ON A FAMILY OF JACOBI TYPE POLYNOMIALS AS EIGENFUNCTIONS OF 2X2 HYPERGEOMETRIC OPERATORS. STRUCTURAL FORMULAS

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OUTLINE

- A brief Introduction about Matrix Orthogonal Polynomials

- A family of Jacobi type Polynomials associated to a differential opertor of hypergeometric type

- Structural formulas:
  - Rodrigues formula
  - Pearson equations

- The sequence of derivatives of our family of MOP:
  - Shift operators
  - Rodrigues formula

- Some comments about the algebra D(W) of matrix-valued differential operators associated to our matrix weight W

INTRODUCTION

Given a self-adjoint positive definite matrix valued weight function $W(t)$ (of **dimension** $N \times N$) consider the skew symmetric bilinear form defined for any pair of matrix valued functions $P(t)$ and $Q(t)$ by the numerical matrix

$$\langle P, Q\rangle = \langle P, Q\rangle_W = \int_{\mathbb{R}} P(t)W(t)Q^*(t)dt,$$

where $Q^*(t)$ denotes the conjugate transpose of $Q(t)$.

There exists a sequence $(P_n)_n$ of matrix polynomials, orthonormal with respect to W and with P_n of degree n.

The sequence $(P_n)_n$ is unique up to a product with a unitary matrix.

INTRODUCTION

Any sequence of orthonormal matrix valued polynomials $(P_n)_n$ satisfies a three term recurrence relation

Hermitian Nonsingular

$$A_n^* P_{n-1}(t) + B_n P_n(t) + A_{n+1} P_{n+1}(t) = t P_n(t),$$

where P_{-1} is the zero matrix and P_0 is non singular.

Considering **possible applications** of **MOP** it is natural to concentrate on those cases where some **extra property** holds.

MATRIX ORTHOGONAL POLYNOMIALS AND DIFFERENTIAL EQUATIONS

In the nineties, A. Duran, *Rocky Mountain J. Math* (1997) raises the problem of characterizing **MOP** which satisfy *second order differential equations.*

The matrix Bochner Problem

Characterize all families of **MOP** satisfying

$$P_n \ell_{2,R} = P_n^{''} F_2(t) + P_n^{'} F_1(t) + P_n F_0(t) = \Lambda_n P_n(t), \;\; n \geq 0$$

Right hand side differential operator

$$\ell_{2,R} = D^2 F_2(t) + D^1 F_1(t) + D^0 F_0(t).$$

P_n eigenfunctions, Λ_n eigenvalues:

$$P_n \ell_{2,R} = \Lambda_n P_n$$

MATRIX ORTHOGONAL POLYNOMIALS AND DIFFERENTIAL EQUATIONS

What does this mean?

The **first** examples of **MOP** non reducible to scalar satisfying $2nd$ order differential equations appeared using representations of matrix valued spherical functions associated to symmetric spaces F.A. Grünbaum, I. Pacharoni, J. Tirao (2003).

In the framework of the general theory of orthogonal polynomials appeared **first** in

- Durán-Grünbaum , *Orthogonal Matrix Polynomials satisfying differential equations* Int. Math Res. Not. 2004.

Search for an orthogonality weight W and a differential operator D such that the pair (W, D) *"does not reduce to scalar"*.

The pair (W, D) *"reduces to scalar"* if there exists a nonsingular matrix S (independent of t) for which:

$$W(t) = S\widetilde{W}(t)S^*, \quad (S^*)^{-1}\widetilde{D}S^*$$

Diff op. with diagonal coefficients

Diagonal Matrix Weight

MATRIX ORTHOGONAL POLYNOMIALS AND DIFFERENTIAL EQUATIONS

The collection of examples of MOP in connection with differential equations has been growing in the last 20 years (see for instance a series of papers by several different authors: A. Durán, A. Grünbaum, A. Tirao, I. Pacharoni, M.C., M.D. de la Iglesia, P.Román, I. Zurrián, E. Koelink, M. van Pruijssen, A.M. de los Ríos…)

The problem of giving a general classification of these families of matrix-valued orthogonal plynomials as solutions of the so called *Matrix Bochner Problem* has been also recently addressed in R. Casper and M. Yakimov, *The matrix Bochner problem*, Amer. J. Math. (2021), to appear, arXiv:1803.04405.

A new family of matrix-valued orthogonal polynomials of size 2x2 was introduced in:

C. Calderón, Y. González, I. Pacharoni, S. Simondi, and I. Zurrián, *2x2 hypergeometric operators with diagonal eigenvalues*, J. Approx. Theory, 248:105299, 17 pp (2019).

which are common eigenfunctions of a differential operator of hypergeometric type, in the sense defined by A. Tirao in *The matrix-valued hypergeometric equation,* Proc. Natl. Acad. Sci. U.S.A., 100(14), (2003).

THE FAMILY OF MATRIX ORTHOGONAL POLYNOMIALS ASSOCIATED TO 2X2 HYPERGEOMETRIC OPERATORS WIT DIAGONAL EIGENVALUES

A new family of matrix-valued orthogonal polynomials of size 2x2 introduced in:

C. Calderónn, Y. González, I. Pacharoni, S. Simondi, and I. Zurrián, *2x2 hypergeometric operators with diagonal eigenvalues*, J. Approx. Theory, 248:105299, 17 pp (2019).

which are common eigenfunctions of a differential operator of hypergeometric type, in the sense defined by A. Tirao (2003):

$$D = \frac{d^2}{dt^2} t(1-t) + \frac{d}{dt}(C - tU) - V, \qquad \text{with } U, V, C \in \mathbb{C}^{2\times 2}.$$

Jacobi parameters

In particular, the polynomials $(P_n^{(\alpha,\beta,v)})_{n\geq 0}$ introduced by C. Calderón et al. are orthogonal with respect to a weight matrix $W^{(\alpha,\beta,v)}$ are common eigenfunctions of an hypergeometric operator with matrix eigenvalues Λ_n, which are diagonal matrices with no repetition in their entries:

$$\Lambda_n = \begin{pmatrix} \lambda_n & 0 \\ 0 & \mu_n \end{pmatrix}, \qquad \begin{aligned} \lambda_n &= -n(n-1) - n(\alpha+\beta+4) - v, \\ \mu_n &= -n(n-1) - n(\alpha+\beta+4). \end{aligned}$$

The commutativity of the matrix-valued eigenvalues could play an important role in the context of *time-and-band limiting*

The weight matrix

Consider: $$W^{(\alpha,\beta,v)}(t) = t^{\alpha}(1-t)^{\beta}\,\widetilde{W}^{(\alpha,\beta,v)}(t)\,, \quad \text{for } t \in (0,1),$$

where: $$\alpha,\ \beta,\ v \in \mathbb{R},\ \alpha, \beta > -1 \text{ and } |\alpha-\beta| < |v| < \alpha+\beta+2.$$

$$\widetilde{W}^{(\alpha,\beta,v)}(t) = \begin{pmatrix} \frac{v(\kappa_{v,\beta}+2)}{\kappa_{v,-\beta}}t^2 - (\kappa_{v,\beta}+2)\,t + (\alpha+1) & (\alpha+\beta+2)t-(\alpha+1) \\ (\alpha+\beta+2)t-(\alpha+1) & -\frac{v(\kappa_{-v,\beta}+2)}{\kappa_{-v,-\beta}}t^2 - (\kappa_{-v,\beta}+2)\,t + (\alpha+1) \end{pmatrix},$$

for the sake of clearness we will use the notation: $$\kappa_{\pm v,\pm\beta} = \alpha \pm v \pm \beta\ .$$

$W^{(\alpha,\beta,v)}$ is an irreducible matrix-weight and the hypergeometric type differential operator given by

$$D = \frac{d^2}{dt^2}t(1-t) + \frac{d}{dt}(C-tU) - V, \qquad \text{with } U, V, C \in \mathbb{C}^{2\times 2}.$$

where:

$$C = \begin{pmatrix} \alpha+1-\frac{\kappa_{-v,-\beta}}{v} & \frac{\kappa_{v,-\beta}}{v} \\ -\frac{\kappa_{-v,-\beta}}{v} & \alpha+1+\frac{\kappa_{v,-\beta}}{v} \end{pmatrix},\ U = (\alpha+\beta+4)\ \text{I and } V = \begin{pmatrix} v & 0 \\ 0 & 0 \end{pmatrix},$$

is symmetric with respect to the matrix-weight $W^{(\alpha,\beta,v)}$.

SYMMETRIC OPERATORS

$$D = \frac{d^2}{dt^2} F_2(t) + \frac{d}{dt} F_1(t) + F_0$$

The differential operator D is symmetric with respect to W if

$$\langle PD, Q \rangle_W = \langle P, QD \rangle_W \,, \text{ for all } P, Q \in \mathbb{C}^{N \times N}[t].$$

The differential operator D is symmetric with respect to W if and only if (Durán-Grunbaum, 2004)

Symmetry Equations

$$\begin{aligned} F_2 W &= W F_2^* \\ 2(F_2 W)' &= W F_1^* + F_1 W \\ (F_2 W)'' - (F_1 W)' + F_0 W &= W F_0^* \end{aligned}$$

with the boundary conditions

$$\lim_{t \to \pm\infty} t^n F_2(t) W(t) = 0, \quad \lim_{t \to \pm\infty} t^n \left((F_2(t) W(t))' - F_1(t) W(t) \right) = 0$$

THE RODRIGUES FORMULA

Useful tool

$$P_n = R_n^{(n)} W^{-1}$$

Theorem, A. Durán, Int. Math. Research Notices (2009) Let F_2, F_1 and F_0 be matrix polynomials of degrees not larger than 2, 1, and 0, respectively. Let W, R_n be $N \times N$ matrix functions twice and n times differentiable, respectively, in an open set Ω of the real line. Assume that $W(t)$ is nonsingular for $t \in \Omega$ and that satisfies the *symmetry equations*.

If for a matrix Λ_n, the function R_n satisfies

$$(R_n F_2^*)'' - \left(R_n \left[F_1^* + n(F_2^*)'\right]\right)' + R_n \left[F_0^* + n(F_1^*)' + \binom{n}{2}(F_2^*)''\right] = \Lambda_n R_n.$$

then P_n satisfies

$$P_n''(t)F_2(t) + P_n'(t)F_1(t) + P_n(t)F_0 = \Lambda_n P_n(t).$$

THE RODRIGUES FORMULA

$\kappa_{\pm v,\pm\beta} = \alpha \pm v \pm \beta$.

Theorem (C. Calderón, M.C)

Consider the matrix-weight $W(t) = W^{(\alpha,\beta,v)}(t)$ given by the expression above. Consider the matrix-valued functions $(P_n)_{n\geq 0}$ and $(R_n)_{n\geq 0}$ defined by

$$P_n(t) = (R_n(t))^{(n)}(W(t))^{-1},$$

$$R_n(t) = R_n^{(\alpha,\beta,v)}(t) = t^{n+\alpha}(1-t)^{n+\beta}\left(R_{n,2}^{(\alpha,\beta,v)}t^2 + R_{n,1}^{(\alpha,\beta,v)}t + R_{n,0}^{(\alpha,\beta,v)}\right),$$ where

$$R_{n,2}^{(\alpha,\beta,v)} = R_{n,2} = \begin{pmatrix} c_n & 0 \\ 0 & d_n \end{pmatrix},$$

$$R_{n,1}^{(\alpha,\beta,v)} = R_{n,1} = \frac{1}{v}\begin{pmatrix} -c_n\kappa_{v,-\beta} & \dfrac{c_n(\alpha+2n+2+\beta)\kappa_{v,-\beta}}{(\kappa_{v,\beta}+2n+2)} \\ -\dfrac{d_n(\alpha+2n+2+\beta)\kappa_{-v,-\beta}}{(\kappa_{-v,\beta}+2n+2)} & d_n\kappa_{-v,-\beta} \end{pmatrix},$$

$$R_{n,0}^{(\alpha,\beta,v)} = R_{n,0} = \frac{1+n+\alpha}{v}\begin{pmatrix} c_n\dfrac{\kappa_{v,-\beta}}{(\kappa_{v,\beta}+2n+2)} & -c_n\dfrac{\kappa_{v,-\beta}}{(\kappa_{v,\beta}+2n+2)} \\ d_n\dfrac{\kappa_{-v,-\beta}}{(\kappa_{-v,\beta}+2n+2)} & -d_n\dfrac{\kappa_{-v,-\beta}}{(\kappa_{-v,\beta}+2n+2)} \end{pmatrix},$$

where $(c_n)_n$ and $(d_n)_n$ are arbitrary sequences of complex numbers. Then $P_n(t)$ is a polynomial of degree n with nonsingular leading coefficient equal to

$|\alpha-\beta| < |v| < \alpha+\beta+2.$

$$\begin{pmatrix} \dfrac{\kappa_{v,-\beta}(\alpha+\beta+n+3)_n}{(-1)^n v(\kappa_{v,\beta}+2)}c_n & 0 \\ 0 & \dfrac{\kappa_{-v,-\beta}(\alpha+\beta+n+3)_n}{(-1)^{n+1} v(\kappa_{-v,\beta}+2)}d_n \end{pmatrix},$$

$(P_n(t))_n$ is a sequence of MOP with respect to W

THE RODRIGUES FORMULA

Theorem (C. Calderón, M.C)

Consider the matrix-weight $W(t) = W^{(\alpha,\beta,v)}(t)$ given by the expression above. Consider the matrix-valued functions $(P_n)_{n\geq 0}$ and $(R_n)_{n\geq 0}$ defined by

$$\begin{aligned} P_n(t) &= (R_n(t))^{(n)} (W(t))^{-1}, \\ R_n(t) &= R_n^{(\alpha,\beta,v)}(t) = t^{n+\alpha}(1-t)^{n+\beta}\left(R_{n,2}^{(\alpha,\beta,v)} t^2 + R_{n,1}^{(\alpha,\beta,v)} t + R_{n,0}^{(\alpha,\beta,v)}\right), \end{aligned}$$

Main Tools in the Proof:

We use the following Rodrigues formula for the classical Jacobi polynomial $p_n^{(\alpha,\beta)}(t)$:

$$\frac{d^n}{dt^n}\left[t^{n+\alpha}(1-t)^{n+\beta}\right] = n! t^{\alpha}(1-t)^{\beta} p_n^{(\alpha,\beta)}(1-2t),$$

where

$$p_n^{(\alpha,\beta)}(1-2t) = \frac{\Gamma(n+\alpha+1)}{n!\Gamma(n+\alpha+\beta+1)} \sum_{j=0}^{n} \binom{n}{j} \frac{\Gamma(n+\alpha+\beta+1+j)}{\Gamma(j+\alpha+1)} (-1)^j t^j.$$

Thus, we obtain

$$R_n^{(n)}(t) = n! t^{\alpha}(1-t)^{\beta}\left(p_n^{(\alpha+2,\beta)}(1-2t) R_{n,2} t^2 + p_n^{(\alpha+1,\beta)}(1-2t) R_{n,1} t + p_n^{(\alpha,\beta)}(1-2t) R_{n,0}\right).$$

The ortogonality of P_n follows from this expression

We write $(W(t))^{-1} = t^{-\alpha-2}(1-t)^{-\beta-2}\left(J_2 t^2 + J_1 t + J_0\right)$,

one verifies that $P_n(t) = (R_n(t))^{(n)} (W(t))^{-1}$ is a polynomial of degree n

THE RODRIGUES FORMULA

$$\kappa_{\pm v,\pm\beta} = \alpha \pm v \pm \beta\ .$$

We are considering the Jacobi type weight -matrix:

$$W^{(\alpha,\beta,v)}(t) = t^{\alpha}\,(1-t)^{\beta}\,\widetilde{W}^{(\alpha,\beta,v)}(t)\,, \quad \text{for } t\in(0,1),$$

where:

$$\widetilde{W}^{(\alpha,\beta,v)}(t) = \begin{pmatrix} \frac{v(\kappa_{v,\beta}+2)}{\kappa_{v,-\beta}}t^2 - (\kappa_{v,\beta}+2)\,t + (\alpha+1) & (\alpha+\beta+2)t-(\alpha+1) \\ (\alpha+\beta+2)t-(\alpha+1) & -\frac{v(\kappa_{-v,\beta}+2)}{\kappa_{-v,-\beta}}t^2 - (\kappa_{-v,\beta}+2)\,t + (\alpha+1) \end{pmatrix},$$

Collorary: The sequence of monic polynomials, orthogonal w.r.t $W^{(\alpha,\beta,v)}(t)$ defined by the Rodrigues formula: $P_n^{(\alpha,\beta,v)}(t) = (R_n^{(\alpha,\beta,v)}(t))^{(n)}\left(W^{(\alpha,\beta,v)}(t)\right)^{-1}$, can be written as:

$$P_n\,(t) = n!\left(p_n^{(\alpha,\beta)}(1-2t)\mathcal{C}_{n,2}^{(\alpha,\beta,v)} + p_{n+1}^{(\alpha,\beta)}(1-2t)\mathcal{C}_{n,1}^{(\alpha,\beta,v)} + p_{n+2}^{(\alpha,\beta)}(1-2t)\mathcal{C}_{n,0}^{(\alpha,\beta,v)}\right)(\widetilde{W}^{(\alpha,\beta,v)}\,(t))^{-1}$$

for certain matrix valued entries $\mathcal{C}_{n,i}^{(\alpha,\beta,v)}$, $i=0,1,2$.

ORTHONORMAL POLYNOMIALS

$\kappa_{\pm v,\pm\beta} = \alpha \pm v \pm \beta\ .$

Rodrigues formula allows us to compute the norm of the sequence of monic matrix- valued OP:

$$\left\| P_n^{(\alpha,\beta,v)} \right\|^2 = \frac{n! v B(\alpha+n+2, \beta+n+2)}{(\alpha+n+3+\beta)_n} \begin{pmatrix} \frac{(\kappa_{v,\beta}+2)(\kappa_{-v,\beta}+2n+4)}{\kappa_{v,-\beta}(\kappa_{v,\beta}+2n+2)} & 0 \\ 0 & -\frac{(\kappa_{-v,\beta}+2)(\kappa_{v,\beta}+2n+4)}{\kappa_{-v,-\beta}(\kappa_{-v,\beta}+2n+2)} \end{pmatrix}.$$

Having the norm of monic OP one can write the recurrence relation for the sequence of orthonormal polynomials:

$$t\widetilde{P}_n^{(\alpha,\beta,v)}(t) = \widetilde{A}_{n+1}^{(\alpha,\beta,v)} \widetilde{P}_{n+1}^{(\alpha,\beta,v)}(t) + \widetilde{B}_n^{(\alpha,\beta,v)} \widetilde{P}_n^{(\alpha,\beta,v)}(t) + \left(\widetilde{A}_n^{(\alpha,\beta,v)}\right)^* \widetilde{P}_{n-1}^{(\alpha,\beta,v)}(t),$$

with:

$$\begin{aligned} \widetilde{A}_{n+1}^{(\alpha,\beta,v)} &= \left\| P_n^{(\alpha,\beta,v)} \right\|^{-1} \left\| P_{n+1}^{(\alpha,\beta,v)} \right\|, \\ \widetilde{B}_n^{(\alpha,\beta,v)} &= \left\| P_n^{(\alpha,\beta,v)} \right\|^{-1} B_n^{(\alpha,\beta,v)} \left\| P_n^{(\alpha,\beta,v)} \right\|, \end{aligned}$$

$B_n^{(\alpha,\beta,v)}$ are the entries of the recurrence for the monic OP already given in C. Calderón et al., JAT, (2019):

$$tP_n^{(\alpha,\beta,v)} = P_{n+1}^{(\alpha,\beta,v)} + B_n^{(\alpha,\beta,v)} P_n^{(\alpha,\beta,v)} + A_n^{(\alpha,\beta,v)} P_{n-1}^{(\alpha,\beta,v)}$$

Having the recurrence relation for the orthonormal OP one can write the **C-D** formula:

$$(x-y)\sum_{k=0}^{n} \left(\widetilde{P}_k^{(\alpha,\beta,v)}\right)^*(y)\, \widetilde{P}_k^{(\alpha,\beta,v)}(x) = \\ \left(\widetilde{P}_n^{(\alpha,\beta,v)}\right)^*(y) \left(\widetilde{A}_{n+1}^{(\alpha,\beta,v)}\right)^* \widetilde{P}_{n+1}^{(\alpha,\beta,v)}(x) - \left(\widetilde{P}_{n+1}^{(\alpha,\beta,v)}\right)^*(y)\, \widetilde{A}_{n+1}^{(\alpha,\beta,v)} \widetilde{P}_n^{(\alpha,\beta,v)}(x)$$

THE SEQUENCE OF DERIVATIVES OF THE MOP

It is very well known that in the case of classical orthogonal polynomials (P_n), can be characterized by the orthogonality of their derivatives (P'_{n+1}):

Classical orthogonal polynomials (P_n), can be characterized equivalently by a linear relation between P_n and P'_{n+1}, P'_n, P'_{n-1}:

These properties are also equivalent to a **Pearson type equation** for the orthogonality functional:

$$D(u\Phi) = u\Psi, \quad deg(\Phi) \leq 2, \; deg(\Psi) = 1$$

- T. S. Chihara, a*n introduction to Orthogonal Polynomials*, Gordon and Breach, NY, 1978
- S. Bonan, D. S. Lubinsky, P. Nevai, T. S. Chihara, *orthogonal polynomials and their derivatives*, SIAM J. Math. Anal. 18 (1987)
- P. Maroni, *Connected problems, Variations arround classical orthogonal polynomials, J. Comput.* Appl. Math. 48 (1-2) (1993) 133-155.
- F. Marcellán, A. Branquinho, J. Petronilho, *Classical orthogonal polynomials: a functional approach*, Acta Appl. Math. 34 (3) (1994) 283-303.

THE SEQUENCE OF DERIVATIVES OF THE MOP

A first step to determine whether or not this characterizations hold in the matrix setting was given in:

- A. Durán, F.A. Grünbaum, *Orthogonal Polynomials*, *scalar type Rodriges formulas and Pearson equations,* J. Approx. Theory 134 (2005).

Here one may see an example of MOP satisfying a second order differential equation but not the required Pearson-type equation in order for the sequence of derivatives to be orthogonal thus not all families of MOP have orthogonal derivatives.

THE SEQUENCE OF DERIVATIVES OF THE MOP

A nice characterization of these properties for the matrix setting was given in:

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Matrix orthogonal polynomials whose derivatives are also orthogonal☆

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Abstract

In this paper we prove some characterizations of the matrix orthogonal polynomials whose derivatives are also orthogonal, which generalize other known ones in the scalar case. In particular, we prove that the corresponding orthogonality matrix functional is characterized by a Pearson-type equation with two matrix polynomials of degree not greater than 2 and 1. The proofs are given for a general sequence of matrix orthogonal polynomials, not necessarily associated with a hermitian functional. We give several examples of non-diagonalizable positive definite weight matrices satisfying a Pearson-type equation, which show that the previous results are non-trivial even in the positive definite case.

A detailed analysis is made for the class of matrix functionals which satisfy a Pearson-type equation whose polynomial of degree not greater than 2 is scalar. We characterize the Pearson-type equations of this kind that yield a sequence of matrix orthogonal polynomials, and we prove that these matrix orthogonal polynomials satisfy a second order differential equation even in the non-hermitian case. Finally, we prove and improve a conjecture of Durán and Grünbaum concerning the triviality of this class in the positive definite case, while some examples show the non-triviality for hermitian functionals which are not positive definite.

In particular, the authors show that if a matrix-valued functional satisfies a Pearson type equation then the sequence of derivatives of the corresponding MOP is also orthogonal.

u is a $\mathcal{P}_{2,1}$ functional if there exist matrix-valued poynomials Φ, $deg(\Phi) \leq 2$, Ψ, $deg(\Psi) \leq 1$, with $det(\Phi) \neq 0$ such that $D(u\Phi) = u\Psi$

"Matrix-valued weights belonging to $\mathcal{P}_{2,1}$ class can be considered as matrix generalizations of the classical scalar orthogonal polynomials"

THE SEQUENCE OF DERIVATIVES OF THE MOP

We prove that polynomials in the sequence of derivatives of the orthogonal matrix polynomials $\left(P_n^{(\alpha,\beta,v)}\right)_{n\geq 0}$ are also orthogonal by obtaining a Pearson equation for the weight matrix $W^{(\alpha,\beta,v)}(t)$.

Consider the sequence of monic polynomials corresponding to the derivative of order k of the monic polynomial $P_n^{(\alpha,\beta,v)}(t)$, for $n \geq k$:

$$P_n^{(\alpha,\beta,v,k)}(t) = \frac{(n-k)!}{n!}\frac{d^k}{dt^k}P_n^{(\alpha,\beta,v)}(t)$$

THE PEARSON EQUATION

$\kappa_{\pm v,\pm\beta} = \alpha \pm v \pm \beta$.

Let $\alpha, \beta > -(k+1)$ and $|\alpha - \beta| < |v| < \alpha + \beta + 2(k+1)$. We consider the weight matrix

$$W^{(k)}(t) = W^{(\alpha,\beta,v,k)}(t) = t^{\alpha+k}(1-t)^{\beta+k}\widetilde{W}^{(\alpha,\beta,v,k)}(t)\,, \text{ where } \widetilde{W}^{(\alpha,\beta,v,k)}(t) = W_2^{(k)}t^2 + W_1^{(k)}t + W_0^{(k)}$$

with

$$W_2^{(k)} = v\begin{pmatrix} \dfrac{\kappa_{v,\beta}+2(k+1)}{\kappa_{v,-\beta}} & 0 \\ 0 & -\dfrac{\kappa_{-v,\beta}+2(k+1)}{\kappa_{-v,-\beta}} \end{pmatrix}, \quad W_0^{(k)} = (\alpha+k+1)\begin{pmatrix} 1 & -1 \\ -1 & 1. \end{pmatrix},$$

$$W_1^{(k)} = \begin{pmatrix} -\kappa_{v,\beta} & \alpha+\beta \\ \alpha+\beta & -\kappa_{-v,\beta} \end{pmatrix} + 2(k+1)\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Theorem (M.C, C. Calderón) The matrix-weight $W^{(k)}$ satisfies the following Pearson equation,

$$\left(W^{(k)}(t)\,\Phi^{(k)}(t)\right)' = W^{(k)}(t)\,\Psi^{(k)}(t)\,,\;\; k \geq 0 \quad \text{with}$$

$$\Phi^{(k)}(t) = \mathcal{A}_2^k t^2 + \mathcal{A}_1^k t + \mathcal{A}_0^k \text{ and } \Psi^{(k)}(t) = \mathcal{B}_1^k t + \mathcal{B}_0^k,$$

Taking into account that $\deg\left(\Phi^{(k)}(t)\right) = 2$ and $\deg\left(\Psi^{(k)}(t)\right) = 1$, we obtain from [CMV, corollary 3.10] the following

Corollary The sequence of polynomials $\left(P_n^{(\alpha,\beta,v,k)}\right)_{n\geq k}$ is orthogonal with respect to the matrix-valued weight $W^{(k)}$, $k \geq 1$.

THE PEARSON EQUATION

$$\kappa_{\pm v,\pm\beta} = \alpha \pm v \pm \beta\,.$$

$$W^{(k)}(t) = W^{(\alpha,\beta,v,k)}(t) = t^{\alpha+k}\,(1-t)^{\beta+k}\,\widetilde{W}^{(\alpha,\beta,v,k)}\,(t)$$

Theorem The matrix-weight $W^{(k)}$ satisfies the following Pearson equation,

$$\left(W^{(k)}\,(t)\,\Phi^{(k)}\,(t)\right)' = W^{(k)}\,(t)\,\Psi^{(k)}\,(t)\,,\ \ k\geq 0 \quad \text{with}$$

$$\Phi^{(k)}\,(t) = \mathcal{A}_2^k t^2 + \mathcal{A}_1^k t + \mathcal{A}_0^k \text{ and } \Psi^{(k)}\,(t) = \mathcal{B}_1^k t + \mathcal{B}_0^k,$$

$$\mathcal{A}_2^k = \begin{pmatrix} -\dfrac{\kappa_{v,\beta}+2(k+2)}{\kappa_{v,\beta}+2(k+1)} & 0 \\ 0 & -\dfrac{\kappa_{-v,\beta}+2(k+2)}{\kappa_{-v,\beta}+2(k+1)} \end{pmatrix},$$

$$\mathcal{A}_1^k = \frac{2}{(\kappa_{-v,\beta}+2(k+1))(\kappa_{v,\beta}+2(k+1))}\begin{pmatrix} 0 & \kappa_{v,-\beta} \\ \kappa_{-v,-\beta} & 0 \end{pmatrix} - \mathcal{A}_2^k,$$

$$\mathcal{A}_0^k = \frac{\kappa_{v,-\beta}\kappa_{-v,-\beta}}{v(\kappa_{-v,\beta}+2(k+1))(\kappa_{v,\beta}+2(k+1))}\begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix},$$

$$\mathcal{B}_1^k = (\alpha+\beta+4+2k)\mathcal{A}_2^k,$$

$$\mathcal{B}_0^k = \left(-(\alpha+k+1)I - \frac{1}{v}\begin{pmatrix} -\kappa_{-v,-\beta} & 0 \\ 0 & \kappa_{v,-\beta} \end{pmatrix}\right)\mathcal{A}_2^k$$

$$+\frac{1}{2v}\left(\frac{\alpha+\beta+2k+4}{v}\mathcal{A}_1^k + \mathcal{B}_1^k\right)\begin{pmatrix} -\kappa_{-v,\beta}-2(k+1) & 0 \\ 0 & \kappa_{v,\beta}+2(k+1) \end{pmatrix}.$$

THE SEQUENCE OF DERIVATIVES OF THE MOP

$\kappa_{\pm v,\pm\beta} = \alpha \pm v \pm \beta$.

Consider the sequence of monic polynomials corresponding to the derivative of order k of the monic polynomial $P_n^{(\alpha,\beta,v)}(t)$, for $n \geq k$:

$$P_n^{(\alpha,\beta,v,k)}(t) = \frac{(n-k)!}{n!}\frac{d^k}{dt^k}P_n^{(\alpha,\beta,v)}(t)$$

One has that

$$P_n^{(\alpha,\beta,v,k)}D^{(\alpha,\beta,v,k)} = \Lambda_n P_n^{(\alpha,\beta,v,k)}, \quad n \geq k,$$

where

$$D^{(\alpha,\beta,v,k)} = \frac{d^2}{dt^2}t(1-t) + \frac{d}{dt}((C^{(k)})^* - tU^{(k)}) - V$$

with:

$$C^{(k)} = \begin{pmatrix} \alpha + 1 + k - \dfrac{\kappa_{-v,-\beta}}{v} & \dfrac{\kappa_{v,-\beta}}{v} \\ -\dfrac{\kappa_{-v,-\beta}}{v} & \alpha + 1 + k + \dfrac{\kappa_{v,-\beta}}{v} \end{pmatrix}, \qquad U^{(k)} = (\alpha+\beta+2(2+k))\,\mathrm{I}, \quad V = \begin{pmatrix} v & 0 \\ 0 & 0 \end{pmatrix}.$$

$$\Lambda_n = \begin{pmatrix} \lambda_n & 0 \\ 0 & \mu_n \end{pmatrix}, \qquad \begin{aligned} \lambda_n &= -n(n-1) - n\,(\alpha+\beta+4) - v, \\ \mu_n &= -n(n-1) - n\,(\alpha+\beta+4)\,. \end{aligned}$$

THE SEQUENCE OF DERIVATIVES OF THE MOP

$\kappa_{\pm v,\pm\beta} = \alpha \pm v \pm \beta$.

Let $\alpha, \beta > -(k+1)$ and $|\alpha - \beta| < |v| < \alpha + \beta + 2\,(k+1)$. We consider the weight matrix

$$W^{(k)}(t) = W^{(\alpha,\beta,v,k)}(t) = t^{\alpha+k}\,(1-t)^{\beta+k}\,\widetilde{W}^{(\alpha,\beta,v,k)}\,(t)\,, \text{where } \widetilde{W}^{(\alpha,\beta,v,k)}\,(t) = W_2^{(k)}t^2 + W_1^{(k)}t + W_0^{(k)},$$

with

$$W_2^{(k)} = v\begin{pmatrix} \dfrac{\kappa_{v,\beta} + 2\,(k+1)}{\kappa_{v,-\beta}} & 0 \\ 0 & -\dfrac{\kappa_{-v,\beta} + 2(k+1)}{\kappa_{-v,-\beta}} \end{pmatrix}, \quad W_0^{(k)} = (\alpha + k + 1)\begin{pmatrix} 1 & -1 \\ -1 & 1. \end{pmatrix},$$

$$W_1^{(k)} = \begin{pmatrix} -\kappa_{v,\beta} & \alpha+\beta \\ \alpha+\beta & -\kappa_{-v,\beta} \end{pmatrix} + 2\,(k+1)\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Proposition $W^{(k)}$ is an irreducible matrix-weight and the differential hypergeometric operator $D^{(\alpha,\beta,v,k)}$ is symmetric with respect to the matrix-weight $W^{(k)}$.

THE SEQUENCE OF DERIVATIVES OF THE MOP

We have the following explicit expression for the sequence of polynomials $\left(P_n^{(\alpha,\beta,v,k)}\right)_{n\geq k}$ in terms of hypergeometric function ${}_2H_1\left(U,V,C;t\right)$ defined by J. A. Tirao in *The matrix-valued hypergeometric equation,* Proc. Natl. Acad. Sci. U.S.A., *100(14):8138–8141 (2003).*

$$\left(P_n^{(\alpha,\beta,v,k)}\left(t\right)\right)^* = {}_2H_1\left(U^{(k)},V+\lambda_n,C^{(k)};t\right)(n-k)!\left[C^{(k)},U^{(k)},V+\lambda_n\right]_{n-k}^{-1}\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}+ \tag{1}$$

$$ {}_2H_1\left(U^{(k)},V+\mu_n,C^{(k)};t\right)(n-k)!\left[C^{(k)},U^{(k)},V+\mu_n\right]_{n-k}^{-1}\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}.$$

where:

$$ {}_2H_1\left(U,V,C;t\right)=\sum_{j\geq 0}\left[C,U,V\right]_j\mathcal{F}_0\frac{t^j}{j!},\mathcal{F}_0\in\mathbb{C}^2,$$

and $\left[C,U,V\right]_j$ is defined inductively as $\left[C,U,V\right]_0=I$ and
$\left[C,U,V\right]_{j+1}=\left(C+j\right)^{-1}\left(j\left(j-1\right)I+jU+V\right)\left[C,U,V\right]_j.$

THE SEQUENCE OF DERIVATIVES OF THE MOP

$$\left(P_n^{(\alpha,\beta,v,k)}(t)\right)^* = {}_2H_1\left(U^{(k)}, V+\lambda_n, C^{(k)}; t\right)(n-k)!\left[C^{(k)}, U^{(k)}, V+\lambda_n\right]_{n-k}^{-1}\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix} + {}_2H_1\left(U^{(k)}, V+\mu_n, C^{(k)}; t\right)(n-k)!\left[C^{(k)}, U^{(k)}, V+\mu_n\right]_{n-k}^{-1}\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}.$$

Indeed, the polynomials $\left(P_n^{(\alpha,\beta,v,k)}\right)_{n\geq k}$ are common eigenfuncions of the matrix hypergeometric type operator

$$D^{(\alpha,\beta,v,k)} = \frac{d^2}{dt^2}t(1-t) + \frac{d}{dt}((C^{(k)})^* - tU^{(k)}) - V,$$

with diagonal eigenvalue Λ_n

The fact that the eigenvalue is diagonal implies that the matrix equation can be written as two vectorial hypergeometric equations as in (Theorem 5, J. Tirao, The matrix hypergeometric equation, 2003) and the solutions of these equations are the columns of $\left(P_n^{(\alpha,\beta,v,k)}\right)_{n\geq k}$.

Since the eigenvalues of the matrices $C^{(k)}$, $3+\alpha+k$ and $1+\alpha+k$, are non negative integers for all $k \geq 1$, then these solutions are hypergeometric vector functions.

SHIFT OPERATORS

Following the ideas in:

- E.Koelink, A. de los Ríos, and P.Román, *Matrix-valued Gegenbauer-type polynomials*, Constr. Approx., 46(3):459--487 (2017).

Consider the sequence of monic polynomials corresponding to the derivative of order k of the monic polynomial $P_n^{(\alpha,\beta,v)}(t)$, for $n \geq k$:

$$P_n^{(\alpha,\beta,v,k)}(t) = \frac{(n-k)!}{n!} \frac{d^k}{dt^k} P_n^{(\alpha,\beta,v)}(t)$$

Consider the monic n-degree polynomials $P_{n+k}^{(\alpha,\beta,v,k)}(t)$, $n \geq 0$, orthogonal w.r.t $W^{(k)}$.

We use Pearson equation to give explicit *lowering and rising operators* for the polynomials $\left(P_{n+k}^{(\alpha,\beta,v,k)}\right)_{n\geq 0}$

Moreover, from the existence of the shift operators we deduce a Rodrigues formula for the sequence of derivatives $\left(P_{n+k}^{(\alpha,\beta,v,k)}\right)_{n\geq 0}$, and we find a matrix-valued differential operator for which these matrix-valued polynomials are eigenfunctions in terms of the entries of Pearson equation.

SHIFT OPERATORS

$\kappa_{\pm v,\pm\beta} = \alpha \pm v \pm \beta$.

For any pair of matrix-valued functions P and Q, we denote

$$\langle P, Q\rangle_k = \int_0^1 P(t)\, W^{(k)}(t)\, Q^*(t)\, dt.$$

Proposition Let $\eta^{(k)}$ be the first order matrix-valued right differential operator

$$\eta^{(k)} = \frac{d}{dt}(\Phi^{(k)}(t))^* + (\Psi^{(k)}(t))^*.$$

Then $\frac{d}{dt} : L^2\left(W^{(k)}\right) \to L^2\left(W^{(k+1)}\right)$ and $\eta^k : L^2\left(W^{(k+1)}\right) \to L^2\left(W^{(k)}\right)$ satisfy

$$\left\langle \frac{dP}{dt}, Q \right\rangle_{k+1} = -\left\langle P, Q\eta^{(k)} \right\rangle_k .$$

lemma The following identity holds true

$$I\eta^{(k+n-1)} \cdots \eta^{(k+1)}\eta^{(k)} = \mathcal{C}_n^k P_{n+k}^{(\alpha,\beta,v,k)}, \quad n \geq 1, \text{ for a given } k \geq 0$$

$$\mathcal{C}_n^k = (-1)^n\,(\alpha+\beta+3+2k+n)_n \begin{pmatrix} \frac{(\kappa_{v,\beta}+2(k+1+n))}{(\kappa_{v,\beta}+2(k+1))} & 0 \\ 0 & \frac{(\kappa_{-v,\beta}+2(k+1+n))}{(\kappa_{-v,\beta}+2(k+1))} \end{pmatrix}, \quad n \geq 1.$$

Some basic facts:

$$\frac{d}{dt}P_{n+k}^{(\alpha,\beta,v,k)}(t) = nP_{n+k}^{(\alpha,\beta,v,k+1)}(t) .$$

$P_{n+k}^{(\alpha,\beta,v,k+1)}\eta^{(k)}$ is a multiple of $P_{n+k}^{(\alpha,\beta,v,k)}$.

RODRIGUES FORMULA

Theorem

The polynomials $\left(P_{n+k}^{(\alpha,\beta,v,k)}\right)_{n\geq 0}$, $n\geq 1$, satisfy the following Rodrigues formula

$$P_{n+k}^{(\alpha,\beta,v,k)}(t) = \left(\mathcal{C}_n^k\right)^{-1}\left(\frac{d^n}{dt^n}W^{(k+n)}(t)\right)\left(W^{(k)}(t)\right)^{-1},$$

Proof: For any matrix-valued function Q we write

$$Q\eta^{(k)} = \frac{dQ}{dt}(\Phi^{(k)})^* + Q(\Psi^{(k)})^*.$$

On has the Pearson equation:

$$\left(W^{(k)}(t)\,\Phi^{(k)}(t)\right)' = W^{(k)}(t)\,\Psi^{(k)}(t)\,,\;\; k\geq 0$$

and the identities:

$$W^{(\alpha,\beta,v,k+1)}(t) = W^{(\alpha,\beta,v,k)}(t)\,\Phi^{(k)}(t)\,,$$

$$\left(W^{(\alpha,\beta,v,k+1)}(t)\right)' = W^{(\alpha,\beta,v,k)}(t)\,\Psi^{(k)}(t)$$

we obtain

$$Q\eta^k = \frac{d}{dt}\left(QW^{(k+1)}\right)\left(W^{(k)}\right)^{-1}.$$

Iterating, it gives

$$Q\eta^{(k+n-1)}\cdots\eta^{(k+1)}\eta^{(k)} = \frac{d^n}{dt^n}\left(QW^{(k+n)}\right)\left(W^{(k)}\right)^{-1}.$$

- ◇ put $Q(t) = I$
- ◇ apply previous lemma

THE DIFFERENTIAL OPERATOR

$$W^{(k)}(t)=W^{(\alpha,\beta,v,k)}(t)=t^{\alpha+k}\,(1-t)^{\beta+k}\,\widetilde{W}^{(\alpha,\beta,v,k)}\,(t)$$

Corollary the differential operator

$$E^{(k)}=\eta^{(k)}\circ\frac{d}{dt}=\frac{d^2}{dt^2}(\Phi^{(k)}\,(t))^*+\frac{d}{dt}(\Psi^{(k)}\,(t))^*$$

is symmetric with respect to $W^{(k)}\,(t)$ for all $k\in\mathbb{N}_0$.

Moreover, the polynomials $\left(P_{n+k}^{(\alpha,\beta,v,k)}\right)_{n\geq 0}$ are eigenfunctions of the operator $E^{(k)}$ with eigenvalue

$$\Lambda_n\left(E^{(k)}\right)=n(n+\alpha+\beta+3+2k)\begin{pmatrix}-\dfrac{\kappa_{v,\beta}+2(k+2)}{\kappa_{v,\beta}+2(k+1)} & 0\\ 0 & -\dfrac{\kappa_{-v,\beta}+2(k+2)}{\kappa_{-v,\beta}+2(k+1)}\end{pmatrix}$$

One also has the associated second order differential operator of hypergeometric type

$$D^{(\alpha,\beta,v,k)}=\frac{d^2}{dt^2}t(1-t)+\frac{d}{dt}((C^{(k)})^*-tU^{(k)})-V$$

with diagonal eigenvalue $\Lambda_n(D^{(k)})$

◊ The operators $E^{(k)}$ and $D^{(k)}$ commute.

◊ The Darboux transform $\widetilde{E}^{(k)}=\dfrac{d}{dt}\circ\eta^{(k)}$ of the operator $E^{(k)}$ is not symmetric with respect to $W^{(k)}$.

THE ALGEBRA OF DIFFERENTIAL OPERATORS $D(W)$

$$\kappa_{\pm v,\pm\beta} = \alpha \pm v \pm \beta \,.$$

Coming back to the Jacobi type weight-matrix we are considering:

$$W^{(\alpha,\beta,v)}(t) = t^{\alpha}\,(1-t)^{\beta}\,\widetilde{W}^{(\alpha,\beta,v)}(t)\,, \quad \text{for } t \in (0,1),$$

where:

$$\widetilde{W}^{(\alpha,\beta,v)}(t) = \begin{pmatrix} \frac{v(\kappa_{v,\beta}+2)}{\kappa_{v,-\beta}}t^2 - (\kappa_{v,\beta}+2)\,t + (\alpha+1) & (\alpha+\beta+2)t - (\alpha+1) \\ (\alpha+\beta+2)t - (\alpha+1) & -\frac{v(\kappa_{-v,\beta}+2)}{\kappa_{-v,-\beta}}t^2 - (\kappa_{-v,\beta}+2)\,t + (\alpha+1) \end{pmatrix},$$

We consider the algebra of matrix differential operators having as eigenfunctions a sequence of polynomials $(P_n)_{n\geq 0}$, orthogonal with respect to the weight matrix $W = W^{(\alpha,\beta,v)}$. i.e.

$$D\,(W) = \left\{D : P_n D = \Lambda_n\,(D)\,P_n,\ \Lambda_n\,(D) \in \mathbb{C}^{N\times N} \text{ for all } n \geq 0\right\}.$$

The definition of $D(W)$ does not depend on the particular sequence of orthogonal polynomials (Grünbaum-Tirao, 2007).

We show that the dimension of the complex vector space $\mathcal{D}_2$ of differential operators in $D(W)$ of order at most two is $\dim \mathcal{D}_2 = 5$.

THE ALGEBRA OF DIFFERENTIAL OPERATORS *D(W)*

$$\kappa_{\pm v,\pm\beta} = \alpha \pm v \pm \beta\,.$$

We exhibit a set of symmetric operators $\{D_1, D_2, D_3, D_4, I\}$ which is a basis for the differential operators of order at most two in $D(W)$. The corresponding eigenvalues for the differential operators D_1, D_2, D_3 and D_4 are

$$\Lambda_n(D_1) = \frac{1}{4}\begin{pmatrix} (\kappa_{v,\beta}+2(n+1))(\kappa_{-v,\beta}+2(n+2)) & 0 \\ 0 & 0 \end{pmatrix},$$

$$\Lambda_n(D_2) = \begin{pmatrix} -\frac{1}{4}(\kappa_{-v,\beta}+2)(\kappa_{v,\beta}+4) & 0 \\ 0 & (n+\alpha+\beta+3)\,n \end{pmatrix},$$

$$\Lambda_n(D_3) = \frac{1}{4}(\kappa_{-v,\beta}+2(1+n))(\kappa_{-v,\beta}+2(2+n))\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} - \frac{(\kappa_{v,\beta}+2(1+n))(\kappa_{v,\beta}+2(2+n))(\kappa_{-v,\beta}+2)\,\kappa_{v,-\beta}}{4\kappa_{-v,-\beta}(\kappa_{v,\beta}+2)}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

$$\Lambda_n(iD_4) = -\frac{1}{4}(\kappa_{-v,\beta}+2(1+n))(\kappa_{-v,\beta}+2(2+n))\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} - \frac{(\kappa_{v,\beta}+2(1+n))(\kappa_{v,\beta}+2(2+n))(\kappa_{-v,\beta}+2)\,\kappa_{v,-\beta}}{4\kappa_{-v,-\beta}(\kappa_{v,\beta}+2)}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

◇ The algebra $D(W)$ is not commutative.

◇ There are no operators of order one in the algebra $D(W)$.

The existence of operators of order one associated to a matrix valued weight *W(x)* was initially considered by M. C. - A. Grünbaum (J. *Nonlinear Math. Phys., 2005) and A. Durán- M. D. de la Iglesia, (J. Approx Theory, 2008).*

THE ALGEBRA OF DIFFERENTIAL OPERATORS *D(W)*

For a given matrix-valued weight W, the analysis of the algebra $D(W)$ of all differential operators that have a sequence of matrix-valued orthogonal polynomials with respect to W as eigenfunctions has received much attention in the literature in the last fifteen years

- M.M. C. and F.A. Grünbaum, The algebra of differential operators associated to a family of matrix-valued orthogonal polynomials: five instructive examples, *Int. Math. Res. Not.*, 7, 1–33 (2006).
- F. A. Grünbaum and J. Tirao, The algebra of differential operators associated to a weight matrix, *Integral Equations Operator Theory*, 58(4):449–475 (2007).
- J. Tirao, The algebra of differential operators associated to a weight matrix: a first example, Polcino Milies, César (ed.), Groups, algebras and applications. XVIII Latin American algebra colloquium, São Pedro, Brazil, August 3–8, 2009. Proceedings. Providence, RI: American Mathematical Society (AMS). Contemporary Mathematics 537, 291-324, (2011).
- I. Pacharoni and I. Zurrián, Matrix Gegenbauer Polynomials: The 2×2 Fundamental Cases, *Constr. Approx.*, 43(2):253–271 (2016).
- I. Zurrián, The Algebra of Differential Operators for a Gegenbauer Weight Matrix, *Int. Math. Res. Not.*, 8, 2402-2430 (2016).

THE ALGEBRA OF DIFFERENTIAL OPERATORS *D(W)*

More recently:

- W. R. Casper, Elementary examples of solutions to Bochner's problem for matrix differential operators. *J. Approx. Theory*, 229, 36–71 (2018).
- W. R. Casper and M. Yakimov, The matrix Bochner problem, Amer. J. Math. (2021), to appear, *arXiv:1803.04405*.
- W. R. Casper, The symmetric 2×2 hypergeometric matrix differential operators, (2019), *arXiv:1907.12703*.

The author gives a classification of all 2x2 real hypergeometric Bochner pairs *(W(x),D)*, where *D* is symmetric with respect to the inner product defined by *W(x)*.

FURTHER WORK

To use the family of OP studied here in the context of time-and-band limiting, where the commutativity of the matrix valued eigenvalues Λ_n could play an important role.

Consider the sequence of *orthonormal* polynomials Q_n w.r.t. $W(x)$.

One considers the Integral kernel

$$K_N(x, y) = \sum_{n=0}^{N} Q_n^*(x) Q_n(y).$$

time-limiting parameter

It defines an integral operator I_K acting on any function $F \in L^2(W(x))$ as

$$I_K(F) = \int_0^{\Omega} F(s) W(s) (K_N(x, s))^* ds, \qquad \Omega \in (0, 1]$$

band-limiting parameter

One searches for an operator

$$\widetilde{D} = \frac{d^2}{dx^2} E_2(x) + \frac{d}{dx} E_1(x) + \frac{d^0}{dx} E_0(x)$$

such that

$$I_K \widetilde{D} = \widetilde{D} I_K.$$

FURTHER WORK

The sequence of monic polynomials $P_n^{(\alpha,\beta,v)}(t)$, orthogonal w.r.t $W(t)$ satisfies the differential equation

$$P_n^{(\alpha,\beta,v)}(t)D = \Lambda_n P_n^{(\alpha,\beta,v)}$$

According to a result of A, Grünbaum, I. Pacharoni and I. Zurrián, IMRN (2018)

Assuming the following hypothesis on the weight W and the differential operator D:

There exists a matrix $\widetilde{M}$, independent of the variables x,n and the band parameter Ω, but possibly depending on the time parameter N such that:

$$(\widetilde{M} - x(\wedge_{N+1} + \wedge_N)W(x) - W(x)(\widetilde{M} - x(\wedge_{N+1} + \wedge_N))^* = 0,$$

then for the time-band-limiting integral operator given by

$$I_K(F) = \int_a^\Omega F(s)W(s)(K_N(x,s))^* ds, \qquad \Omega \in (a,b]$$

the **commuting differential operator** can be written as

$$T = xD + Dx - 2\Omega D - (\wedge_{N+1} + \wedge_N)x + \widetilde{M}.$$

- Do we have such a matrix $\widetilde{M}$ in this case?