eA) is a τ-tilting pair. Conversely, if (M, P) is a τ-tilting pair then we have that $P = eA$ for some idempotent $e \in A$. Then M is a support τ-tilting module with associated idempotent e. The list of all τ-tilting pairs of the algebra in Example 3.7 can be found in Table 2

τ-tilting pair
$\left(\begin{smallmatrix}1\\2\end{smallmatrix} \oplus \begin{smallmatrix}2\\3\end{smallmatrix} \oplus \begin{smallmatrix}3\\1\end{smallmatrix}, 0\right)$
$\left(\begin{smallmatrix}1\\2\end{smallmatrix} \oplus \begin{smallmatrix}2\\3\end{smallmatrix} \oplus \begin{smallmatrix}2\end{smallmatrix}, 0\right)$
$\left(\begin{smallmatrix}1\\2\end{smallmatrix} \oplus \begin{smallmatrix}3\\1\end{smallmatrix} \oplus \begin{smallmatrix}1\end{smallmatrix}, 0\right)$
$\left(\begin{smallmatrix}2\\3\end{smallmatrix} \oplus \begin{smallmatrix}3\\1\end{smallmatrix} \oplus \begin{smallmatrix}3\end{smallmatrix}, 0\right)$
$\left(\begin{smallmatrix}3\\1\end{smallmatrix} \oplus \begin{smallmatrix}3\end{smallmatrix}, \begin{smallmatrix}2\\3\end{smallmatrix}\right)$
$\left(\begin{smallmatrix}3\\1\end{smallmatrix} \oplus \begin{smallmatrix}1\end{smallmatrix}, \begin{smallmatrix}2\\3\end{smallmatrix}\right)$
$\left(\begin{smallmatrix}1\\2\end{smallmatrix} \oplus \begin{smallmatrix}1\end{smallmatrix}, \begin{smallmatrix}3\\1\end{smallmatrix}\right)$
$\left(\begin{smallmatrix}1\\2\end{smallmatrix} \oplus \begin{smallmatrix}2\end{smallmatrix}, \begin{smallmatrix}3\\1\end{smallmatrix}\right)$
$\left(\begin{smallmatrix}2\\3\end{smallmatrix} \oplus \begin{smallmatrix}3\end{smallmatrix}, \begin{smallmatrix}1\\2\end{smallmatrix}\right)$
$\left(\begin{smallmatrix}2\\3\end{smallmatrix} \oplus \begin{smallmatrix}2\end{smallmatrix}, \begin{smallmatrix}1\\2\end{smallmatrix}\right)$
$\left(\begin{smallmatrix}1\end{smallmatrix}, \begin{smallmatrix}2\\3\end{smallmatrix} \oplus \begin{smallmatrix}3\\1\end{smallmatrix}\right)$
$\left(\begin{smallmatrix}2\end{smallmatrix}, \begin{smallmatrix}1\\2\end{smallmatrix} \oplus \begin{smallmatrix}3\\1\end{smallmatrix}\right)$
$\left(\begin{smallmatrix}3\end{smallmatrix}, \begin{smallmatrix}1\\2\end{smallmatrix} \oplus \begin{smallmatrix}2\\3\end{smallmatrix}\right)$
$\left(\begin{smallmatrix}0\end{smallmatrix}, \begin{smallmatrix}1\\2\end{smallmatrix} \oplus \begin{smallmatrix}2\\3\end{smallmatrix} \oplus \begin{smallmatrix}3\\1\end{smallmatrix}\right)$

TABLE 2. τ-tilting pairs in mod$A$

4. τ-TILTING PAIRS AND TORSION CLASSES

In this section, after recalling the definition of torsion pairs and their basic properties, we will investigate the deep relation existing between τ-tilting theory and torsion classes.

4.1. **Torsion pairs and torsion classes.** The notion of torsion pairs, also known as torsion theories, started almost with the introduction of abelian categories as a generalisation of a well-known phenomenon in the category of finitely generated abelian groups, one of the most iconic example of abelian categories.

A classical classification result states that given a finitely generated abelian group, up to isomorphism, has a unique torsion subgroup such that the resulting factor group is torsion free. The extension of this fact to every abelian category was done by Dickson in [] as follows.

Definition 4.1. [35] Let $\mathcal{A}$ be an abelian category and let $(\mathcal{T}, \mathcal{F})$ be a pair of subcategories of $\mathcal{A}$. We say that $(\mathcal{T}, \mathcal{F})$ is a torsion pair in $\mathcal{A}$ if the following holds.

(1) $\operatorname{Hom}_{\mathcal{A}}(X, Y) = 0$ for all $X \in \mathcal{T}$ and $Y \in \mathcal{F}$.
(2) For all object $M \in \mathcal{A}$ there exists, up to isomorphism, a unique short exact sequence
$$0 \to tM \to M \to fM \to 0$$
such that tM is an object of $\mathcal{T}$ and fM is an object of $\mathcal{F}$.

If $(\mathcal{T}, \mathcal{F})$ is a torsion pair in $\mathcal{A}$ we say that $\mathcal{T}$ is a torsion class and $\mathcal{F}$ is a torsion free class. Moreover, for each object M of $\mathcal{A}$, we say that
$$0 \to tM \to M \to fM \to 0$$
is the canonical short exact sequence of M and that tM is the torsion object of M with respect to the torsion pair $(\mathcal{T}, \mathcal{F})$.

The previous definition is valid for an arbitrary abelian category. However, in these notes we are interested in a particular class of abelian categories, namely the category of finitely generated modules over a finitely dimensional algebra. These categories have many extra properties (for example they are length categories) that allow us to describe more precisely the torsion pairs in this category.

Proposition 4.2. *Let A be an algebra. Then the following holds true.*

(1) *A subcategory $\mathcal{T}$ of* $\operatorname{mod}A$ *is a torsion class if and only if $\mathcal{T}$ is closed under quotients and extensions. Moreover, in this case the torsion free class associated to $\mathcal{T}$ is*

$$\mathcal{F} = \{Y \in \operatorname{mod}A : \operatorname{Hom}_A(X, Y) = 0 \text{ for all } X \in \mathcal{T}\}.$$

(2) *A subcategory $\mathcal{F}$ of* $\operatorname{mod}A$ *is a torsion free class if and only if $\mathcal{F}$ is closed under subobjects and extensions. Moreover, in this case the torsion free class associated to $\mathcal{F}$ is*

$$\mathcal{T} := \{X \in \operatorname{mod}A : \operatorname{Hom}_A(X, Y) = 0 \text{ for all } Y \in \mathcal{F}\}.$$

Suppose that M is an A-module. Then we can ask the following: Is there a minimal torsion class in $\operatorname{mod}A$ containing M? The following result answers this question affirmatively.

Proposition 4.3. *Let A be an algebra. Then the intersection of arbitrarily many torsion classes is a torsion class. Likewise, the intersection of arbitrarily many torsion free classes is a torsion free class.*

Then the minimal torsion class containing a given object M of $\operatorname{mod}A$ is simply the intersection of all torsion classes containing M. Clearly, M is in this intersection and, by the precious proposition, this is also a torsion class.

Now, there is a more descriptive answer to this question, but to give that answer we need to introduce some notation.

Let $\mathcal{X}$ be a subcategory of $\operatorname{mod}A$. The category $\operatorname{Filt}(\mathcal{X})$ of objects filtered by $\mathcal{X}$ is defined as the category of all the objects Y in $\operatorname{mod}A$ that admit a filtration

$$0 = Y_0 \subset Y_1 \subset \cdots \subset Y_{r-1} \subset Y_r = Y$$

such that the successive quotients Y_i/Y_{i-1} are objects in $\mathcal{X}$.

We define the category $\operatorname{Fac}\mathcal{X}$ as the category of objects Y such that there exists an object X in $\mathcal{X}$ and an epimorphism $p : X \to Y \to 0$. Often times in the notes, the category $\mathcal{X}$ we will the additive category $addM$ additively generated by a module M. In this case, by abuse of notation we will write $\operatorname{Fac}M$ instead of $\operatorname{Fac}(addM)$. Note that $\operatorname{Fac}M$ can be described as

$$\operatorname{Fac}M = \{Y \in \operatorname{mod}A : \text{ there is an epimorphism} p : M^r \to Y \to 0 \text{ for some } r \in \mathbb{N}\}.$$

Now we are able to give a better description of the minimal torsion class containing M.

Proposition 4.4. *Let A be an algebra and M be an A-module. Then* $\operatorname{Filt}(\operatorname{Fac}M)$ *is the minimal torsion class containing M.*

Remark 4.5. Note that, in general, $\operatorname{Fac}(\operatorname{Filt}M)$ is **not** a torsion class since it might not be closed under extensions.

4.2. **Torsion pairs and τ-rigid modules.** From the previous subsection we have that to get the minimal torsion class containing M one needs to first calculate $\operatorname{Fac}M$ and then make the extension closure of this category. However, sometimes $\operatorname{Fac}M$ is closed under extensions, which makes the second step of the construction superfluous.

The following theorem, originally proved by Auslander and Smalø in [20], is arguably the first result on τ-tilting theory, even if this theory was formally introduced thirty years later.

Theorem 4.6. *Let A be an algebra and M be an object in* $\operatorname{mod}A$. *Then* $\operatorname{Fac}M$ *is a torsion class if and only if M is τ-rigid, that is* $\operatorname{Hom}_A(M, \tau M) = 0$. *Moreover, in this case*

$$M^{\perp} := \{X \in \operatorname{mod}A : \operatorname{Hom}_A(M, X) = 0\}$$

is the torsion free class such that $(\operatorname{Fac}M, M^{\perp})$ *is a torsion pair in* $\operatorname{mod}A$.

As we just said, several years passed between the publication of this result and the start of τ-tilting theory as an independent subject in representation theory. However, this was not the only result that worked with τ-rigid objects. In fact, a well-established technique used in classical tilting to determine if an object is tilting was to show that the candidate M was a τ-rigid module such that $pdM \leq 1$ and $|M| = n$. The interested reader is encouraged to surf the literature to look for such examples.

4.3. **Functorially finite torsion pairs and τ-tilting.** From a torsion theoretic point of view, the breakthrough made by Adachi, Iyama and Reiten in [3] is that they showed that τ-tilting pairs characterised a particular class of torsion classes, the functorially finite torsion classes.

Let $\mathcal{X}$ be a subcategory of an abelian category $\mathcal{A}$ and suppose that X is an object of $\mathcal{X}$ and M is an arbitrary object of $\mathcal{A}$. A morphism $f : X \to M$ in is called a *right $\mathcal{X}$ -approximation* of M if any map $f' : X' \to M$ with $X' \in X$ factors through f. Dually, a morphism $g : M \to X$ in is called a *left $\mathcal{X}$ -approximation* of M if any map $g' : M \to X'$ with $X' \in X$ factors through g. We say that $\mathcal{X}$ is *contravariantly finite* (resp. *covariantly finite*) if any object M in $\mathcal{A}$ admits a right (resp left) $\mathcal{X}$ -approximation. We say that $\mathcal{X}$ is functorially finite if it is both contravariantly finite and covariantly finite.

An important consequence of the uniqueness up to isomorphism of the canonical exact sequence of an object with respect to a torsion pair is the following.

Proposition 4.7. *Let $(\mathcal{T}, \mathcal{F})$ be a torsion pair in an abelian category $\mathcal{A}$ and let M be an object of $\mathcal{A}$. If*

$$0 \to tM \to M \to fM \to 0$$

is the short exact sequence of M with respect to $(\mathcal{T}, \mathcal{F})$ then the canonical inclusion $i : tM \to M$ is a right $\mathcal{T}$-approximation. Dually, the canonical projection $p : M \to fM$ is a left $\mathcal{F}$-approximation. In particular every torsion class in $\mathcal{A}$ is contravariantly finite and every torsion free class $\mathcal{F}$ in $\mathcal{A}$ is covariantly finite.

Given a τ-rigid module M, we know by Theorem 4.6 that Fac$M$ is a torsion class. It is not hard to convince oneself that Fac$M$ is also a functorially finite. In part, the result of Adachi, Iyama and Reiten is telling us that these are all the functorially finite torsion classes. To give the precise statement of this result let us denote by τ-tp-A the set of all τ-tilting pairs and by ftors-A the set of all functorially finite torsion classes in mod$A$. Moreover we recall that an object T in a category $\mathcal{X}$ is said to be Ext-projective in $\mathcal{F}$ if $\operatorname{Ext}^1_A(T, X) = 0$ for all $X \in \mathcal{X}$.

Theorem 4.8. [3] *Let A be an algebra then the map $\Phi : \tau$-tp-$A \to$ ftors-A defined by*

$$\Phi(M, P) = \operatorname{Fac} M$$

is a bijection. Moreover, the inverse Φ^{-1} : ftors-$A \to \tau$-tp-A is defined as

$$\Phi^{-1}(\mathcal{T}) = (\mathcal{P}(\mathcal{T}), {}^{\perp_P}\mathcal{T})$$

where $\mathcal{P}(\mathcal{T})$ is a basic additive generator of the Ext*-projectives of $\mathcal{T}$ and ${}^{\perp_P}\mathcal{T}$ is a basic additive generator of the category of projective modules P such that* $\operatorname{Hom}_A(P, T) = 0$ *for all $T \in \mathcal{T}$.*

4.4. **Completing τ-rigid modules and τ-rigid pairs.** The choice of taking τ-rigid modules to develop τ-tilting theory is completely arbitrary. In fact, one can develop the completely dual τ^{-1}-tilting theory, where we study τ^{-1}-rigid modules, that is, modules N such that $\operatorname{Hom}_A(\tau^{-1}N, N) = 0$. In this case, dual of Theorem 4.6 reads as follows.

Theorem 4.9. [20] *Let A be an algebra and N be an object in* mod$A$*. Then the category*

$$\operatorname{Sub} N := \{Y \in \operatorname{mod} A : \text{ there is an monomorphismi} : 0 \to Y \to N^r \text{ for some } r \in \mathbb{N}\}$$

is a torsion free class if and only in N *is* τ^{-1}*-rigid. Moreover, in this case*

$$^{\perp}N := \{X \in \text{mod}A : \text{Hom}_A(X, N) = 0\}$$

is the torsion class such that $(\text{Sub}N, {}^{\perp}N)$ *is a torsion pair in* mod$A$.

Now, take a τ-rigid pair M. Then it is easy to see that τM is τ^{-1}-rigid. Indeed,

$$\text{Hom}_A(\tau^{-1}\tau M, \tau M) = \text{Hom}_A(M, \tau M) = 0.$$

Hence there are two torsion classes naturally associated to M, namely Fac$M$ and ${}^{\perp}\tau M$. In the following result we give some results regarding the relation between these two torsion classes and M, all of which appeared already in [3].

Theorem 4.10. [3] *Let* A *be an algebra and* M *be a* τ*-rigid* A*-module. Then the following holds.*

(1) $\text{Fac}M \subset {}^{\perp}\tau M$.
(2) The torsion classes $M^{\perp}$ *and* ${}^{\perp}\tau M$ *coincide if and only if* M *is* τ*-tilting.*
(3) Suppose that $\mathcal{T}$ *is a functorially finite torsion class. Then* M *is a direct summand of* $\mathcal{P}(\mathcal{T})$ *if and only if* $\text{Fac}M \subset \mathcal{T} \subset {}^{\perp}\tau M$.

From the previous theorem we have that ${}^{\perp}\tau M$ is the maximal torsion class having M as an Ext-projective, which makes the module $\mathcal{P}({}^{\perp}\tau M)$ especial enough to have a name. We say that $\mathcal{P}({}^{\perp}\tau M)$ is the *Bongartz completion* of M. This name was chosen in honour to Bongartz who showed in [24] that if T is partial tilting, then $\mathcal{P}({}^{\perp}\tau T)$ is a tilting module having T as a direct summand. In other words, Bongartz showed that every partial tilting module can be completed to a tilting module.

If we use the language of τ-rigid pairs instead of τ-rigid modules we can be more precise in our statements. Here we say that a τ-rigid pair (M, P) is a direct summand of (M', P') if M is a direct summand of M and P is a direct summand of P'.

Theorem 4.11. [3] *Let* A *be an algebra and* (M, P) *be a* τ*-rigid pair in* mod$A$*. Then the following holds.*

(1) ${}^{\perp}\tau M \cap P^{\perp}$ *is a torsion class and* $\text{Fac}M \subset {}^{\perp}\tau M \cap P^{\perp}$.
(2) The torsion classes $M^{\perp}$ *and* ${}^{\perp}\tau M \cap P^{\perp}$ *coincide if and only if* (M, P) *is a* τ*-tilting pair.*
(3) Suppose that $\mathcal{T}$ *is a functorially finite torsion class. Then* (M, P) *is a direct summand of* $\Phi^{-1}(\mathcal{T})$ *if and only if* $\text{Fac}M \subset \mathcal{T} \subset {}^{\perp}\tau M \cap P^{\perp}$.

As for τ-rigid modules, we say that $\Phi^{-1}({}^{\perp}\tau M \cap P^{\perp})$ is the Bongartz completion of (M, P). But now we can also compute the τ-tilting pair $\Phi^{-1}(\text{Fac}M)$ which is the τ-tilting pair generating the smallest torsion class containing (M, P). In this case, we say that $\Phi^{-1}(\text{Fac}M)$ is the Bongatrz cocompletion of (M, P).

4.5. **Mutation of τ-tilting pairs and torsion classes.** As we said in in the introduction of these notes, τ-tilting theory was conceived with the goal to complete the classical tilting theory with respect to mutation. The achievement of this goal in the following result.

Theorem 4.12. [3] *Let* A *be an algebra and let* (M, P) *be an almost complete* τ*-tilting pair, that is a* τ*-rigid pair such that* $|M| + |P| = n - 1$*. Then there is no functorially finite torsion class* $\mathcal{T}$ *such that* $\text{Fac}M \subsetneq \mathcal{T} \subsetneq {}^{\perp}\tau M \cap P^{\perp}$.

In other words, for every almost complete τ*-tilting pair* (M, P) *there are exactly two* τ*-tilting pairs* (M_1, P_1) *and* (M_2, P_2) *having* (M, P) *as a direct summand.*

The previous theorem allow us to give the following definition.

Definition 4.13. Let (M_1, P_1) and (M_2, P_2) be two τ-tilting pairs. We say that (M_1, P_1) is a mutation of (M_2, P_2) if there is an almost complete τ-tilting pair (M, P) which is a direct summand of (M_1, P_1) and (M_2, P_2). By abuse of notation we also say that FacM_1 is a mutation of FacM_2 if (M_1, P_1) is a mutation of (M_2, P_2).

Remark 4.14. Note that (M_1, P_1) is a mutation of (M_2, P_2) if and only if (M_2, P_2) is a mutation of (M_1, P_1). Also note that if (M_1, P_1) is a mutation of (M_2, P_2) then either Fac$M_1 \subset$ FacM_2 or Fac$M_2 \subset$ FacM_1. In particular, Fac$M_1 \neq$ FacM_2.

In fact, there are explicit homological formulas to construct (M_1, P_1) from (M_2, P_2) and back but we will not explicit them here. The interested reader is encouraged to see [3] for more details on the matter.

In Theorem 4.12 we said that given two torsion classes FacM_1 and FacM_2 which are mutation of each other, then there are no functorially finite torsion classes in between them. In fact, it was shown by Demonet, Iyama and Jasso in [29] that there are no torsion classes of any kind in between FacM_1 and FacM_2. This is a consequence of the following stronger result.

Theorem 4.15. [29] *Let A be an algebra (M, P) be a τ-tilting pair and $\mathcal{T}$ be a torsion class in* mod$A$. *Then the following hold.*

(1) If $\mathcal{T} \subsetneq$ Fac$M$ then there exists a mutation (M', P') such that $\mathcal{T} \subset$ Fac$M' \subsetneq$ Fac$M$.
(2) If Fac$M \subsetneq \mathcal{T}$ then there exists a mutation (M'', P'') such that Fac$M \subsetneq$ Fac$M'' \subset \mathcal{T}$.

4.6. **Maximal green sequences.** In the module category of any algebra A there are always at least two torsion classes, some times called the trivial torsion classes, which are the whole mod$A$ and the torsion class $\{0\}$ containing only the objects that are isomorphic to the 0 object. Both these torsion classes are functorially finite and they are generated by the τ-tilting pairs $(A, 0)$ and $(0, A)$, respectively.

Clearly $\{0\} \subsetneq$ mod$A$. So we can apply Theorem 4.15.1 and obtain a τ-tilting pair (M_1, P_1) which is a mutation of $(A, 0)$ such that $\{0\} \subset$ Fac$M_1 \subsetneq$ mod$A$. If FacM_1 is not equal to $\{0\}$ we can repeat the process to obtain a mutation (M_2, P_2) of (M_1, P_1) such that $\{0\} \subset$ Fac$M_2 \subsetneq$ Fac$M_1 \subsetneq$ mod$A$. We could repeat this process inductively to obtain a decreasing chain of torsion classes

$$\{0\} \subset \cdots \subsetneq \mathrm{Fac}M_3 \subsetneq \mathrm{Fac}M_2 \subsetneq \mathrm{Fac}M_1 \subsetneq \mathrm{mod}A$$

which in general can continue forever. However, in some cases this process stops. In other words, there is a finite set of τ-tilting pairs $\{(M_i, P_i) : 0 \leq i \leq t\}$ such that $(M_0, P_0) = (A, 0)$, $(M_t, P_t) = (0, A)$ and (M_i, P_i) is a mutation of (M_{i-1}, P_{i-1}). In this case, we say that the chain of torsion classes

$$\{0\} = \mathrm{Fac}M_t \subsetneq \cdots \subsetneq \mathrm{Fac}M_3 \subsetneq \mathrm{Fac}M_2 \subsetneq \mathrm{Fac}M_1 \subsetneq \mathrm{mod}A$$

is a *maximal green sequence.*

Remark 4.16. Note that a maximal green sequence is a non refinable chain of torsion classes, that is, if

$$\{0\} = \mathrm{Fac}M_t \subsetneq \cdots \subsetneq \mathrm{Fac}M_3 \subsetneq \mathrm{Fac}M_2 \subsetneq \mathrm{Fac}M_1 \subsetneq \mathrm{mod}A$$

is a maximal green sequence and $\mathcal{T}$ is a torsion class such that Fac$M_i \subset \mathcal{T} \subset$ FacM_{i-1} for some $1 \leq i \leq t$, then $\mathcal{T} =$ FacM_i or Fac$M_{i-1} = \mathcal{T}$.

Maximal green sequences were originally introduced by Keller in [] in the context of cluster algebras to give a combinatorial method to calculate certain geometric invariants known as Donaldson-Thomas invariants. The definition can be considered as a generalisation to the setting of τ-tilting theory of the original definition, since there are many examples of algebras which do not have a cluster counter-part.

Remark 4.17. Note that the word *green* in the name maximal green sequence does **not** make reference to any mathematician of name Green. Instead this words makes reference to the classical colouring in the traffic lights.

The reason for this is that in cluster algebras there is no evident reason to say that a mutation is going forward or backwards. However, Keller needed to impose such a direction to mutations in order to get the desired calculation. Then, he came up with a colouring of the vertices of the quiver associated to the cluster algebra in which a vertex is either green or red, which indicates if we are allowed to mutate at the given vertex or not, respectively. In this colouring, every vertex in the quiver of the initial seed is green and we are allowed to mutate at one green vertex at a time. The process finishes if after a finite number of mutations all the vertices in the quiver are red.

There has been a lot of interest in the representation theory community on the study of maximal green sequences. To learn more about this rich subject, please see [31].

4.7. **τ-tilting reduction and torsion classes.** For the moment we have only seen that an almost complete τ-tilting pair can be completed in exactly two ways to a τ-tilting pair. More generally, one can consider the problem of finding all τ-tilting pairs having a given τ-rigid pair (M, P) as a direct summand. This problem was solved by Jasso in [46] using a procedure that is now known as *τ-tilting reduction.* Here we give a brief summary of that process.

By Theorem 4.11 one knows that (M, P) yields the torsion classes $\mathrm{Fac}M$ and ${}^{\perp}(\tau M) \cap P^{\perp}$. Moreover, Theorem 4.11 states the existence of a τ-tilting pair of the form $(M \oplus M', P)$ such that $\mathrm{Fac}(M \oplus M') ={}^{\perp} (\tau M) \cap P^{\perp}$.

Now define $B_{(M,P)} = \mathrm{End}_A(M \oplus M')$ to be the endomorphism algebra of $M \oplus M'$. In the algebra $B_{(M,P)} = \mathrm{End}_A(M \oplus M')$, there is an idempotent element $e_{(M,P)}$ associated to the $B_{(M,P)}$-projective module $\mathrm{Hom}_A(M \oplus M', M)$. We define the algebra $C_{(M,P)}$ as the quotient of $B_{(M,P)}$ by the ideal generated by $e_{(M,P)}$, that is,

$$C_{(M,P)} := B_{(M,P)}/B_{(M,P)}e_{(M,P)}B_{(M,P)}.$$

Now we are able to state one of the main theorems of [46].

Theorem 4.18. [46] *Let (M, P) be a τ-rigid pair in* $\mathrm{mod}A$*. Then the functor*

$$\mathrm{Hom}_A(M \oplus M', -) : \mathrm{mod}A \to \mathrm{mod}B_{(M,P)}$$

induces an equivalence of categories

$$F : M^{\perp} \cap {}^{\perp}\tau M \cap P^{\perp} \to \mathrm{mod}C_{(M,P)}$$

between the perpendicular category $M^{\perp} \cap {}^{\perp}\tau M \cap P^{\perp}$ of (M, P) and the module category $\mathrm{mod}C_{(M,P)}$.

A direct consequence of Theorem 4.18 and Theorem 4.11 we obtain the following result.

Theorem 4.19. [46] *Let (M, P) be a τ-rigid pair in* $\mathrm{mod}A$ *and $C_{(M,P)}$ as above. Then the the functor*

$$\mathrm{Hom}_A(M \oplus M', -) : \mathrm{mod}A \to \mathrm{mod}B_{(M,P)}$$

induces a bijection between the torsion classes $\mathcal{T}$ in $\mathrm{mod}A$ *such that* $\mathrm{Fac}M \subset \mathcal{T} \subset {}^{\perp}\tau M \cap P^{\perp}$ *and the torsion classes in* $\mathrm{mod}C_{(M,P)}$.

In particular the functor

$$\mathrm{Hom}_A(M \oplus M', -) : \mathrm{mod}A \to \mathrm{mod}B_{(M,P)}$$

induces a bijection between the τ-tilting pairs in $\mathrm{mod}A$ *having (M, P) as a direct summand and the τ-tilting pairs in* $\mathrm{mod}C_{(M,P)}$.

Remark 4.20. Note that Theorem 4.19 does not give a specific number of completions of a given τ-rigid pair (M, P). This is due to the fact that the number of τ-tilting pairs in two algebras might differ hugely.

However, we can still recover Theorem 4.12 as a consequence of Theorem 4.19. Indeed, one can verify that for all almost complete τ-tilting pair (M, P) the algebra $C_{(M,P)}$ is local, which implies that there is only one isomorphism class of simple modules in $\text{mod}C_{(M,P)}$. As a consequence of this, if S is a simple module in $\text{mod}C_{(M,P)}$ we have that $\text{Hom}_{C_{(M,P)}}(X, S) = 0$ implies that X is isomorphic to 0.

Let S be a simple module in $\text{mod}C_{(M,P)}$ and $\mathcal{T}$ be a torsion class. Then we have two options either $S \in \mathcal{T}$ or $S \notin \mathcal{T}$. If $S \in \mathcal{T}$, then $X \in \mathcal{T}$ for all $X \in \text{mod}C_{(M,P)}$ since torsion classes are closed under extensions. Otherwise, we have that $\mathcal{T} = \{0\}$ by the argument above.

This shows that every local algebra, no matter how complicated its representation theory, has exactly two torsion classes in its module category, which are the trivial torsion classes. In particular, this implies that there are exactly two τ-tilting pairs in $\text{mod}C_{(M,P)}$ if (M, P) is an almost complete τ-tilting pair. Hence Theorem 4.12 follows from Theorem 4.19.

Note that the perpendicular category $M^{\perp} \cap {}^{\perp}\tau M \cap P^{\perp}$ defined by Jasso is a the intersection of the torsion class ${}^{\perp}\tau M \cap P^{\perp}$ with the torsion free class $M^{\perp}$. Moreover, in this case $M^{\perp\perp}\tau M \cap P^{\perp}$ is what is called a *wide subcategory* of $\text{mod}A$. A subcategory $\mathcal{X}$ is called wide when is closed under kernels, cokernels and extensions. In particular, this implies that $\mathcal{X}$ is an abelian category. Then Asai and Pfeiffer found in [6] the following generalisation of Theorem 4.19.

Theorem 4.21. [6] *Let* $(\mathcal{T}_1, \mathcal{F}_1)$ *and* $(\mathcal{T}_2, \mathcal{F}_2)$ *be two torsion pairs in* $\text{mod}A$ *such that* $\mathcal{T}_1 \subset \mathcal{T}_2$. *Suppose moreover that* $\mathcal{T}_2 \cap \mathcal{F}_1$ *is a wide subcategory of* $\text{mod}A$. *Then there is a bijection between the torsion classes* $\mathcal{T}$ *in* $\text{mod}A$ *such that* $\mathcal{T}_1 \subset \mathcal{T} \subset \mathcal{T}_2$ *and the torsion classes in* $\mathcal{T}_2 \cap \mathcal{F}_1$ *given by map* $\mathcal{T} \mapsto \mathcal{T} \cap \mathcal{T}_1$.

However, the intersection of a torsion class with a torsion free class is not always a wide subcategory. However it has some structure, the intersection of a torsion class and a torsion free class is always a *quasi-abelian* subcategory. The definition of quasi-abelian subcategories is a bit technical and it will be skipped.

However, it is worth mentioning that Tattar showed in [59] that there is a well-defined notion of torsion classes in quasi-abelian subcategories. Moreover he showed that Theorem 4.19 can be generalised to this setting as follows.

Theorem 4.22. [59] *Let* $(\mathcal{T}_1, \mathcal{F}_1)$ *and* $(\mathcal{T}_2, \mathcal{F}_2)$ *be two torsion pairs in* $\text{mod}A$ *such that* $\mathcal{T}_1 \subset \mathcal{T}_2$. *Then there is a bijection between the torsion classes* $\mathcal{T}$ *in* $\text{mod}A$ *such that* $\mathcal{T}_1 \subset \mathcal{T} \subset \mathcal{T}_2$ *and the torsion classes in* $\mathcal{T}_2 \cap \mathcal{F}_1$ *given by map* $\mathcal{T} \mapsto \mathcal{T} \cap \mathcal{T}_1$.

4.8. **τ-tilting finite algebras.** To finish this section we speak about a new class of algebras that originated with the study of τ-tilting theory, the so-called τ-tilting finite algebras. They were introduced by Demonet, Iyama and Jasso as follows.

Definition 4.23. [29] An algebra A is τ-tilting finite if there are only finitely many τ-tilting pairs in $\text{mod}A$.

Even if the class of τ-tilting finite algebras has been recently introduced, they have received a lot of attention. In the following theorem we compile a series of characterisations of τ-tilting finite algebras. For that, recall that an A-module M is called a *brick* if its endomorphism algebra is a division algebra.

Theorem 4.24. *Let* A *be an algebra. Then the following are equivalent.*

(1) A *is* τ*-tilting finite.*

(2) *There are finitely many indecomposable τ-rigid objects in* mod$A$.
(3) [29] *There are finitely many torsion classes in* mod$A$.
(4) [30] *There are finitely many bricks in* mod$A$.
(5) [55] *The length of all bricks in* mod$A$ *is bounded.*

Remark 4.25. Note that by Theorem 4.24.3 we have that all torsion classes in the module category of a τ-tilting finite algebra is functorially finite.

As we can see in Theorem 4.24, τ-tilting finite algebras have module categories that are somehow manageable for a torsion theoretic perspective, even if they are wild. As a consequence, there is an ongoing informal programme that aims to classify all the τ-tilting finite algebras. This problem has been attacked by several people in different families of algebras. The following is a list of families of algebras where some progress to understanding on the problem has been made. Note that this list is not exhaustive nor efficient, since some families are included in others.

- [39] Hereditary algebras.
- [1] Nakayama algebras.
- [37, 27] Cluster-tilted algebras.
- [60] Auslander algebras.
- [50] Preprojective algebras.
- [51] Gentle algebras.
- [2] Brauer graph algebras.
- [56] Special biserial algebras.

5. Integer vectors and τ-tilting theory

In the introduction of these notes we said that many developments that occurred in representation theory in the twenty first century, including τ-tilting theory, were aiming to *categorify* cluster algebras to some extent.

In loose terms, the term *categorification* refers to the process of explaining some combinatorial phenomena by showing the existence some categorical phenomena. For instance, the bijection between the indecomposable τ-rigid modules in the module category of an hereditary algebra of Dynkin type and the number of non-initial variables in the cluster algebra of the corresponding Dynkin type is a categorification of cluster variables.

But as there is a process of categorification, there is also a process of *decategorification*, a process where you start with a category and you find some combinatorial or numerical data that reflects the phenomena occurring at the categorical level.

In this section we focus on different ways that one can decategorify the τ-tilting theory of an algebra using integer vectors.

5.1. The Grothendieck group of an algebra.

The most classical decategorification using integer vectors of the representation theory of an algebra is the *Grothendieck group* of an algebra. We start recalling their definition, which we define for arbitrary abelian categories.

Definition 5.1. Let $\mathcal{A}$ be an abelian category. The Grothendieck group $K_0(\mathcal{A})$ of $\mathcal{A}$ is the defined as the quotient of the free abelian group generated by the isomorphism classes $[M]$ of all objects $M \in \mathcal{A}$ modulo the ideal generated by the short exact sequences as follows.

$$K_0(\mathcal{A}) = \frac{\langle [M] : M \in \mathcal{A} \rangle}{\langle [M] - [L] - [N] : 0 \to L \to M \to N \to 0 \text{ is a short exact sequence in } \mathcal{A} \rangle}$$

In these notes we are interested only in the module categories mod$A$ of finite dimensional algebras A over an algebraically closed field. By abuse of notation, the Grothendieck group $K_0(\text{mod}A)$ of mod$A$ will be denoted by $K_0(A)$ and we refer to it as the Grothendieck group of

A. An immediate consequence of the Jordan-Hölder theorem for module categories we have the following result.

Theorem 5.2. *Let A be an algebra. Then $K_0(A)$ is isomorphic to $\mathbb{Z}^n$, where n is the number of isomorphism classes of simple modules in* mod$A$.

From now on, we fix a complete set $\{[S(1)], \dots, [S(n)]\}$ of isomorphism classes of simple A-modules. Clearly, $\{[S(1)], \dots, [S(n)]\}$ forms a basis of $K_0(A)$. However, this is not the only basis of $K_0(A)$. For instance is well-known that the set

$$\{[P(1)], \dots, [P(n)] : P(i) \text{ is the projective cover of } [S(i)]\}$$

forms another basis of $K_0(A)$.

In these notes, when we speak about the Grothendieck group of A we always assume that the basis chosen to represent our vectors is the basis given by the simple modules with a fixed order.

Theorem 5.3. *Let A be an algebra and $K_0(A)$ be its Grothendieck group having as canonical basis the set $\{[S(1)], \dots, [S(n)]\}$ of isomorphism classes of simple A-modules. Then for every object $M \in$ mod$A$ we have that*

$$[M] = [\dim_{\mathbb{K}}(\mathrm{Hom}_A(P(1), M)), \dots, \dim_{\mathbb{K}}(\mathrm{Hom}_A(P(n), M))]$$
$$[M] = [\dim_{\mathbb{K}}(\mathrm{Hom}_A(M, I(1)), \dots, \dim_{\mathbb{K}}(\mathrm{Hom}_A(M, I(n)))]$$

where $P(i)$ and $I(i)$ are the projective cover and the injective envelope of the simple $S(i)$, respectively, for all $1 \leq i \leq n$.

The previous result justifies that the element of the Grothendieck group $[M]$ associated to M is often called the *dimension vector* of M, terminology that we adopt in these notes as well.

Sometimes in the literature one finds the notation $\underline{\dim} M$ for the dimension vector, reserving $[M]$ for the abstract class of M in the Grothendieck group with no prefered basis of $K_0(A)$.

Remark 5.4. In the previous result we are actually using the hypothesis that A is an algebra over an algebraically closed field. Otherwise, the result is not true in general. We warn the reader that this remark is also valid for several other results in this section.

5.2. **g-vectors.** Another set of integer vectors that can be associated to the category of finitely presented A-modules are the g-vectors.

Although the idea of g-vectors has been around for several decades, their systematic study is rather recent since the main motivation behind its study, as the might be guessing already, lyes on the categorification of cluster algebras.

In fact, the name g-vector itself comes from cluster theory. The g-vectors were introduced by Fomin and Zelevinsky in [38], where they conjectured that cluster variables could be parametrised using g-vectors. Later on, it was shown that g-vectors encoded the projective presentation of τ-rigid A-modules. Their definition is the following.

Definition 5.5. Let M be an A-module. Choose the minimal projective presentation

$$P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

of M, where $P_0 = \bigoplus_{i=1}^{n} P(i)^{a_i}$ and $P_1 = \bigoplus_{i=1}^{n} P(i)^{b_i}$. Then the g-vector of M is defined as

$$g^M = (a_1 - b_1, a_2 - b_2, \dots, a_n - b_n).$$

The g-vector of a τ-rigid pair (M, P) is defined as $g^M - g^P$.

In general there are many A-modules having the same projective presentation, which implies that g-vectors are in some sense ambiguous. However this ambiguity disappears when we restrict ourselves to τ-tilting theory.

Theorem 5.6. *Let A be an algebra and let M and M' be two τ-rigid A-modules. Then $g^M = g^{M'}$ if and only if M is isomorphic to M'.*

Although the spirit of the previous result can be found already in the work of Auslander and Reiten [18], the first appearance of this result in this form was in the work on 2-Calabi-Yau categories of Dehy and Keller [28]. Later, this result was adapted to the context of τ-tilting theory in the works of Adachi, Iyama and Reiten [3] and later extended by Demonet, Iyama and Jasso in [29] as follows.

Theorem 5.7. *Let A be an algebra and let M and M' be two τ-rigid A-modules. Suppose that $(g^M)_i \leq (g^{M'})_i$ for every $1 \leq i \leq n$, then M is a quotient of M'. In particular $g^M = g^{M'}$ if and only if M is isomorphic to M'.*

In order to state the next result we need to fix some notation. Given a τ-tilting pair (M, P) we fix a decomposition $M = \bigoplus_{i=1}^k M_i$ and $P = \bigoplus_{j=k+1}^n P_j$ of M and P, respectively.

Theorem 5.8. *Let (M, P) be a τ-tilting pair. Then the set of g-vectors*

$$\{g^{M_1}, \dots, g^{M_k}, -g^{P_{k+1}}, \dots, -g^{P_n}\}$$

of the indecomposable direct summands of M and P form a basis of $\mathbb{Z}^n$.

5.3. **g-vectors, dimension vectors end the Euler form.** Given a finite dimensional algebra A, one can always associate to it a square matrix known as the Cartan matrix of the algebra as follows.

Definition 5.9. Let A be an algebra and $\{P(1), \dots, P(n)\}$ be a complete set of non-isomorphic indecomposable projective A-modules. The Cartan matrix $\mathbf{C}_A$ of A is the $n \times n$ matrix

$$\mathbf{C}_A := ([P(1)]|[P(2)]| \dots |[P(n)])$$

where the i-th column corresponds to the dimension vector $[P(i)]$ of $P(i)$ for all $1 \leq i \leq n$.

The Euler characteristic of A is a $\mathbb{Z}$-bilinear form

$$\langle -, - \rangle_A : K_0(A) \times K_0(A) \to \mathbb{Z}$$

defined as $\langle [M], [N] \rangle_A = [M]^T \mathbf{C}_A^{-1} [N]$, where $[M]$ and $[N]$ are though as column vectors.

Remark 5.10. The fact that the Euler form is well-defined is due to the fact that $\{[P(1)], \dots, [P(n)]\}$ forms a basis of $\mathbb{Z}^n$.

An important property of the Euler characteristic of an algebra is that provides important homological information, as shown in the following proposition

Proposition 5.11. *Let A be an algebra of finite global dimension s and let M, N be two A-modules. Then*

$$\langle [M], [N] \rangle_A = \sum_{i=0}^{s} (-1)^i \dim_{\mathbb{K}}(\operatorname{Ext}_A^i(M, N))$$

where $\operatorname{Ext}_A^0(M, N)$ stands for $\operatorname{Hom}_A(M, N)$. In particular, if A is a hereditary algebra we have that

$$\langle [M], [N] \rangle_A = \dim_{\mathbb{K}}(\operatorname{Hom}_A(M, N)) - \dim_{\mathbb{K}}(\operatorname{Ext}^1(M, N)).$$

The following theorem was proven at the beginning of the 1980's by Auslander and Reiten in [18] but went unnoticed for several decades. Recently, with the development of τ-tilting theory this result came to light again and it is playing a key role in some of the latest developments of this theory. To state the theorem, we denote by $\langle -, - \rangle : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ the classical dot product in $\mathbb{R}$

Theorem 5.12. [18] *Let M and N be modules over an algebra A. Then*

$$\langle g^M, [N] \rangle = \dim_{\mathbb{K}}(\mathrm{Hom}_A(M,N)) - \dim_{\mathbb{K}}(\mathrm{Hom}(N, \tau M)).$$

As a direct consequence of the previous result and the classical Auslander-Reiten formula we have the following corollary.

Corollary 5.13. *Let A be a hereditary algebra and let M and N be two A-modules. Then*

$$\langle g^M, [N] \rangle = \langle [M], [N] \rangle_A.$$

Based on this last corollary, the author is of the opinion that the pairing between g-vectors and dimension vectors of modules is a τ-tilting version of the Euler form of the algebra.

5.4. **The g-vector fan of the algebra.** We know that each g-vector is a vector with n integer coordinates. In this last subsection of the lecture notes, we explore the distribution of the g-vectors of the indecomposable τ-rigid objectes in $\mathbb{R}^n$.

In order to do that, we first need to associate a cone in $\mathbb{R}^n$ to each τ-rigid pair.

Definition 5.14. Let A be an algebra and let (M,P) be a τ-rigid pair whose set of g-vectors is

$$\{g^{M_1}, \dots, g^{M_k}, -g^{P_{k+1}}, \dots, -g^{P_n}\}.$$

Then we define the cone $\mathcal{C}_{(M,P)}$ as

$$\mathcal{C}_{(M,P)} = \left\{ \sum_{i=1}^{k} \alpha_i g^{M_i} - \sum_{j=k+1}^{t} \alpha_j g^{P_j} : \alpha_i \geq 0 \text{ for every } 1 \leq i \leq t \right\}.$$

The following result was shown by Demonet, Iyama and Jasso in [29].

Theorem 5.15. [29] *Let A be an algebra and let (M_1, P_1) and (M_2, P_2) be two τ-rigid pairs. Then $\mathcal{C}_{(M_1,P_1)} \cap \mathcal{C}_{(M_2,P_2)} \neq \{0\}$ if and only if there is a τ-rigid pair (M,P) which is a direct summand of both (M_1,P_1) and (M_2,P_2). Moreover, if (M,P) is the maximal common direct summand of (M_1,P_1) and (M_2,P_2) then $\mathcal{C}_{(M_1,P_1)} \cap \mathcal{C}_{(M_2,P_2)} = \mathcal{C}_{(M,P)}$.*

An important consequence of the previous result is that the g-vectors of indecomposable τ-rigid pairs have a geometrical structure known as *polyhedral fan*. Hence, it is common to refer the set of all g-vectors of indecomposable τ-rigid pairs as the g-vector fan of the algebra.

In particular, if our algebra is τ-tilting algebras we have that the g-vector fan of the algebra has finitely many cones spanned by its g-vectors. Moreover, these cones of g-vectors fit together very well, as was shown by Demonet, Iyama and Jasso in [29].

Theorem 5.16. [29] *Let A be an algebra. Then A is τ-tilting finite if and only if*

$$\mathbb{R}^n = \bigcup_{(M,P)\ \tau\text{-rigid}} \mathcal{C}_{(M,P)}.$$

Note that the previous result characterises τ-tilting finite algebras, so we can complete Theorem 4.24 as follows.

Theorem 5.17. *Let A be an algebra. Then the following are equivalent.*

(1) A is τ-tilting finite.
(2) There are finitely many indecomposable τ-rigid objects in mod$A$.
(3) [29] *There are finitely many torsion classes in* mod$A$.
(4) [30] *There are finitely many bricks in* mod$A$.
(5) [55] *The length of all bricks in* mod$A$ *is bounded.*
(6) [29] *The g-vector fan of A spans the whole $\mathbb{R}^n$.*

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Hipolito Treffinger
hipolito1988@gmail.com