From ketchup to concentration-driven convex integration

Jan Burczak

UNIVERSITÄT LEIPZIG

Edinburgh, 11.11.2021

Continuum mechanics

$$\partial_t v + v \cdot \nabla v + \nabla p = \operatorname{div} \mathring{T}$$

$\mathring{T}(Dv) = \alpha I + \beta Dv + \gamma (Dv)^2$

α, β, γ scalars depending on invariants of Dv: $tr, \frac{1}{2}((tr)^2 - tr(^2)), det$

- $\alpha = \beta = \gamma = 0$ Euler
- $\alpha = \gamma = 0,\ \beta = \nu_0$ Navier-Stokes
- $\alpha = \gamma = 0,\ \beta = (\nu_0 + \nu_1 |Dv|)^{q-2}$ non-Newtonian

non-Newtonian fluids

$$\partial_t v + v \cdot \nabla v - \operatorname{div} \left((\nu_0 + \nu_1 |Dv|)^{q-2} Dv \right) + \nabla p = 0, \qquad \operatorname{div} v = 0$$

'viscosity changes under applied forces (shear)'

forces decrease viscosity	$q < 2$	forces increase viscosity	$q > 2$
peanut butter	1.07	corn starch+water	
whipped cream	1.12	silicone solutions	
high impact polystyrene	1.2		
ketchup	1.24		
paints	$1.6 - 1.85$		
blood	1.9		

Chabra&Richardson 2000

1929 Norton for molten steel, Ostwald for polymers
1966 Ladyzhenskaya ICM

Classics

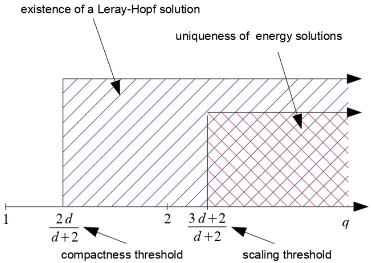
$$\partial_t v + v \cdot \nabla v - \operatorname{div}\left(|Dv|^{q-2} Dv\right) + \nabla p = 0, \qquad \operatorname{div} v = 0$$

- $E(v) = \int |v|^2(t) + 2\int_0^t \int |Dv|^q$

  $$W^{1,q} \subset\subset L^2 \;\; (q > \tfrac{2d}{d+2}) \implies \text{existence of energy solution}$$

- $v_\lambda := \lambda^\alpha v(\lambda x, \lambda^{\alpha+1} t) \quad \alpha = \frac{q-1}{3-q}$

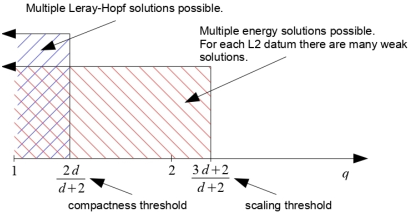
  $$E(v_\lambda) \overset{\lambda\to\infty}{\to} \infty \;\; q > \tfrac{3d+2}{d+2} \implies \text{uniqueness}$$



Ladyzhenskaya, Nečas, Malek, Frehse, Diening, Buliček 1969–2020

our 'dual picture'

$$\partial_t v + v \cdot \nabla v - \operatorname{div} \left((\nu_0 + \nu_1 |Dv|)^{q-2} Dv \right) + \nabla p = 0, \qquad \operatorname{div} v = 0$$



(A) for $1 < q < \frac{2d}{d+2}$: non-unique solutions $v \in C(L^2) \cap C(W^{1,q})$ with prescribed (total) energy $E(v)$

(B) for $1 < q < \frac{3d+2}{d+2}$: non-unique solutions $v \in C(L^2) \cap C(W^{1,(2d/(d+2))^-})$ with prescribed kinetic energy $\int |v|^2(t)$

(C) for $1 < q < \frac{3d+2}{d+2}$: For any initial datum $v_0 \in L^2$ there are infinitely many solutions $v \in C(L^2) \cap L^{(2d/(d+2))^-}(W^{1,(2d/(d+2))^-})$ of the Cauchy problem.

B., Modena, Székelyhidi, Comm. Math. Phys. 2021

convex integration

De Lellis & Székelyhidi: introducing conv. int. to fluid dynamics, Euler, C^α

$\downarrow$

Buckmaster & Vicol: (very weak) NSE, L^p via intermittency (Fourier side)

$\downarrow$

Modena & Székelyhidi (&Sattig): transport, intermittency $\rightsquigarrow$ concentration (real side)

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non-Newtonian: back to fluid dynamics

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non-Newtonian: back to fluid dynamics

take (u_0, q_0, R_0) solving Non-Newtonian-Reynolds

$$\partial_t u_0 + \mathrm{div}\,(u_0 \otimes u_0) - \mathrm{div}\, A(Du_0) + \nabla p = -\mathrm{div}\, R_0$$

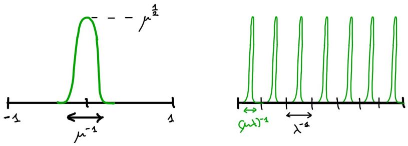
aim: produce (u_1, q_1, R_1) via $u_1 := u_0 + u_p$ $\quad u_p = \sqrt{|R_0|}W^k$

such that $R_1 = \mathrm{div}^{-1}\mathrm{div}\,(u_p \otimes u_p - R_0)$ small via fast oscillations of $W^k(\lambda\cdot)$,

$\mathrm{div}^{-1} \rightsquigarrow \lambda^{-1}$

Building blocks W^k

if all L^p norms behave same, then $1/3$ differentiability at best



$$f_\mu = \mu^{1/2} f(\mu) \implies |f_\mu|_{L^q(\mathbb{R})} \sim \mu^{1/2-1/q} \qquad |\nabla (f_\mu)_\lambda|_{L^q(\mathbb{T})} \sim (\mu\lambda)\mu^{1/2-1/q}$$

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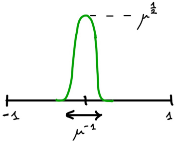

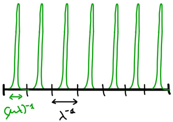

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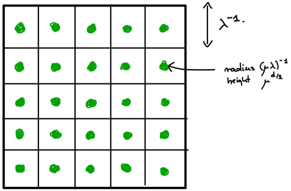


$$f_\mu = \mu^{d/2} f(\mu) \implies |f_\mu|_{L^q(\mathbb{R}^d)} \sim \mu^{d/2-d/q} \qquad |\nabla (f_\mu)_\lambda|_{L^{\frac{2d}{d+2}-}(\mathbb{T}^d)} \le \lambda \mu^{-\epsilon}$$

Iteration Step

Fix $e \in C^\infty([0,1];[\frac{1}{2},1])$. (u_0, π_0, R_0) solves Non-Newtonian-Reynolds

$$\partial_t u_0 + \operatorname{div}(u_0 \otimes u_0) - \operatorname{div} A(Du_0) + \nabla \pi_0 = -\operatorname{div} R_0, \quad \operatorname{div} u_0 = 0.$$

Take any $\delta, \eta \in (0,1]$. Assume

$$\tfrac{3}{4}\delta e(t) \leq e(t) - \left(\int_{\mathbb{T}^d} |u_0|^2(t) + 2\int_0^t \int_{\mathbb{T}^d} A(Du_0)Du_0 \right) \leq \tfrac{5}{4}\delta e(t)$$

$$|R_0(t)|_{L^1} \leq \frac{\delta}{2^7 d}.$$

Then, $\exists$ solution (u_1, π_1, R_1) to Non-Newtonian-Reynolds such that

$$|(u_1 - u_0)(t)|_{L^2} \leq M|R_0(t)|_{L^1}^{\frac{1}{2}}$$

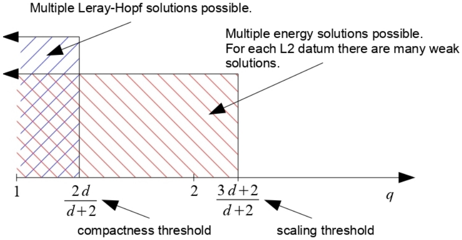
$$|R_1(t)|_{L^1} \leq \eta$$

$$|(u_1 - u_0)(t)|_{W^{1,(6/5)-}} \leq \eta$$

$$\tfrac{3}{8}\delta e(t) \leq e(t) - \left(\int_{\mathbb{T}^d} |u_1|^2(t) + 2\int_0^t \int_{\mathbb{T}^d} A(Du_1)Du_1 \right) \leq \tfrac{5}{8}\delta e(t).$$

Summary

- non-uniqueness picture, sharp in powers



- improves regularity of NSE non-unique weak solutions by Buckmaster&Vicol
- avoids Fourier side
- avoids meticulous control of decays
- introduces concentration mechanism into fluid-dynamics convex integration

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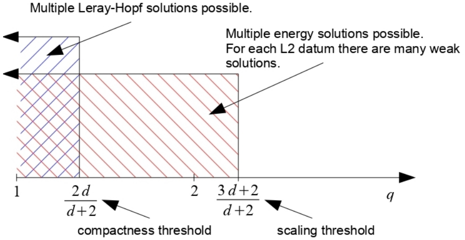

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THANK YOU