# INTRODUCTION TO QUANTUM GROUPS
# SPECIAL LMS POSTGRADUATE SCHOOL IN ALGEBRA, AUTUMN 2020

BART VLAAR, HERIOT-WATT UNIVERSITY

## Contents



## 1. Introduction

In these lecture notes we give an introductory account of quantum groups and the notion of quasitriangularity. Note that categories of modules of a bialgebra $A$ have a tensor product structure. Given two $A$-modules $V, W$, in general it is not clear that $V \otimes W$ is isomorphic to $W \otimes V$. If $A$ possesses a quasitriangular structure, i.e. a universal R-matrix, then there exists a natural isomorphism which is compatible with the tensor structure. Here we will discuss the bialgebra $U_q\mathfrak{g}$, a nontrivial deformation of the universal enveloping algebra $U\mathfrak{g}$ of a finite-dimensional semisimple Lie algebra $\mathfrak{g}$, and explain how to construct the universal R-matrix. This construction underpins the interest in quantum groups from the point of view of low-dimensional topology and knot theory, as well as quantum integrability (simultaneous diagonalizability of Hamiltonians and related operators via solutions of the Yang-Baxter equation). Other motivations for the study of quantum groups include q-deformed harmonic analysis (i.e. the theory of special functions depending on a deformation parameter) and noncommutative geometry (more precisely, quantized algebras of functions).

Quantum groups were discovered in the 1980s in the context of quantum integrability, initially in [KR83], and provide a rich family of noncommutative non-cocommutative Hopf algebras. There is not a single all-encompassing definition of quantum group; rather, there is a collection of related types of non-(co)commutative Hopf algebras, each deforming a (co)commutative Hopf algebra. Since the representation theory of quantized enveloping algebras $U_q\mathfrak{g}$, for $q$ not a root of unity, stays fairly close to that of $\mathfrak{g}$, we will focus on this special class. We will also briefly discuss the associated dual quantum groups (quantized algebras of functions on matrices) and Kac-Moody generalizations of Drinfeld-Jimbo quantum groups. Unfortunately there is no time to discuss other incarnations of quantum groups such as compact quantum groups, bicrossproduct quantum groups, Yangians, RTT algebras, and many more. Also certain important aspects of quantum group theory such as Lie bialgebra quantization and canonical bases are omitted. We hope nevertheless that these notes spur the reader on to a deeper investigation in this rich variety of topics. Good textbook resources for quantum groups are for instance [CP95, Ja96, Lu94].

1.1. **Outline.** First of all we will review the key concepts of Hopf algebras and their representations, focusing initially on the (co)commutative case and then discussing quasitriangular Hopf algebras, thereby providing some background, motivation and notation for the rest of the course. In Section 4 we recall some basic theory of enveloping algebras and in Section 5 we discuss their quantizations: Drinfeld-Jimbo quantum groups. Finally in Section 6 we discuss quasitriangularity for Drinfeld-Jimbo quantum groups, paying special attention to the $\mathfrak{sl}_2$-case.

1.2. **Acknowledgements.** These notes and the associated lecture series, as part of the LMS Autumn Algebra School 2020, were supported by the London Mathematical Society, the European Research Council and the International Centre for Mathematical Sciences in Edinburgh, who are hosting the event online. The author is grateful to Stefan Kolb for useful comments.

## 2. Hopf algebras

We fix a field $k$; linearity and tensor products will always be with respect to $k$ (later on we will assume that $k$ is of characteristic zero and algebraically closed). We will always assume that algebras (except Lie algebras) are unital and associative and similarly that coalgebras are counital and coassociative.

2.1. **Definition and basic properties.** A vector space $A$ is called a *bialgebra* if it is simultaneously an algebra, with multiplication map $m : A \otimes A \to A$ and unit map $\eta : k \to A$ (embedding of scalars, sending $x \in k$ to $x \cdot 1_A$), and a coalgebra, with comultiplication $\Delta : A \to A \otimes A$ and counit $\epsilon : A \to k$, in such a way that the two structures are compatible: the comultiplication and counit are algebra homomorphisms or, equivalently, the multiplication map and unit map are coalgebra homomorphisms. If $A$ and $B$ are bialgebras, a linear map $\phi : A \to B$ which is both an algebra and a coalgebra homomorphism is called a *bialgebra homomorphism*.

A *Hopf algebra* is a bialgebra $A$ equipped with an invertible[1] linear map $S : A \to A$ (antipode) such that the following diagram commutes:

(2.1)
$$\begin{tikzcd}
 & A \otimes A \arrow[rr, "S \otimes \mathrm{id}"] & & A \otimes A \arrow[dr, "m"] & \\
A \arrow[ur, "\Delta"] \arrow[rr, "\epsilon"] \arrow[dr, "\Delta"'] & & k \arrow[rr, "\eta"] & & A \\
 & A \otimes A \arrow[rr, "\mathrm{id} \otimes S"] & & A \otimes A \arrow[ur, "m"'] &
\end{tikzcd}$$

We now state some basic facts and further terminology involving Hopf algebras, referring to [CP95, Sec. 4.1] and references therein for proofs.

(1) If a bialgebra has an antipode satisfying (2.1) then it is unique.
(2) If $A, B$ are Hopf algebras with antipodes $S_A$ and $S_B$ and $\phi : A \to B$ is a bialgebra homomorphism, then automatically $\phi \circ S_A = S_B \circ \phi$ , and we call $\phi$ a *Hopf algebra homomorphism*.
(3) Let $A$ be a Hopf algebra; consider the linear map $\sigma$ on $A \otimes A$ defined by

$$\sigma\left(\sum_r a_r \otimes b_r\right) := \sum_r b_r \otimes a_r. \tag{2.2}$$

Denoting the multiplication by $m$, comultiplication by $\Delta$ and antipode by $S$, automatically we have two more Hopf algebras:
- the opposite bialgebra $A^{\mathrm{op}}$, with multiplication $m^{\mathrm{op}} := m \circ \sigma$, antipode $S^{-1}$ and the other structure maps unchanged;
- the co-opposite bialgebra $A^{\mathrm{cop}}$, with comultiplication $\Delta^{\mathrm{op}} := \sigma \circ \Delta$, antipode $S^{-1}$ and the other structure maps unchanged.

A Hopf algebra $A$ is called *commutative* if $A^{\mathrm{op}} = A$ (i.e. if the underlying algebra is commutative) and *co-commutative* if $A^{\mathrm{cop}} = A$ (i.e. if $\Delta^{\mathrm{op}} = \Delta$).
(4) The antipode $S$ is an anti-automorphism of both the underlying algebra and the coalgebra structure, in other words $S : A \to A^{\mathrm{op,cop}}$ is a Hopf algebra isomorphism.
(5) If $f : A \to A$ is an algebra automorphism, then we can twist the Hopf algebra structure of $A$ to obtain a new Hopf algebra $A_f$ with the same underlying algebra. More precisely, the structure maps $\Delta$, $\epsilon$ and $S$ have been replaced by

$$\Delta_f := (f \otimes f) \circ \Delta \circ f^{-1}, \qquad \epsilon_f := \epsilon \circ f^{-1}, \qquad S_f := f \circ S \circ f^{-1}. \tag{2.3}$$

(6) We call $h \in A$ *grouplike* if $h \neq 0$ and $\Delta(h) = h \otimes h$; automatically $\epsilon(h) = 1$, $h$ is invertible, and $S(h) = h^{-1}$; the set of all grouplike elements forms a group.
(7) We call $h \in A$ *primitive* if $\Delta(h) = h \otimes 1 + 1 \otimes h$; automatically $\epsilon(h) = 0$ and $S(h) = -h$; the set of all primitive elements forms a Lie algebra, with bracket given by the commutator.

[1] A weaker (perhaps more customary) definition allows for a non-invertible antipode. In this more general setting one can prove that $S^2 = \mathrm{id}_A$ if the underlying algebra is commutative or the underlying coalgebra is co-commutative. Since we are interested in "deformations" of these cases it is natural to require $S$ to be bijective from the start.

2.2. **Some module theory for bialgebras and Hopf algebras.** Recall that if $A$ is an algebra then a *(left) $A$-module* is a vector space $V$ together with a linear map $\lambda_V : A \otimes V \to V$, called *(left) action map*, such that the following diagrams commute:

(2.4)
$$\begin{tikzcd} A \otimes A \otimes V \arrow[r, "m\otimes \mathrm{id}_V"] \arrow[d, "\mathrm{id}_A \otimes \lambda_V"'] & A \otimes V \arrow[d, "\lambda_V"] \\ A \otimes V \arrow[r, "\lambda_V"'] & V \end{tikzcd} \qquad \begin{tikzcd} k \otimes V \arrow[r, "\eta\otimes \mathrm{id}_V"] \arrow[dr, "\cong"'] & A \otimes V \arrow[d, "\lambda_V"] \\ & V \end{tikzcd}$$

In other words, $\pi_V : A \to \mathrm{End}(V)$ given by $\pi_V(a) := \lambda_V(a \otimes \cdot)$ is an algebra map, i.e. $(\pi_V, V)$ is a representation of $A$. We will often denote $\lambda_V(a \otimes v)$ by $a \cdot v$ for all $a \in A$ and all $v \in V$. If $V, W$ are $A$-modules, then we call a linear map $\phi : V \to W$ an *$A$-intertwiner* (or $A$-module homomorphism, or $A$-equivariant map) if $\phi$ commutes with the action of $A$, i.e. if the following diagram commutes:

(2.5)
$$\begin{tikzcd} A \otimes V \arrow[r, "\mathrm{id}\otimes\phi"] \arrow[d, "\lambda_V"'] & A \otimes W \arrow[d, "\lambda_W"] \\ V \arrow[r, "\phi"'] & W \end{tikzcd},$$

Finally, if $V$ is an $A$-module we call $v \in V$ *$A$-invariant* if $\lambda_V(a \otimes v) = \epsilon(a)v$ for all $a \in A$; any vector space $V$ becomes a $A$-module in a trivial manner by stipulating that all elements of $V$ are $A$-invariant.

By definition, the objects of the category $\mathrm{Rep}(A)$ are (left) $A$-modules and the arrows are $A$-intertwiners. There is a natural forgetful functor from $\mathrm{Rep}(A)$ to the category $\mathrm{Vect}$, whose objects are vector spaces and whose arrows are linear maps, consisting of ignoring the $A$-module structure. Note that the category of vector spaces $\mathrm{Vect}$ is a tensor category; that is, there is an associative (up to natural isomorphisms) bifunctor $\otimes$ and a "multiplicative identity" with respect to this bifunctor, namely $k$. Given an algebra $A$, one may wonder if the category $\mathrm{Rep}(A)$ is a tensor category; in particular, do tensor products of $A$-modules naturally possess an $A$-module structure themselves?

In the case of a bialgebra, this is guaranteed, and the comultiplication and counit provide the action on tensor products and on $k$, respectively. More precisely, for all $a \in A$ there exist $a_{(1),n}, a_{(2),n} \in A$ such that $\Delta(a) = \sum_n a_{(1),n} \otimes a_{(2),n}$ and we set

(2.6) $$a \cdot (v \otimes w) = \sum_n (a_{(1),n} \cdot v) \otimes (a_{(2),n} \cdot w) \qquad \text{for all } v \in V, w \in W,\ V, W \in \mathrm{Rep}(A),$$

(2.7) $$a \cdot v = \epsilon(a)v \qquad \text{for all } v \in k.$$

In terms of representations, we have $\pi_{V\otimes W} = (\pi_V \otimes \pi_W) \circ \Delta$ and $\pi_k = \epsilon$.

Note that if $A$ is co-commutative then the tensor category $\mathrm{Rep}(A)$ is *symmetric*: for all $V, W \in \mathrm{Rep}(A)$ there is an arrow from the object $V \otimes W$ to the object $W \otimes V$, given by the $A$-intertwiner $P_{V,W} : V \otimes W \to W \otimes V$ uniquely determined by

(2.8) $$P_{V,W}(v \otimes w) = w \otimes v \qquad \text{for all } v \in V, w \in W.$$

The fact that $P_{V,W}$ is an intertwiner is equivalent to the statement that $\Delta = \Delta^{\mathrm{op}}$.

If $A$ is in addition a Hopf algebra then the dual $V^* = \mathrm{Hom}(V, k)$ of $V \in \mathrm{Rep}(A)$ becomes an $A$-module by setting, for all $a \in A$,

(2.9) $$(a \cdot f)(v) = f(S(a) \cdot v) \qquad \text{for all } f \in V^*, v \in V.$$

The corresponding categorical notion is *rigidity*: in the tensor category of $A$-modules every object has a natural dual object.

2.3. **Some group-related examples.** At this point, let us mention two key classes of examples depending on a group $G$.

(1) The group algebra $kG$ (which is by definition an algebra) is a co-commutative Hopf algebra if we set

(2.10) $$\Delta(g) = g \otimes g, \qquad \epsilon(g) = 1, \qquad S(g) = g^{-1} \qquad \text{for all } g \in G$$

and extend linearly.

(2) Dually, for a finite group $G$, consider the algebra $k^G$ of functions $f : G \to k$ (with pointwise addition and multiplication). We may identify $k^G \otimes k^G$ with $k^{G\times G}$, so that $k^G$ is a commutative Hopf algebra if we set

$$\Delta(f)(g,h) = f(gh), \qquad \epsilon(f) = f(1_G), \qquad S(f)(g) = f(g^{-1}) \qquad \text{for all } f \in k^G, g,h \in G. \tag{2.11}$$

Similarly, say for $k = \mathbb{C}$, the algebra of continuous $\mathbb{C}$-valued functions on a compact topological group (for instance a Lie group) becomes a Hopf algebra with these assignments, as does the algebra of regular functions on an algebraic group.

We will see that quantum groups arise in a certain way as non-(co-)commutative variations of the above Hopf algebras. Note especially that in non-commutative geometry one studies algebraic groups via their algebras of regular functions; it is natural to consider a "modified (or quantized) algebraic group" by making the algebra of regular functions non-commutative. This is the origin of the name "quantum group".

In order to make this concrete, we will deform the universal enveloping algebra of a Lie algebra associated to a (connected, complex, semisimple) Lie group. Furthermore, we want to do the deformation in a controlled way, effectively leading to a natural weakening of the notion of co-commutativity, namely quasitriangularity. In the next two sections we recall some basic theory involving quasitriangular Hopf algebras and universal enveloping algebras. Then we will be ready to define a particular type of quantum group: quantized universal enveloping algebras, also known as Drinfeld-Jimbo quantum groups.

## 3. Quasitriangular Hopf algebras and braided tensor categories

3.1. **Quasitriangularity: definition and basic properties.** Before we define the key notion of quasitriangularity, we need an extra piece of notation which naturally arises in the context of tensor products. Fix $L \in \mathbb{Z}_{\geq 2}$ and consider algebras $A_1, A_2, \ldots, A_L$. For $1 \leq i < j \leq L$ consider the algebra embedding of $A_i \otimes A_j$ into $A_1 \otimes A_2 \otimes \cdots \otimes A_L$, mapping $X = \sum_n a_n \otimes b_n \in A_i \otimes A_j$ to

$$X_{ij} = \sum_n 1_{A_1 \otimes \cdots \otimes A_{i-1}} \otimes a_n \otimes 1_{A_{i+1} \otimes \cdots \otimes A_{j-1}} \otimes b_n \otimes 1_{A_{j+1} \otimes \cdots \otimes A_L}. \tag{3.1}$$

(i.e. the first tensor factor is mapped into the $i$-th factor and the second factor is mapped into the $j$-th factor). For now we will use this in the special case $L = 3$ and $A_1 = A_2 = A_3$.

**Definition 3.1.** Let $A$ be a Hopf algebra or bialgebra and $\mathcal{R} \in A \otimes A$ an invertible element. The pair $(A, \mathcal{R})$ is called *quasitriangular* and $\mathcal{R}$ a *(universal) R-matrix* for $A$ if

$$\mathcal{R}\Delta(a) = \Delta^{\mathrm{op}}(a)\mathcal{R} \qquad \text{for all } a \in A, \tag{3.2}$$

$$(\Delta \otimes \mathrm{id})(\mathcal{R}) = \mathcal{R}_{13}\mathcal{R}_{23}, \tag{3.3}$$

$$(\mathrm{id} \otimes \Delta)(\mathcal{R}) = \mathcal{R}_{13}\mathcal{R}_{12}. \tag{3.4}$$

If $A$ is co-commutative then it is quasitriangular with $\mathcal{R} = 1 \otimes 1$; clearly, (3.2) is a weakening of co-commutativity. Since $A^{\mathrm{cop}}$ itself is another bialgebra, there are natural restrictions on the element $\mathcal{R}$ induced by coassociativity. Namely, let $a$ be an arbitrary element of a bialgebra $A$ satisfying (3.2) for some $\mathcal{R} \in (A \otimes A)^\times$. We have

$$\begin{aligned} \big((\Delta^{\mathrm{op}} \otimes \mathrm{id}) \circ \Delta^{\mathrm{op}}\big)(a) &= \mathcal{R}_{12}(\Delta \otimes \mathrm{id})\big(\mathcal{R}\Delta(a)\mathcal{R}^{-1}\big)\mathcal{R}_{12}^{-1} \\ &= \mathcal{R}_{12}(\Delta \otimes \mathrm{id})(\mathcal{R})(\Delta \otimes \mathrm{id})(\Delta(a))(\Delta \otimes \mathrm{id})(\mathcal{R}^{-1})\mathcal{R}_{12}^{-1} \\ &= \mathcal{R}_{12}(\Delta \otimes \mathrm{id})(\mathcal{R})(\Delta \otimes \mathrm{id})(\Delta(a))\big(\mathcal{R}_{12}(\Delta \otimes \mathrm{id})(\mathcal{R})\big)^{-1}. \end{aligned} \tag{3.5}$$

In an analogous way we obtain

$$\big((\mathrm{id} \otimes \Delta^{\mathrm{op}}) \circ \Delta^{\mathrm{op}}\big)(a) = \mathcal{R}_{23}(\mathrm{id} \otimes \Delta)(\mathcal{R})(\mathrm{id} \otimes \Delta)(\Delta(a))\big(\mathcal{R}_{23}(\mathrm{id} \otimes \Delta)(\mathcal{R})\big)^{-1}. \tag{3.6}$$

For both $A$ and $A^{\mathrm{op}}$ to be coassociative it is certainly sufficient to impose $\mathcal{R}_{12}(\Delta \otimes \mathrm{id})(\mathcal{R}) = \mathcal{R}_{23}(\mathrm{id} \otimes \Delta)(\mathcal{R})$. Similarly, the counit axioms for both $A$ and $A^{\mathrm{op}}$ are equivalent to $(\epsilon \otimes \mathrm{id})(\mathcal{R})$ and $(\mathrm{id} \otimes \epsilon)(\mathcal{R})$ being central elements in $A$. The axioms (3.3-3.4) in fact allow us to derive desirable results which are slightly stronger than these constraints on $\mathcal{R}$, namely properties (2) and (3) in the following result.

**Proposition 3.2.** *Let $(A, \mathcal{R})$ be a quasitriangular bialgebra.*

*(1) Also $(A, \sigma(\mathcal{R})^{-1})$, $(A_{\mathrm{op}}, \mathcal{R}_{21})$ and $(A_{\mathrm{cop}}, \mathcal{R}_{21})$ are quasitriangular.*

*(2) The* (universal) (quantum) Yang-Baxter equation *is satisfied:*

$$\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12} \qquad \in A \otimes A \otimes A. \tag{3.7}$$

*(3) We have*

$$(\epsilon \otimes \mathrm{id})(\mathcal{R}) = (\mathrm{id} \otimes \epsilon)(\mathcal{R}) = 1. \tag{3.8}$$

*(4) If $A$ is a Hopf algebra with antipode $S$ then*

$$(S \otimes \mathrm{id})(\mathcal{R}) = \mathcal{R}^{-1} = (\mathrm{id} \otimes S^{-1})(\mathcal{R}), \qquad (S \otimes S)(\mathcal{R}) = \mathcal{R}. \tag{3.9}$$

*Proof.*

(1) These follow from straightforwardly checking conditions (3.2-3.4) for the three candidate quasitriangular bialgebras in question.

(2) This follows from conditions (3.2) and (3.3) (or alternatively (3.2) and (3.4)):

$$\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} = \mathcal{R}_{12}(\Delta \otimes \mathrm{id})(\mathcal{R}) = (\Delta^{\mathrm{op}} \otimes \mathrm{id})(\mathcal{R})\mathcal{R}_{12} = (\sigma \otimes \mathrm{id})(\mathcal{R}_{13}\mathcal{R}_{23})\mathcal{R}_{12} = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12}. \tag{3.10}$$

(3) These relations follow by applying $\mathrm{id} \otimes \epsilon \otimes \mathrm{id}$ to (3.3-3.4), respectively, and using coassociativity and invertibility of $\mathcal{R}$.

(4) To establish the first identity, we will show that $\mathcal{R}(S \otimes \mathrm{id})(\mathcal{R}) = 1 \otimes 1$. Note that we may write

$$\mathcal{R}(S \otimes \mathrm{id})(\mathcal{R}) = (m \otimes \mathrm{id})(\mathcal{R}_{13}(S \otimes \mathrm{id})(\mathcal{R})_{23}) = \big((m \otimes \mathrm{id}) \circ (\mathrm{id} \otimes S \otimes \mathrm{id})\big)(\mathcal{R}_{13}\mathcal{R}_{23}) \tag{3.11}$$

where $m : A \otimes A \to A$ is the multiplication. From condition (3.3) we obtain

$$\mathcal{R}(S \otimes \mathrm{id})(\mathcal{R}) = \big((m \otimes \mathrm{id}) \circ (\mathrm{id} \otimes S \otimes \mathrm{id}) \circ (\Delta \otimes \mathrm{id})\big)(\mathcal{R}). \tag{3.12}$$

Now applying the "upper half" of the antipode axiom (2.1) and the counit property (3.8) we arrive at the desired result:

$$\mathcal{R}(S \otimes \mathrm{id})(\mathcal{R}) = \big((\eta \circ \epsilon) \otimes \mathrm{id}\big)(\mathcal{R}) = (\eta \otimes \mathrm{id})(1) = 1 \otimes 1. \tag{3.13}$$

In order to prove the second identity, apply the same reasoning to the quasitriangular Hopf algebra $(A^{\mathrm{op}}, \sigma(\mathcal{R}))$. The third identity follows readily if we combine the first two. □

The following is very useful when proving that a particular bialgebra is quasitriangular.

**Lemma 3.3.** *Let $A$ be a bialgebra such that we have a map $\omega : A \to A$ which is at the same time an algebra automorphism and coalgebra anti-automorphism. If $\mathcal{R} \in (A \otimes A)^\times$ satisfies*

$$(\omega \otimes \omega)(\mathcal{R}) = \sigma(\mathcal{R}) \tag{3.14}$$

*then conditions (3.3) and (3.4) are equivalent.*

*Proof.* This follows from the identities $\mathrm{id}\otimes\Delta = (\sigma\otimes\mathrm{id})\circ(\mathrm{id}\otimes\sigma)\circ(\Delta\otimes\mathrm{id})\circ\sigma$ and $(\omega\otimes\omega)\circ\sigma = \sigma\circ(\omega\otimes\omega)$. □

3.2. **Braided tensor categories.** The main point of having a quasitriangular structure on a bialgebra is that the category of (left) $A$-modules is not just a tensor category, but that tensor products of $A$-modules are naturally isomorphic as $A$-modules, thereby preserving the key property of symmetric tensor categories. Moreover the category of $A$-modules carries a natural braided structure.

More precisely, let $V$ and $W$ be two $A$-modules with corresponding representation maps $\pi_V : A \to \mathrm{End}(V)$, $\pi_W : A \to \mathrm{End}(W)$. Denote by $R_{V,W} = (\pi_V \otimes \pi_W)(\mathcal{R})$ the linear map on $V \otimes W$ corresponding to the action of $\mathcal{R}$. Recall the linear map $P_{V,W} : V \otimes W \to W \otimes V$ defined in (2.8) and consider the linear map $\check{R}_{V,W} : V \otimes W \to W \otimes V$ defined by

$$\check{R}_{V,W} := P_{V,W} \circ R_{V,W}. \tag{3.15}$$

The axiom (3.2) implies

$$R_{V,W}(\pi_V \otimes \pi_W)(\Delta(a)) = (\pi_V \otimes \pi_W)(\Delta^{\mathrm{op}}(a))R_{V,W} \qquad \text{for all } a \in A \tag{3.16}$$

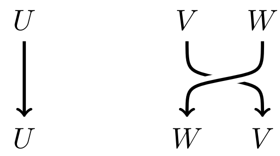
and hence $\check{R}_{V,W}$ intertwines the modules $V \otimes W$ and $W \otimes V$:

$$\check{R}_{V,W}\pi_{V\otimes W}(a) = \pi_{W\otimes V}(a)\check{R}_{V,W} \qquad \text{for all } a \in A, \tag{3.17}$$

But this means precisely that $V \otimes W$ and $W \otimes V$ are isomorphic as $H$-modules.

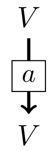
It is possible, see [RT90], to represent the category $\mathrm{Rep}(A)$ using a diagrammatical calculus, with $A$-intertwiners $\phi : U_1 \otimes \cdots \otimes U_m \to V_1 \otimes \cdots \otimes V_n$ corresponding to diagrams with $m$ incoming arrows and $n$ outgoing arrows, labelled by the corresponding modules. Furthermore taking tensor products corresponds to horizontal juxtaposition, and composition of intertwiners corresponds to vertical juxtaposition (we use the convention that composition is downward, which is also indicated by arrows). In particular, the intertwiners $\mathrm{id}_U$ and $\check{R}_{V,W}$ are represented by a single strand and a braiding:

(3.18)



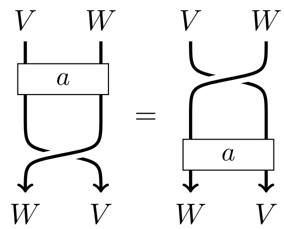

We also represent the action of an element $a \in A$ on $V \in \mathrm{Rep}(A)$ by a decoration, marked by $a$, of the strand labelled by $V$:

(3.19)




In particular, (3.17) corresponds to

(3.20)




Also the coproduct axioms (3.3-3.4) correspond to very natural categorical and topological points of view, providing us with another motivation. Namely, assume that $U, V, W \in \mathrm{Rep}(A)$ and apply $\pi_U \otimes \pi_V \otimes \pi_W$ to (3.3). It yields

$$R_{U\otimes V,W} = (R_{U,W})_{13}(R_{V,W})_{23} \tag{3.21}$$

from which we deduce, by left-multiplying by $(P_{U,W})_{12}(P_{V,W})_{23} = P_{U\otimes V,W}$,

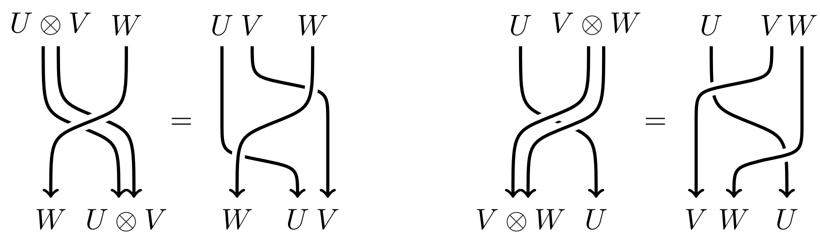
$$\check{R}_{U\otimes V,W} = (\check{R}_{U,W} \otimes \mathrm{id}_V)(\mathrm{id}_U \otimes \check{R}_{V,W}) \in \mathrm{Hom}_A(U\otimes V\otimes W, W\otimes U\otimes V). \tag{3.22}$$

In the same way, from (3.4) we obtain

$$\check{R}_{U,V\otimes W} = (\mathrm{id}_V \otimes \check{R}_{U,W})(\check{R}_{U,V} \otimes \mathrm{id}_W) \in \mathrm{Hom}_A(U\otimes V\otimes W, V\otimes W\otimes U). \tag{3.23}$$

In terms of the diagrammatical calculus (3.22-3.23) correspond to the topological identities

(3.24)




To complete the description of the braided structure on $\mathrm{Rep}(A)$, suppose $U, V, W \in \mathrm{Rep}(A)$ and apply $\pi_U \otimes \pi_V \otimes \pi_W$ to (3.7). We obtain the (quantum) (matrix) Yang-Baxter equation

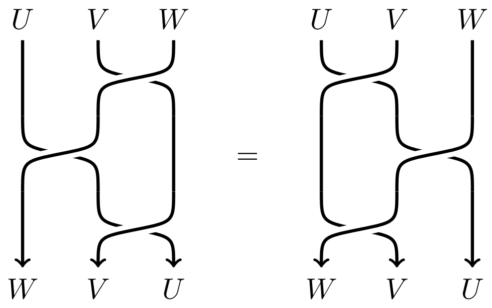
$$(R_{U,V})_{12}(R_{U,W})_{13}(R_{V,W})_{23} = (R_{V,W})_{23}(R_{U,W})_{13}(R_{U,V})_{12} \tag{3.25}$$

an equation of linear maps from $U\otimes V\otimes W$ to itself, or equivalently, its braided formulation

$$(\mathrm{id}_W \otimes \check{R}_{U,V})(\check{R}_{U,W} \otimes \mathrm{id}_V)(\mathrm{id}_U \otimes \check{R}_{V,W}) = (\check{R}_{V,W} \otimes \mathrm{id}_U)(\mathrm{id}_V \otimes \check{R}_{U,W})(\check{R}_{U,V} \otimes \mathrm{id}_W) \tag{3.26}$$

an equation of linear maps from $U \otimes V \otimes W$ to $W \otimes V \otimes U$; the latter corresponds diagrammatically to

(3.27)



Finally, let $L \in \mathbb{Z}_{\geq 0}$ and $V \in \mathrm{Rep}(A)$ be arbitrary. Consider the braid group

$$B_L := \big\langle b_1, \ldots, b_{L-1} \,\big|\, b_i b_{i+1} b_i = b_{i+1} b_i b_{i+1}, b_i b_j = b_j b_i \text{ if } |i-j| > 1 \big\rangle. \tag{3.28}$$

We obtain a representation of $B_L$ on $V^{\otimes L}$, given by

$$b_i \mapsto (\check{R}_{V,V})_{i\,i+1}. \tag{3.29}$$

Note that if $\sigma(\mathcal{R})\mathcal{R} = 1 \otimes 1$ then for all $V, W \in \mathrm{Rep}(A)$ we have $P_{W,V} R_{W,V} P_{V,W} R_{V,W} = \mathrm{Id}_{V \otimes W}$ and hence $\check{R}_{W,V} \check{R}_{V,W} = \mathrm{Id}_{V \otimes W}$; in this case the above representation of $B_L$ factors through a representation of the symmetric group $S_L$.

3.3. **Sweedler's Hopf algebra - a warm-up exercise.** In addition to the many co-commutative Hopf algebras, it would be nice to have quasitriangular Hopf algebras with a nontrivial R-matrix. Before we discuss (Drinfeld-Jimbo) quantum groups in detail, let us start with a finite-dimensional example.

Assume that the characteristic of $k$ is not 2 and consider Sweedler's Hopf algebra, see [Sw69], i.e. the algebra $A$ generated by symbols $f$ and $g$ subject to the relations

$$f^2 = 0, \qquad g^2 = 1, \qquad fg = -gf. \tag{3.30}$$

**Proposition 3.4.** *The assignments*

$$\Delta(f) = f \otimes g + 1 \otimes f, \qquad \epsilon(f) = 0, \qquad S(f) = -fg^{-1} \quad (= gf) \tag{3.31}$$

$$\Delta(g) = g \otimes g, \qquad \epsilon(g) = 1, \qquad S(g) = g^{-1} \quad (= g) \tag{3.32}$$

*define a Hopf algebra structure on $A$.*

*Proof.* A straightforward check on generators. □

Note that $g$ is a group-like element and, up to a $g$-dependent correction, $f$ is a primitive element. The algebra $A$ is the smallest non-commutative non-cocommutative Hopf algebra and hence provides the simplest nontrivial setting to showcase the concepts we have discussed in abstract. We will see later that certain typical properties of Drinfeld-Jimbo quantum groups are foreshadowed by analogues statements for $A$.

In order to construct a quasitriangular structure on $A$, let $\beta \in k$ be arbitrary and set

$$\mathcal{R}_\beta = \tfrac{1}{2}(1 \otimes 1 + 1 \otimes g + g \otimes 1 - g \otimes g)(1 \otimes 1 + \beta f \otimes gf) \in A \otimes A. \tag{3.33}$$

Before we prove that $(A, \mathcal{R}_\beta)$ is a quasitriangular, we make some observations. It is useful to consider the elements $\mathcal{R}_0$ and

$$\widetilde{\mathcal{R}}_\beta := \mathcal{R}_0^{-1} \mathcal{R}_\beta = 1 \otimes 1 + \beta f \otimes gf. \tag{3.34}$$

By writing $\mathcal{R}_0 = 1 \otimes 1 - 2(\frac{1-g}{2})^{\otimes 2}$ and noting that $\frac{1-g}{2} \in A$ is an idempotent, we deduce that $\mathcal{R}_0^2 = 1 \otimes 1$. We also observe that $\widetilde{\mathcal{R}}_\beta$ is a power series in $f \otimes gf$ with invertible constant term. Hence $\mathcal{R}_0$ and $\widetilde{\mathcal{R}}_\beta$ are both invertible, so that $\mathcal{R}_\beta$ is invertible. Moreover by a direct computation, we obtain

$$\mathcal{R}_0(f \otimes g) = (f \otimes 1)\mathcal{R}_0, \qquad \mathcal{R}_0(1 \otimes f) = (g \otimes f)\mathcal{R}_0 \tag{3.35}$$

(one can be obtained from the other by applying $\sigma$ and the involutiveness of $\mathcal{R}_0$).

Consider the linear involution $\omega : A \to A$ fixing the group-like elements $1$ and $g$ pointwise and swapping $f$ and $fg$.

**Lemma 3.5.** *The map $\omega$ is an algebra automorphism of $A$, as well as a coalgebra anti-automorphism of $A$. Moreover, $\mathcal{R}_\beta$ satisfies* (3.14).

*Proof.* The first two statements can be verified on the generators. The last statement follows from $\sigma(\mathcal{R}_0) = \mathcal{R}_0 = (\omega \otimes \omega)(\mathcal{R}_0)$ and $\sigma(\widetilde{\mathcal{R}}_\beta) = (\omega \otimes \omega)(\widetilde{\mathcal{R}}_\beta)$. □

Now we are ready to state and prove the quasitriangularity property of $(A, \mathcal{R})$:

**Proposition 3.6.** *The Hopf algebra $(A, \mathcal{R}_\beta)$ is quasitriangular (for any $\beta \in k$).*

*Proof.* This is essentially a computation, but it is instructive to highlight some salient points. For the axiom (3.2), it suffices to prove

$$\widetilde{\mathcal{R}}_\beta \Delta(a) = \Delta(a)\widetilde{\mathcal{R}}_\beta, \qquad \mathcal{R}_0\Delta(a) = \Delta^{\mathrm{op}}(a)\mathcal{R}_0 \qquad \text{for all } a \in A. \tag{3.36}$$

(This is the main reason for insisting on the factorization $\mathcal{R}_\beta = \mathcal{R}_0\widetilde{\mathcal{R}}_\beta$.) In turn, it suffices to verify these statements for $a \in \{f, g\}$, which is a straightforward consequence of (3.35).

By Lemma 3.3 it now suffices to prove the axiom (3.3). A direct computation shows that

$$(\Delta \otimes \mathrm{id})(\mathcal{R}_0) = (\mathcal{R}_0)_{13}(\mathcal{R}_0)_{23}, \tag{3.37}$$

so that it remains to prove that

$$(\Delta \otimes \mathrm{id})(\widetilde{\mathcal{R}}_\beta) = (\mathcal{R}_0)_{23}(\widetilde{\mathcal{R}}_\beta)_{13}(\mathcal{R}_0)_{23}(\widetilde{\mathcal{R}}_\beta)_{23}. \tag{3.38}$$

It suffices to prove this on the level of the coefficients of the powers of $\beta$, i.e.

$$(\Delta \otimes \mathrm{id})(1 \otimes 1) = (\mathcal{R}_0)_{23}(\mathcal{R}_0)_{23} \tag{3.39}$$

$$(\Delta \otimes \mathrm{id})(f \otimes gf) = (\mathcal{R}_0)_{23}(f \otimes 1 \otimes gf)(\mathcal{R}_0)_{23} + 1 \otimes f \otimes gf \tag{3.40}$$

$$0 = (\mathcal{R}_0)_{23}(f \otimes 1 \otimes gf)(\mathcal{R}_0)_{23}(1 \otimes f \otimes gf). \tag{3.41}$$

Note that the first equation is trivial. The second equation follows by combining (3.35) with the coproduct formulas for $f$ and $g$. Finally, the third equation follows by combining (3.35) with $f^2 = 0$. □

Consider the two-dimensional $A$-modules $V^\pm$ defined by:

$$\pi^\pm(f) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad \pi^\pm(g) = \begin{pmatrix} \pm 1 & 0 \\ 0 & \mp 1 \end{pmatrix} \tag{3.42}$$

with respect to a given ordered basis $(v_1^\pm, v_2^\pm)$. Let us focus for now on $\pi^+$. With respect to the ordered basis $(v_1^+ \otimes v_1^+, v_1^+ \otimes v_2^+, v_2^+ \otimes v_1^+, v_2^+ \otimes v_2^+)$ of $V^+ \otimes V^+$, we have

$$(\pi^+ \otimes \pi^+)(\mathcal{R}_0) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \qquad (\pi^+ \otimes \pi^+)(\widetilde{\mathcal{R}}_\beta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta & 0 & 0 & 1 \end{pmatrix} \tag{3.43}$$

and hence

$$R_\beta := (\pi^+ \otimes \pi^+)(\mathcal{R}_\beta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \beta & 0 & 0 & -1 \end{pmatrix}. \tag{3.44}$$

It follows that $R_\beta \in \mathrm{End}(V^+ \otimes V^+)$ satisfies the Yang-Baxter equation (3.25); we have obtained a natural way of constructing nontrivial solutions of (3.25) in tensor products of $A$-modules. Unfortunately, the representation theory of $A$ is not very rich:

**Proposition 3.7.** *If $k$ is algebraically closed, $A$ has exactly four isomorphism classes of indecomposable modules. More precisely, up to isomorphism there are two one-dimensional modules, given by $\pm\epsilon$ and the two two-dimensional modules defined by* (3.42).

*Proof.* This follows from the fact that we may assume that $g$ acts as a diagonalizable map on the module, which must therefore split up as a direct sum of $\pm 1$-eigenspaces. For more details, see [CP95, 4.2F(g)]. □

On the other hand, semisimple Lie algebras $\mathfrak{g}$ have a very interesting category of modules; moreover their enveloping algebras $U\mathfrak{g}$ are naturally co-commutative (and hence quasitriangular) Hopf algebras. We will now discuss their quantizations $U_q\mathfrak{g}$, which inherit the richness of representations and are quasitriangular Hopf algebras in their own right.

The latter statement is only true "up to completion": the R-matrix will not be an element of $U_q\mathfrak{g} \otimes U_q\mathfrak{g}$ but lies in a larger algebra. In practice this means that one has to restrict to particular types of representations; we will see that the category $\mathcal{O}$ of $\mathfrak{g}$-modules has a direct "quantum" counterpart and the R-matrix a well-defined action in tensor products of such modules.

## 4. Universal enveloping algebras

We review some basic properties of universal enveloping algebras. In the remainder of these notes we assume that $k$ is of characteristic zero and algebraically closed.

4.1. **Tensor algebras.** Recall that any algebra is also a Lie algebra by defining the Lie bracket of two elements to be their commutator. Roughly speaking, the universal enveloping algebra (UEA) of a Lie algebra $\mathfrak{g}$ is the "most general algebra that contains all representations of $\mathfrak{g}$". To make this more precise, let $\mathfrak{g}$ be, for now, any linear space over $k$. We may form its *tensor algebra*

$$T\mathfrak{g} = k \oplus \mathfrak{g} \oplus (\mathfrak{g} \otimes \mathfrak{g}) \oplus \dots . \tag{4.1}$$

It is a free algebra whose multiplication is given by the tensor product and among its elements we find all possible tensor products of all possible elements of $\mathfrak{g}$. There is a canonical embedding of linear spaces $\mathfrak{g} \hookrightarrow T\mathfrak{g}$.

4.2. **Universal enveloping algebras.** Now assume that $\mathfrak{g}$ is in fact a Lie algebra (not necessarily semisimple and finite-dimensional, for now); in order to study representations of $\mathfrak{g}$ using an associative algebra, we would like to have a setup where the linear embedding $\mathfrak{g} \hookrightarrow T\mathfrak{g}$ is replaced by an embedding of Lie algebras. To do this, consider the two-sided ideal $I \subset T\mathfrak{g}$ generated by all elements of the form

$$x \otimes y - y \otimes x - [x, y] \in \mathfrak{g} \oplus (\mathfrak{g} \otimes \mathfrak{g}) \subset T\mathfrak{g}, \qquad x, y \in \mathfrak{g}. \tag{4.2}$$

Now we define the *universal enveloping algebra* of $\mathfrak{g}$ to be $U\mathfrak{g} = T\mathfrak{g}/I$. In other words, the UEA of $\mathfrak{g}$ is obtained if we match the Lie algebra structure coming from $\mathfrak{g}$ with the canonical Lie algebra structure that any tensor algebra $T\mathfrak{g}$ has (the one defined in terms of commutators). Note that $U\mathfrak{g}$ is automatically an algebra.

4.3. **The universal property.** Consider the canonical Lie algebra embedding $\upsilon : \mathfrak{g} \hookrightarrow U\mathfrak{g}$. Let $A$ be an arbitrary algebra. Automatically, any (algebra) map $\rho : U\mathfrak{g} \to A$ induces a (Lie algebra) map $\rho \circ \upsilon : \mathfrak{g} \to A$. But we have a rather strong converse to this statement. Namely, let $\phi : \mathfrak{g} \to A$ be a Lie algebra map (where the Lie bracket of $A$ is given by the commutator). Then there exists a *unique* algebra homomorphism $\widehat{\phi} : U\mathfrak{g} \to A$ such that $\phi = \widehat{\phi} \circ \upsilon$.

Let $V$ be a vector space and consider the algebra $\mathrm{End}(V)$ consisting of endomorphisms of $V$ (with composition providing the multiplication). Recall that a *(Lie algebra) representation* of $\mathfrak{g}$ on $V$ is simply a linear map $\phi : \mathfrak{g} \to \mathrm{End}(V)$ such that

$$\phi([x, y]) = \phi(x) \circ \phi(y) - \phi(y) \circ \phi(x), \qquad \text{for all } x, y \in \mathfrak{g}. \tag{4.3}$$

Automatically, the associated map $\widehat{\phi} : U\mathfrak{g} \to \mathrm{End}(V)$ is an (algebra) representation of $U\mathfrak{g}$ on $V$ and equivalently, we may view $V$ as a $U\mathfrak{g}$-module. In categorical terms, the category of representations of $\mathfrak{g}$ and the category of left $U\mathfrak{g}$-modules are isomorphic.

4.4. **Hopf algebra structure.** A co-commutative Hopf algebra structure can be defined on $U\mathfrak{g}$ (and indeed on $T\mathfrak{g}$) by the assignments

$$\Delta(x) = x \otimes 1 + 1 \otimes x, \qquad \epsilon(x) = 0, \qquad S(x) = -x \qquad \text{for all } x \in \mathfrak{g} \tag{4.4}$$

and extending multiplicatively; since these assignments are compatible with commutators, indeed we find that $\Delta$ and $\epsilon$ are algebra homomorphisms and $S$ an algebra anti-homomorphism. In other words, elements of $\mathfrak{g}$ (viewed via $\upsilon$ as elements of $U\mathfrak{g}$) are primitive elements with respect to the Hopf algebra structure.

4.5. **A key example: $\mathfrak{sl}_2$.** Semisimple Lie algebras have a very rich representation theory; let us study the basic case of $\mathfrak{sl}_2$ in more detail. The Lie algebra $\mathfrak{sl}_2$ of traceless $2 \times 2$-matrices over $k$ has a basis given by

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{4.5}$$

For Lie algebras consisting of square matrices, the standard Lie bracket is given by the commutator, and we see that in this case we have only the following Lie bracket relations between the basis elements

$$[h, e] = 2e, \qquad [h, f] = -2f, \qquad [e, f] = h. \tag{4.6}$$

It follows immediately that $U\mathfrak{sl}_2$ is given by the free algebra over the symbols $E, F, H$ modulo the relations

$$HE - EH = 2E, \qquad HF - HF = -2F, \qquad EF - FE = H, \tag{4.7}$$

with the canonical embedding $\upsilon : \mathfrak{sl}_2 \to U\mathfrak{sl}_2$ given by $e \mapsto E$, $f \mapsto F$ and $h \mapsto H$. In the quantum deformed version we will "keep" the basis elements $E$ and $F$ and "replace" $H$ by a well-chosen linear combination of a group-like element and its inverse.

## 5. Drinfeld-Jimbo quantum groups

First we deal with the $\mathfrak{sl}_2$ case. Let $q$ be an indeterminate[2] and consider the algebra $U_q\mathfrak{sl}_2$ generated over $k(q)$ by symbols $E$, $F$, $t$ and $t^{-1}$ subject to the relations

$$tE = q^2 Et, \qquad tF = q^{-2}Ft, \qquad [E,F] = \frac{t-t^{-1}}{q-q^{-1}}, \qquad tt^{-1} = t^{-1}t = 1. \tag{5.1}$$

The additional Hopf algebra structure maps given by

$$\begin{aligned} \Delta(E) &= E\otimes 1 + t\otimes E, & \epsilon(E) &= 0, & S(E) &= -t^{-1}E \\ \Delta(F) &= F\otimes t^{-1} + 1\otimes F, & \epsilon(F) &= 0, & S(F) &= -Ft \\ \Delta(t^{\pm 1}) &= t^{\pm 1}\otimes t^{\pm 1}, & \epsilon(t^{\pm 1}) &= 1, & S(t^{\pm 1}) &= t^{\mp 1}. \end{aligned} \tag{5.2}$$

5.1. **The topological quantum group $U_{[[h]]}\mathfrak{sl}_2$.** Morally, sending $q\to 1$ should recover the defining relations and Hopf algebra structure of $U\mathfrak{sl}_2$.

*Remark* 5.1. By making the formal substitution

$$t = q^H \tag{5.3}$$

this can indeed be done. For instance, in the right-hand side of the relation $[E,F] = \frac{t-t^{-1}}{q-q^{-1}}$ one may take the formal limit $q\to 1$ and immediately obtain $H$, as required. On the other hand, applying the $q\to 1$ limit directly to $tE = q^2Et$ only yields a tautology; however we may write $t = q^H$ and $q^2t = q^{H+2}$ as formal power series in $\log(q)$, in which case $tE = q^2Et$ is equivalent to

$$\sum_{r\geq 0}\frac{1}{r!}\log(q)^r H^r E = \sum_{r\geq 0}\frac{1}{r!}\log(q)^r E(H+2)^r. \tag{5.4}$$

Since $q$ is an indeterminate, this should be true on the level of the coefficients, yielding the $U\mathfrak{sl}_2$-relations $H^rE = E(H+2)^r$. This suggests a connection between $U_q\mathfrak{sl}_2$ and $U\mathfrak{sl}_2[[\log(q)]]$. ∅

To make this rigorous, choose a new indeterminate $h = \log(q)$. The $h$-adic topology on a vector space $V$ over $k[[h]]$ is defined by stipulating that

(1) $\{h^nV \mid n\in\mathbb{Z}_{\geq 0}\}$ is a base of the neighbourhoods of $0$ in $V$,
(2) translations in $V$ are continuous.

It follows then that $k[[h]]$-linear maps are continuous. A *topological Hopf algebra* over $k[[h]]$ is an $h$-adic complete $k[[h]]$-module $A$ equipped with $k[[h]]$-linear structure maps $\eta$, $m$, $\epsilon$, $\Delta$ and $S$ satisfying the Hopf algebra axioms discussed in Section 2, but with algebraic tensor products replaced by $h$-adic completions. We can then consider the topological Hopf algebra $U_{[[h]]}\mathfrak{sl}_2$, defined as follows. Namely, consider the free algebra $\mathcal{P} := k\langle E,F,H\rangle$ and consider the algebra of power series $\mathcal{P}[[h]]$. Consider the two-sided ideal $I$ of $\mathcal{P}[[h]]$ generated by

$$[H,E]-2E, \qquad [H,F]+2F, \qquad [E,F] - \frac{\mathrm{e}^{hH}-\mathrm{e}^{-hH}}{\mathrm{e}^{h}-\mathrm{e}^{-h}} \tag{5.5}$$

and let $I^{\mathrm{cl}}$ be its closure in the $h$-adic topology. Then we can define $U_{[[h]]}\mathfrak{sl}_2 := \mathcal{P}[[h]]/I^{\mathrm{cl}}$. One then can deduce that $U_{[[h]]}\mathfrak{sl}_2 \cong (U\mathfrak{sl}_2)[[h]]$ as algebras over $k[[h]]$, see [CP95, Cor. 6.5.4].

5.2. **Some representations of $U_q\mathfrak{sl}_2$.** It is easy to explicitly construct finite-dimensional representations of $U_q\mathfrak{sl}_2$. We denote, for $m\in\mathbb{Z}$,

$$[m]_q = \frac{q^m-q^{-m}}{q-q^{-1}} \in k(q). \tag{5.6}$$

Consider, for $d\in\mathbb{Z}_{>0}$, the $n$-dimensional vector space

$$V^{(n)} = kv_1^{(n)}\oplus\cdots\oplus kv_n^{(n)} \tag{5.7}$$

[2]Since we are working in characteristic 0, it is equally valid to let $q$ be a scalar, provided we choose it "generically", i.e. avoiding special values. Typical values to avoid are 0 and roots of unity. The study of (suitably defined) quantum groups for these special values is very interesting but outside the scope of this introductory course. For the basic definition one merely needs to avoid $0$, $1$ and $-1$.

and the representation $\pi^{(n)} : U_q(\mathfrak{sl}_2) \to \mathrm{End}(V^{(n)})$ defined by

$$\begin{aligned} \pi^{(n)}(E)(v_i^{(n)}) &= [n-i-1]_q v_{i-1}^{(n)}, \\ \pi^{(n)}(F)(v_i^{(n)}) &= [i]_q v_{i+1}^{(n)}, \\ \pi^{(n)}(t^{\pm 1})(v_i^{(n)}) &= q^{\pm(n-2i+1)} v_i^{(n)} \end{aligned} \tag{5.8}$$

for $i \in \{1, 2, \dots, n\}$, where we have set $v_i^{(n)} = 0$ if $i < 1$ or $i > n$. Note that $\pi^{(0)}$ is the trivial representation of $U_q\mathfrak{sl}_2$ on $k$ determined by the counit.

5.3. **The dual quantum group $F_q(\mathrm{SL}(2))$.** It is useful to mention here also the dual object $F_q(\mathrm{SL}(2))$, the quantized algebra of scalar-valued functions on $\mathrm{SL}(2)$. It is generated over $k(q)$ by elements $a, b, c, d$ subject to

$$ab = qba, \qquad bd = qdb, \qquad ac = qca, \qquad cd = qdc, \qquad bc = cb, \qquad ad - qbc = 1 = da - q^{-1}cb. \tag{5.9}$$

Note that as $q \to 1$ we recover the commutative algebra of functions on $\mathrm{SL}(2)$, where $a$ corresponds to the function returning the $(1,1)$-entry, $b$ to the function returning the $(1,2)$-entry, etc.

The Hopf algebra structure on $F_q(\mathrm{SL}(2))$ is most easily encoded by forming the matrix

$$T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{5.10}$$

and setting

$$\Delta(T_{ij}) = \sum_k T_{ik} \otimes T_{kj}, \qquad \epsilon(T_{ij}) = \delta_{ij}, \qquad S(T_{ij}) = (T^{-1})_{ij}. \tag{5.11}$$

More explicitly, this gives

$$\begin{aligned} \Delta(a) &= a \otimes a + b \otimes c, & \epsilon(a) &= 1, & S(a) &= d \\ \Delta(b) &= a \otimes b + b \otimes d, & \epsilon(b) &= 0, & S(b) &= -q^{-1}b \\ \Delta(c) &= c \otimes a + d \otimes c, & \epsilon(c) &= 0, & S(c) &= -qc \\ \Delta(d) &= c \otimes b + d \otimes d, & \epsilon(d) &= 1, & S(d) &= a. \end{aligned} \tag{5.12}$$

Again, in the limit $q \to 1$ we recover the standard Hopf algebra structure on $k$-valued functions on $\mathrm{SL}(2)$, given by (2.11).

5.4. **Drinfeld-Jimbo quantum groups.** Likewise we are interested in constructing quantum groups $U_q\mathfrak{g}$ for arbitrary finite-dimensional semisimple Lie algebras $\mathfrak{g}$. This is most conveniently done using the Chevalley-Serre presentation of $\mathfrak{g}$ in terms of its Cartan matrix. More precisely, let $A = (a_{ij})_{i,j \in I}$ be an arbitrary Cartan matrix, i.e. $a_{ii} = 2$, $a_{ij} \in \mathbb{Z}_{\geq 0}$, $a_{ij} = 0$ if and only if $a_{ji} = 0$ and finally all principal minors of $A$ are positive (we briefly discuss the Kac-Moody generalization in section 5.5). There exist positive setwise-coprime integers $d_i$ such that $d_i a_{ij} = d_j a_{ji}$ for all $i, j \in I$. Then each semisimple finite-dimensional Lie algebra arises as follows. Consider the Lie algebra $\mathfrak{g} = \mathfrak{g}(A)$ generated by the subalgebras

$$\mathfrak{sl}_{2,i} := \langle e_i, f_i, h_i \rangle \tag{5.13}$$

for all $i \in I$, subject to the $\mathfrak{sl}_2$-relations

$$[h_i, e_i] = 2e_i, \qquad [h_i, f_i] = -2f_i, \qquad [e_i, f_i] = h_i \tag{5.14}$$

and, for $i \neq j$, the relations

$$[h_i, h_j] = 0, \qquad [h_i, e_j] = a_{ij}e_j, \qquad [h_i, f_j] = -a_{ij}f_j, \qquad [e_i, f_j] = 0, \tag{5.15}$$

$$[e_i, [e_i, \dots, [e_i, e_j] \cdots ]] = 0, \qquad [f_i, [f_i, \dots, [f_i, f_j] \cdots ]] = 0 \tag{5.16}$$

where there are $1 - a_{ij}$ nested Lie brackets in the last two relations (Serre relations). We have a triangular decomposition

$$\mathfrak{g} = \mathfrak{n}^+ \oplus \mathfrak{h} \oplus \mathfrak{n}^- \qquad \text{as } \mathfrak{h}\text{-modules} \tag{5.17}$$

where we have introduced the subalgebras

$$\mathfrak{n}^+ = \langle e_i \,|\, i \in I \rangle, \qquad \mathfrak{h} = \langle h_i \,|\, i \in I \rangle \qquad \mathfrak{n}^- = \langle f_i \,|\, i \in I \rangle. \tag{5.18}$$

Denote the root system of $\mathfrak{g}$ (with respect to $\mathfrak{h}$) by $\Phi$ and the positive subsystem by $\Phi^+$. Moreover, we denote the simple roots in $\Phi^+$ by $\alpha_i$ ($i \in I$). Note that we have a symmetric bilinear form $(\, , \,)$ on $\mathfrak{h}^*$ satisfying $(\alpha_i, \alpha_j) = d_i a_{ij} = d_i \alpha_j(h_i)$ for all $i, j \in I$. Consider the weight lattice

$$P = \{\lambda \in \mathfrak{h}^* \,|\, \lambda(h_i) \in \mathbb{Z} \text{ for all } i \in I\}. \tag{5.19}$$

Note that $(\, , \,)$ restricts to a map

$$(\, , \,) : P \times P \to \tfrac{1}{d}\mathbb{Z} \tag{5.20}$$

for some positive integer $d$.

We are interested in representations which have a weight-decomposition with respect to $\mathfrak{h}$. That is, for $V \in \mathrm{Rep}(\mathfrak{g})$ and $\lambda \in \mathfrak{h}^*$ denote

$$V_\lambda = \{v \in V \,|\, h \cdot v = \lambda(h)v \text{ for all } h \in \mathfrak{h}\}. \tag{5.21}$$

The category $\mathcal{O}$ is the full subcategory of $\mathrm{Rep}(\mathfrak{g}) \cong \mathrm{Rep}(U\mathfrak{g})$ whose objects are finitely-generated modules $V$ such that

$$V = \bigoplus_{\lambda \in \mathfrak{h}^*} V_\lambda \tag{5.22}$$

and such that $U\mathfrak{n}^+$ acts locally finitely, i.e. for all $v \in V$ the $U\mathfrak{n}^+$-module generated by $v$ is finite-dimensional.

The subcategory $\mathcal{O}_{\rm int}$ is obtained by additionally assuming that for each $i \in I$ the subalgebra $U\mathfrak{sl}_{2,i} = v(\mathfrak{sl}_{2,i})$ acts locally finitely; by the triangular decomposition for this subalgebra, this is equivalent to $E_i = v(e_i)$ and $F_i = v(f_i)$ acting locally nilpotently: for all $v \in V$ there exists $m \in \mathbb{Z}_{\geq 0}$ such that $E_i^m \cdot v = F_i^m \cdot v = 0$. Then $\mathcal{O}_{\rm int}$ is a tensor category and a semisimple category, with the simple objects given by irreducible highest-weight representations (more precisely, the associated highest weight $\lambda$ is dominant and integral, i.e. $\lambda(h_i) \geq 0$ for all $i \in I$ and $\lambda \in P$). In fact, $\mathcal{O}_{\rm int}$ corresponds to the category of finite-dimensional $\mathfrak{g}$-representations, with, after a suitable choice of basis, each $e_i$ acting as a strict upper triangular matrix and $f_i$ as its transpose.

The definition of the corresponding quantum group is due to independent work by Drinfeld [Dr85] and Jimbo [Ji86] and goes as follows. Let $q$ be an indeterminate and set $q_i := q^{d_i}$. The *Drinfeld-Jimbo quantum group* is the algebra $U_q\mathfrak{g}$ generated over $k(q)$ by subalgebras

$$U_q\mathfrak{sl}_{2,i} := \langle E_i, F_i, t_i, t_i^{-1} \rangle \tag{5.23}$$

for $i \in I$, subject to the $U_q\mathfrak{sl}_2$ relations

$$t_i E_i = q_i^2 E_i t_i, \qquad t_i F_i = q_i^{-2} F_i t_i, \qquad [E_i, F_i] = \frac{t_i - t_i^{-1}}{q_i - q_i^{-1}}, \qquad t_i t_i^{-1} = t_i^{-1} t_i = 1 \tag{5.24}$$

(morally, we may think of $t_i$ as $q_i^{H_i}$), and, for $i \neq j$, the relations

$$[t_i, t_j] = 0, \qquad t_i E_j = q_i^{a_{ij}} E_j t_i, \qquad t_i F_j = q_i^{-a_{ij}} F_j t_i, \qquad [E_i, F_j] = 0, \tag{5.25}$$

$$[E_i, [E_i, \ldots, [E_i, E_j]_{q_i^{a_{ij}}} \cdots]_{q_i^{-a_{ij}-2}}]_{q_i^{-a_{ij}}} = 0, \qquad [F_i, [F_i, \ldots, [F_i, F_j]_{q_i^{-a_{ij}}} \cdots]_{q_i^{a_{ij}+2}}]_{q_i^{a_{ij}}}, \tag{5.26}$$

where we have used the notation $[x, y]_p := xy - pyx$ for the deformed commutator. To be precise, this is the so-called *adjoint form*, where the Cartan subalgebra $U_q\mathfrak{h} = \langle t_i, t_i^{-1} \,|\, i \in I \rangle$ is defined in terms of the root lattice $Q := \mathrm{Sp}_{\mathbb{Z}}\Phi \subset P$. The *simply connected form* is obtained by working with the weight lattice $P$ instead, yielding a larger Cartan subalgebra generated by $t_\lambda$ for $\lambda \in P$.

*Remark* 5.2. In the case $\mathfrak{g} = \mathfrak{sl}_2$, $P = \frac{1}{2}Q$ and the simply connected and adjoint forms are the only relevant forms of $U_q\mathfrak{g}$, with the simply connected form obtained from the adjoint form by adjoining square roots of the generators $t_i$ and $t_i^{-1}$. $\varnothing$

By a straightforward check on generators, one has the following result.

**Proposition 5.3.** *$U_q\mathfrak{g}$ is a (non-cocommutative) Hopf algebra with the additional structure maps given by*

$$\begin{aligned}
\Delta(E_i) &= E_i \otimes 1 + t_i \otimes E_i, & \epsilon(E_i) &= 0, & S(E_i) &= -t_i^{-1} E_i \\
\Delta(F_i) &= F_i \otimes t_i^{-1} + 1 \otimes F_i, & \epsilon(F_i) &= 0, & S(F_i) &= -F_i t_i \\
\Delta(t_i^{\pm}) &= t_i^{\pm} \otimes t_i^{\pm}, & \epsilon(t_i^{\pm}) &= 1, & S(t_i^{\pm}) &= t_i^{\mp}.
\end{aligned}$$

For later convenience it is useful to record the explicit formulas for $S^{-1}$:

$$S^{-1}(E_i) = -E_i t_i^{-1}, \qquad S^{-1}(F_i) = -t_i F_i, \qquad S^{-1}(t_i^{\pm}) = t_i^{\mp}. \tag{5.27}$$

The triangular decomposition induced on $U\mathfrak{g}$ by the multiplication map, namely $U\mathfrak{g} \cong U\mathfrak{n}^+ \otimes U\mathfrak{h} \otimes U\mathfrak{n}^-$ lifts directly to

$$U_q\mathfrak{g} \cong U_q\mathfrak{n}^+ \otimes U_q\mathfrak{h} \otimes U_q\mathfrak{n}^- \tag{5.28}$$

where we have introduced the subalgebras

$$U_q\mathfrak{n}^+ = \langle E_i \,|\, i \in I \rangle, \qquad U_q\mathfrak{n}^- = \langle F_i \,|\, i \in I \rangle. \tag{5.29}$$

For $V \in \mathrm{Rep}(U_q\mathfrak{g})$ and $\lambda \in P$ denote the (quantum) weight space

$$V_\lambda = \{ v \in V \,|\, t_i \cdot v = q_i^{\lambda(h_i)} v = q^{(\alpha_i,\lambda)} v \text{ for all } i \in I \}. \tag{5.30}$$

In particular, we have the root space decompositions

$$U_q\mathfrak{g} = \bigoplus_{\lambda \in Q} (U_q\mathfrak{g})_\lambda, \qquad U_q\mathfrak{n}^\pm = \bigoplus_{\lambda \in \pm Q^+} (U_q\mathfrak{n}^\pm)_{\pm\lambda} \tag{5.31}$$

where $Q^+ = \mathrm{Sp}_{\mathbb{Z}_{\geq 0}} \Phi^+$. Note that $(U_q\mathfrak{n}^+)_{\alpha_i} = k(q)E_i$ and $(U_q\mathfrak{n}^-)_{-\alpha_i} = k(q)F_i$.

The category $\mathcal{O}_q$ is defined as the full subcategory of $\mathrm{Rep}(U_q\mathfrak{g})$ whose objects are finitely-generated modules $V$ such that

$$V = \bigoplus_{\lambda \in \mathfrak{h}^*} V_\lambda \tag{5.32}$$

and such that $U_q\mathfrak{n}^+$ acts locally finitely, i.e. for all $v \in V$ the $U_q\mathfrak{n}^+$-module generated by $v$ is finite-dimensional. Note that the $E_i$-action and $F_i$-action on $V \in \mathcal{O}_q$ satisfy

$$E_i(V_\lambda) \subseteq V_{\lambda+\alpha_i}, \qquad F_i(V_\lambda) \subseteq V_{\lambda-\alpha_i}. \tag{5.33}$$

The subcategory $\mathcal{O}_{q,\mathrm{int}}$ is obtained by additionally assuming that each subalgebra $U_q\mathfrak{sl}_{2,i}$ acts locally finitely. Then $\mathcal{O}_{q,\mathrm{int}}$ is the category of finite-dimensional representations such that each $t_i$ acts diagonalizably with integer powers of $q_i$ as eigenvalues (so-called type-**1** representations). As in the $(q \to 1)$-limit, $\mathcal{O}_{q,\mathrm{int}}$ is a tensor category and a semisimple category, whose simple objects are irreducible highest-weight representations with dominant integral highest weight, see e.g. [Lu94, Cor. 6.2.3] or [CP95, Sec. 10.1].

*Remark* 5.4. In the case $\mathfrak{g} = \mathfrak{sl}_2$, the weight lattice is $P = \frac{1}{2}Q = \frac{\mathbb{Z}}{2}\alpha$, where $\alpha$ is the unique simple root and the representation $\pi^{(n)}$ defined in (5.8) is an irreducible highest weight representation with highest weight vector $v_1^{(n)}$ and highest weight $\frac{n-1}{2}\alpha$. ⌀

5.5. **Kac-Moody generalization.** The definition of the Drinfeld-Jimbo quantum group can be extended to the case where $A = (a_{ij})_{i,j\in I}$ is a symmetrizable generalized Cartan matrix. This means we require $a_{ii} = 2$, $a_{ij} \in \mathbb{Z}_{\geq 0}$, $a_{ij} = 0$ if and only if $a_{ji} = 0$ and the existence of a set of positive setwise-coprime integers $d_i$ such that $d_i a_{ij} = d_j a_{ji}$ for all $i, j \in I$. As in the classical $(q \to 1)$ case, the Cartan subalgebra is larger: $U_q\mathfrak{h}$ is defined in terms of a lattice which as a free abelian group has rank $|I| + \mathrm{cork}(A)$. The category $\mathcal{O}_q$ and the subcategory $\mathcal{O}_{q,\mathrm{int}}$ can be defined as above and are still tensor categories and semisimple categories, whose simple objects are irreducible highest-weight representations with dominant integral highest weight. However $\mathcal{O}_q$ contains no nontrivial finite-dimensional representations (note that the counit map defines a trivial $U_q\mathfrak{g}$-module in $\mathcal{O}_{\mathrm{int},q}$ which is finite-dimensional).

*Remark* 5.5. We say that $A$ is of *affine type* if $\det(A) = 0$ and each proper principal minor of $A$ is positive. If $A$ is of affine type, then $\mathfrak{g}' := \langle e_i, f_i, h_i \,|\, i \in I \rangle$ (but not $\mathfrak{g}$) and similarly $U_q\mathfrak{g}' := \langle E_i, F_i, t_i^{\pm 1} \,|\, i \in I \rangle$ (but not $U_q\mathfrak{g}$) have finite-dimensional representations called *evaluation modules* which arise from the identification of $\mathfrak{g}'$ as a central extension of a loop algebra $\mathfrak{g}[z, z^{-1}]$ of a finite-dimensional Lie algebra. The affine versions of quantum groups are the most relevant in quantum integrability. ⌀

## 6. Quasitriangularity for $U_q\mathfrak{g}$

We will construct the universal R-matrix for $U_q\mathfrak{sl}_2$, roughly following the approach in [Ta92, Sec. 4.3]. From now on we work over the larger field $k(q^{1/d})$, since we want to allow for linear maps acting on objects in $\mathcal{O}_{q,\mathrm{int}} \otimes \mathcal{O}_{q,\mathrm{int}}$ which act as multiplication by such scalars in particular weight spaces (in particular, consider (5.20)). Let us set the stage.

6.1. **The bar involution.** The *bar involution* is an involutive algebra automorphism from $U_q\mathfrak{g}$ to itself denoted by $\overline{\cdot}$ which acts nontrivially on the base field $k(q^{1/d})$: it sends $q^{1/d}$ to $q^{-1/d}$. On the generators it is defined as follows:

$$\overline{E_i} = E_i, \qquad \overline{F_i} = F_i, \qquad \overline{t_i^{\pm 1}} = t_i^{\mp 1}. \tag{6.1}$$

It is straightforward to check that these assignments preserve the defining relations of $U_q\mathfrak{g}$, as required. Note that as $q \to 1$ this bar involution becomes invisible: $\lim_{q\to 1} x = \lim_{q \to 1} \overline{x}$ (whenever these limits exist).

Recall the notion of twisting the Hopf algebra structure by an algebra automorphism, see (2.3). We will approach the construction of the universal R-matrix by considering, in addition to the coproducts $\Delta$ and $\Delta^{\rm op}$, a third coproduct $\overline{\Delta} := \Delta_{\overline{\cdot}} = (\overline{\cdot} \otimes \overline{\cdot}) \circ \Delta \circ \overline{\cdot}$. Explicitly, we have

$$\overline{\Delta}(E_i) = E_i \otimes 1 + t_i^{-1} \otimes E_i, \qquad \overline{\Delta}(F_i) = F_i \otimes t_i + 1 \otimes F_i, \qquad \overline{\Delta}(t_i^{\pm 1}) = t_i^{\pm 1} \otimes t_i^{\pm 1}. \tag{6.2}$$

Why do we care about the third coproduct? We want to show that $U_q\mathfrak{g}$ is quasitriangular; this entails that we construct an invertible element $\mathcal{R}$ which intertwines $\Delta$ with $\Delta^{\rm op}$ as follows:

$$\mathcal{R}\Delta(u) = \Delta^{\rm op}(u)\mathcal{R} \qquad \text{for all } u \in U_q\mathfrak{g}, \tag{6.3}$$

It turns out that $\overline{\Delta}$ is convenient in an intermediate stage of the proof of this. Namely, we will establish (6.3) by constructing two elements $\widetilde{\mathcal{R}}$ and $\kappa$ which intertwine $\Delta$ with $\overline{\Delta}$, and $\overline{\Delta}$ with $\Delta^{\rm op}$, respectively:

$$\widetilde{\mathcal{R}}\Delta(x) = \overline{\Delta}(x)\widetilde{\mathcal{R}}, \qquad \kappa\overline{\Delta}(x) = \Delta^{\rm op}(x)\kappa \qquad \text{for all } x \in U_q\mathfrak{g}. \tag{6.4}$$

From these two equations (6.3) readily follows if we set $\mathcal{R} = \kappa\widetilde{\mathcal{R}}$. Compare this with the situation for Sweedler's Hopf algebra, in particular the proof of Proposition 3.6, where the analogue of the bar involution is the identity automorphism.

6.2. **The Chevalley involution.** The Chevalley involution $\omega$ is the second automorphism of $U_q\mathfrak{g}$ that plays a role in establishing quasitriangularity. It corresponds to the matrix Lie algebra automorphism $x \mapsto -x^{\rm t}$ in representations:

$$\omega(E_i) = -F_i, \qquad \omega(F_i) = -E_i, \qquad \omega(t_i^{\pm 1}) = t_i^{\mp 1}. \tag{6.5}$$

Again, one checks directly that the defining relations of $U_q\mathfrak{g}$ are preserved. Considering Lemma 3.3, the following is relevant for our purpose of constructing a universal R-matrix for $U_q\mathfrak{g}$.

**Lemma 6.1.** *The algebra automorphism $\omega$ maps the Hopf algebra $U_q\mathfrak{g}$ to the Hopf algebra $U_q\mathfrak{g}^{\rm cop}$; in particular it is a coalgebra antiautomorphism.*

*Proof.* We need to show that

$$(\omega \otimes \omega) \circ \Delta = \Delta^{\rm op} \circ \omega, \qquad \epsilon = \epsilon \circ \omega, \qquad \omega \circ S = S^{-1} \circ \omega, \tag{6.6}$$

which is straightforwardly checked on generators. □

6.3. **The completions $\widehat{U}$ and $\widehat{U}^{(2)}$.** In order to construct the universal R-matrix for $U_q\mathfrak{g}$, we will consider an algebra containing $U_q\mathfrak{g} \otimes U_q\mathfrak{g}$. Since $\mathcal{O}_{q,\rm int}$ is a subcategory of $\mathrm{Rep} U_q\mathfrak{g}$, we have a forgetful functor $\mathrm{For} : \mathcal{O}_{q,\rm int} \to \mathrm{Vect}$, which is a tensor functor. Consider now the algebra $\widehat{U}$ of natural transformations from For to itself; more precisely elements of $\widehat{U}$ are tuples $(\phi_V)$, where $V$ ranges through $\mathcal{O}_{q,\rm int}$, consisting of linear maps $\phi_V : \mathrm{For}(V) \to \mathrm{For}(V)$ such that the following diagram commutes:

$$\begin{tikzcd} \mathrm{For}(V) \arrow[r, "\phi_V"] \arrow[d, "\mathrm{For}(f)"'] & \mathrm{For}(V) \arrow[d, "\mathrm{For}(f)"] \\ \mathrm{For}(V) \arrow[r, "\phi_W"'] & \mathrm{For}(V) \end{tikzcd} \tag{6.7}$$

for all $V, W \in \mathcal{O}_{q,\rm int}$ and for all $U_q\mathfrak{g}$-intertwiners $f : V \to W$. Note that $\widehat{U}$ naturally has the structure of a vector space over $k(q^{1/d})$; moreover composition induces a multiplication, which makes $\widehat{U}$ into an algebra. Furthermore, the $U_q\mathfrak{g}$-action on objects of $\mathcal{O}_{q,\rm int}$ produces an algebra homomorphism $U_q\mathfrak{g} \to \widehat{U}$, which is injective [Lu94, Prop. 3.5.4] and [Jan96, 5.11]. We will consider $U_q\mathfrak{g}$ as a subalgebra of $\widehat{U}$.

We also have a functor $\mathrm{For}^{(2)} : \mathcal{O}_{q,\rm int} \times \mathcal{O}_{q,\rm int} \to \mathrm{Vect}$, sending pairs of modules $(V, W)$ to $\mathrm{For}(V \otimes W)$ and pairs of intertwiners $(f, g)$ to $\mathrm{For}(f \otimes g)$. We define $\widehat{U}^{(2)} = \mathrm{End}(\mathrm{For}^{(2)})$, which is an algebra for the same reasons as $\widehat{U}$, and we may view $\widehat{U} \otimes \widehat{U} \subset \widehat{U}^{(2)}$ via $(\phi_V)_V \otimes (\psi_W)_W \mapsto (\phi_V \otimes \psi_W)_{(V,W)}$. Any $\phi \in \widehat{U}$ can be restricted to $\mathrm{For}(V \otimes W)$ for all $V, W \in \mathcal{O}_{q,\mathrm{int}}$; since restriction is compatible with composition and linearity of natural transformations, we obtain an algebra map

$$\Delta : \widehat{U} \to \widehat{U}^{(2)}, \qquad (\phi_M) \mapsto (\phi_{M \otimes N}) \tag{6.8}$$

which we call the coproduct of $\widehat{U}$. It restricts to the usual coproduct of the embedded subalgebra $U_q\mathfrak{g} \subset \widehat{U}$, motivating the notation.

Analogously we can define a completion $\widehat{U}^{(n)}$ for any $n \in \mathbb{Z}_{\geq 0}$ with natural algebra embeddings $\widehat{U}^{(m)} \to \widehat{U}^{(n)}$ whenever $m < n$. We now consider special elements of $\widehat{U}^{(2)}$ which can be used to define a quasitriangular structure on $U_q\mathfrak{g}$. The fact that these elements do not lie in $U_q\mathfrak{g} \otimes U_q\mathfrak{g}$ is the only obstacle for $U_q\mathfrak{g}$ being quasitriangular, so we may say that $U_q\mathfrak{g}$ is quasitriangular "up to completion".

6.4. **A Cartan-like element.** For $V, W \in \mathcal{O}_{q,\mathrm{int}}$ consider

$$\kappa_{V,W} \in \mathrm{End}(V \otimes W), \qquad v \otimes w \mapsto q^{(\mu,\nu)} v \otimes w \text{ for all } v \in V_\mu, w \in W_\nu. \tag{6.9}$$

Then the tuple $\kappa := (\kappa_{V,W})$ lies in $\widehat{U}^{(2)}$ (note that it does not lie in $\widehat{U} \otimes \widehat{U}$). We denote by $\mathrm{Ad}(\kappa)$ the algebra automorphism of $\widehat{U}^{(2)}$ given by conjugation by $\kappa$.

**Lemma 6.2.** *The map $\mathrm{Ad}(\kappa)$ preserves $U_q\mathfrak{g} \otimes U_q\mathfrak{g}$; more precisely*

$$\begin{aligned} \mathrm{Ad}(\kappa)(E_i \otimes 1) &= E_i \otimes t_i, & \mathrm{Ad}(\kappa)(1 \otimes E_i) &= t_i \otimes E_i, \\ \mathrm{Ad}(\kappa)(F_i \otimes 1) &= F_i \otimes t_i^{-1}, & \mathrm{Ad}(\kappa)(1 \otimes F_i) &= t_i^{-1} \otimes F_i, \\ \mathrm{Ad}(\kappa)(t_i^{\pm 1} \otimes 1) &= t_i^{\pm 1} \otimes 1, & \mathrm{Ad}(\kappa)(1 \otimes t_i^{\pm 1}) &= 1 \otimes t_i^{\pm 1}. \end{aligned} \tag{6.10}$$

*Proof.* Note that $U_q\mathfrak{g} \otimes U_q\mathfrak{g}$ is generated by $E_i \otimes 1$, $F_i \otimes 1$, $t_i^{\pm 1} \otimes 1$, $1 \otimes E_i$, $1 \otimes F_i$ and $1 \otimes t_i^{\pm 1}$. Let $V, W \in \mathcal{O}_{q,\mathrm{int}}$ be arbitrary and let $\mu, \nu \in P$. Owing to (5.33), we have

$$\begin{aligned} \mathrm{Ad}(\kappa)(E_i \otimes 1)|_{V_\mu \otimes W_\nu} &= \kappa(E_i \otimes 1)\kappa^{-1}|_{V_\mu \otimes W_\nu} \\ &= q^{-(\mu,\nu)} \kappa(E_i \otimes 1)|_{V_\mu \otimes W_\nu} \\ &= q^{(\mu+\alpha_i,\nu)-(\mu,\nu)}(E_i \otimes 1)|_{V_\mu \otimes W_\nu} \\ &= q^{(\alpha_i,\nu)}(E_i \otimes 1)|_{V_\mu \otimes W_\nu} \\ &= (E_i \otimes t_i)|_{V_\mu \otimes W_\nu}, \end{aligned}$$

as required. The computations for $F_i \otimes 1$, $1 \otimes E_i$ and $1 \otimes F_i$ are entirely similar. Finally, since $t_i^{\pm 1} \otimes 1$ and $1 \otimes t_i^{\pm 1}$ preserve the weight summands of objects in $\mathcal{O}_{q,\mathrm{int}}$, they are fixed by conjugation by $\kappa$. $\square$

The equations (6.10) immediately yield the desired intertwining property of $\kappa$:

**Proposition 6.3.** *We have $\mathrm{Ad}(\kappa) \circ \overline{\Delta} = \Delta^{\mathrm{op}}$.*

We continue our study of the element $\kappa$ with the following result:

**Proposition 6.4.** *We have $(\Delta \otimes \mathrm{id})(\kappa) = \kappa_{13}\kappa_{23}$.*

*Proof.* Let $U, V, W \in \mathcal{O}_{q,\mathrm{int}}$ and $\lambda, \mu, \nu \in P$. From the definition of the coproduct map $\widehat{U} \to \widehat{U}^{(2)}$ and the embedding $\widehat{U}^{(2)} \to \widehat{U}^{(3)}$ we obtain

$$\begin{aligned} (\Delta \otimes \mathrm{id})(\kappa)_{U,V,W}|_{U_\lambda \otimes V_\mu \otimes W_\nu} &= \kappa_{U \otimes V,W}|_{U_\lambda \otimes V_\mu \otimes W_\nu} \\ &= \text{multiplication by } q^{(\lambda+\mu,\nu)}|_{U_\lambda \otimes V_\mu \otimes W_\nu} \\ &= \text{multiplication by } q^{(\lambda,\nu)} q^{(\mu,\nu)}|_{U_\lambda \otimes V_\mu \otimes W_\nu} \\ &= (\kappa_{U,W})_{13}(\kappa_{V,W})_{23}|_{U_\lambda \otimes V_\mu \otimes W_\nu}, \end{aligned}$$

as required. Here we have used that $U_\lambda \otimes V_\mu \subseteq (U \otimes V)_{\lambda+\mu}$, which follows directly from the definition of weight space. $\square$

6.5. **The quasi R-matrix for $U_q\mathfrak{sl}_2$.** We also consider the algebra $\widehat{U}^+ := \prod_{\mu\in Q^+}(U_q\mathfrak{n}^+)_\mu$ and let $(X_\mu)_{\mu\in Q^+} \in \widehat{U}^+$. Note that for all $V \in \mathcal{O}_{q,\mathrm{int}}$ and all $v \in V$, there are only finitely many $\mu \in Q^+$ such that $X_\mu \cdot v$ is nonzero. Hence the expression

$$\sum_{\mu\in Q^+} X_\mu \cdot v \tag{6.11}$$

is well-defined. It can be checked that $(X_\mu)_{\mu\in Q^+}$ defines an element of $\widehat{U}$, so that we may consider $\widehat{U}^+$ as a subalgebra of $\widehat{U}$. Considering the inclusion $U_q\mathfrak{n}^+ \subseteq \widehat{U}^+$, it is safe to write elements of $\widehat{U}^+$ additively as $X = \sum_{\mu\in Q^+} X_\mu$.

Similarly, elements of the form

$$\sum_{\mu,\nu\in Q^+} Y_\nu \otimes X_\mu, \tag{6.12}$$

with $X_\mu \in (U_q\mathfrak{n}^+)_\mu$, $Y_\nu \in (U_q\mathfrak{n}^-)_{-\nu}$ have a well-defined action on $V\otimes W$ for all $V,W \in \mathcal{O}^+_{\mathrm{int}}$ (owing to the finiteness of the $U_q\mathfrak{n}^+$-action), and lie in $\widehat{U}^{(2)}$. The subalgebra of $\widehat{U}^{\otimes 2}$ generated by such elements is denoted $\widehat{U}^-\otimes\widehat{U}^+$. Note that we have analogous subalgebras $\widehat{U}^+\otimes\widehat{U}^+, \widehat{U}^+\otimes\widehat{U}^- \subset \widehat{U}^{(2)}$, but there is no corresponding subalgebra $\widehat{U}^-\otimes\widehat{U}^-$. Consider the subalgebra

$$\widehat{U}^{(2),\pm} := \langle U_q\mathfrak{g}\otimes U_q\mathfrak{g}, \widehat{U}^-\otimes\widehat{U}^+, \widehat{U}^+\otimes\widehat{U}^-, \kappa\rangle \subset \widehat{U}^{(2)}. \tag{6.13}$$

We note that we may extend $\omega\otimes\omega$ from an algebra automorphism of $U_q\mathfrak{g}\otimes U_q\mathfrak{g}$ to $\widehat{U}^{(2),\pm}$; we simply stipulate that $\omega\otimes\omega$ fix $\kappa$ and interchange $\widehat{U}^-\otimes\widehat{U}^+$ and $\widehat{U}^+\otimes\widehat{U}^-$ via

$$\sum_{\mu,\nu\in Q^+} c_{\mu,\nu} Y_\nu\otimes X_\mu \leftrightarrow \sum_{\mu,\nu\in Q^+} c_{\mu,\nu}\omega(Y_\nu)\otimes\omega(X_\mu). \tag{6.14}$$

This is consistent with the relations (6.10) and the natural relations involving series.

We now restrict to the $U_q\mathfrak{sl}_2$ case for simplicity, starting with the following result.

**Proposition 6.5.** *Up to a scalar, there is a unique invertible element in $\widehat{U}^-\otimes\widehat{U}^+$ such that*

$$\mathrm{Ad}(\widetilde{\mathcal{R}})\circ\Delta = \overline{\Delta}. \tag{6.15}$$

*It is given by*

$$\widetilde{\mathcal{R}} = \sum_{r=0}^{\infty} c_r (F\otimes E)^r \in \widetilde{U}^{(2)} \tag{6.16}$$

*with $c_r \in k(q^{1/d})$ such that*

$$c_{r+1} = \frac{q-q^{-1}}{[r+1]_q} q^r c_r, \qquad r\in\mathbb{Z}_{\geq 0}. \tag{6.17}$$

*Proof.* By the defining relations of $U_q\mathfrak{sl}_2$, applying (6.15) to the generator $t$ we infer that (6.16) holds true for some $c_r \in k(q^{1/d})$. It suffices to prove that

$$\widetilde{\mathcal{R}}\Delta(u) = \overline{\Delta}(u)\widetilde{\mathcal{R}} \qquad \text{for } u\in\{E,F\} \tag{6.18}$$

is equivalent to (6.17). By applying $\sigma\circ(\omega\otimes\omega)$ to (6.18) with $u=F$ we obtain (6.18) with $u=E$; this follows from the explicit form of $\widetilde{\mathcal{R}}$ and Lemma 6.1. It suffices therefore to prove that (6.18) with $u=E$ is equivalent to (6.17). Note that (6.18) with $u=E$ is equivalent to

$$\sum_{r\geq 0} c_r (F\otimes E)^r (E\otimes 1 + t\otimes E) = \sum_{r\geq 0} c_r (E\otimes 1 + t^{-1}\otimes E)(F\otimes E)^r, \tag{6.19}$$

i.e.

$$\sum_{r\geq 0} c_r (F^r t - t^{-1}F^r)\otimes E^{r+1} = \sum_{r\geq 0} c_{r+1}[E,F^{r+1}]\otimes E^{r+1}. \tag{6.20}$$

From (5.1) an induction argument with respect to $r\in\mathbb{Z}_{\geq 0}$ yields

$$[E,F^{r+1}] = [r+1]_q \frac{q^r t - q^{-r}t^{-1}}{q-q^{-1}} F^r. \tag{6.21}$$

It follows that (6.18) with $u = E$ is equivalent to

$$\sum_{r\geq 0} q^r c_r (q^r t - q^{-r}t^{-1}) F^r \otimes E^{r+1} = \sum_{r\geq 0} c_{r+1}[r+1]_q \frac{q^r t - q^{-r}t^{-1}}{q-q^{-1}} F^r \otimes E^{r+1}. \tag{6.22}$$

Since the elements $(q^r t - q^{-r}t^{-1})F^r \otimes E^{r+1}$ are linearly independent over $k(q^{1/d})$, it follows that (6.18) with $u = E$ is equivalent to (6.17). This clearly has an essentially unique solution $(c_r)$ in $k(q^{1/d})$. Note that each $c_r$ is nonzero if and only if $c_0$ is nonzero, which is equivalent to the power series $\sum_{r=0}^{\infty} c_r X^r$ being invertible. $\square$

We record the explicit solution of (6.17) for future use:

$$c_r = \frac{(q-q^{-1})^r}{[r]_q!} q^{r(r-1)/2} c_0 \qquad r \in \mathbb{Z}_{\geq 0} \tag{6.23}$$

where we have introduced the notation

$$[r]_q! := [r]_q [r-1]_q \cdots [2]_q [1]_q, \qquad r \in \mathbb{Z}_{\geq 0}. \tag{6.24}$$

**Proposition 6.6.** *The element $\widetilde{\mathcal{R}}$ with $c$ given by (6.23) satisfies*

$$(\Delta \otimes \mathrm{id})(\widetilde{\mathcal{R}}) = \mathrm{Ad}(\kappa_{23}^{-1})(\widetilde{\mathcal{R}}_{13})\widetilde{\mathcal{R}}_{23} \tag{6.25}$$

*if and only if $c_0 = 1$.*

*Proof.* For the right-hand side of (6.25) we have

$$\mathrm{Ad}(\kappa_{23}^{-1})(\widetilde{\mathcal{R}}_{13})\widetilde{\mathcal{R}}_{23} = \sum_{r,s\geq 0} c_r c_s \big(F \otimes \mathrm{Ad}(\kappa^{-1})(1 \otimes E)\big)^r (1 \otimes F^s \otimes E^s). \tag{6.26}$$

By (6.10) we obtain $\mathrm{Ad}(\kappa^{-1})(1\otimes E) = t^{-1} \otimes E$ so that

$$\begin{aligned} \mathrm{Ad}(\kappa_{23}^{-1})(\widetilde{\mathcal{R}}_{13})\widetilde{\mathcal{R}}_{23} &= \sum_{r,s\geq 0} c_r c_s F^r \otimes t^{-r} F^s \otimes E^{r+s} \\ &= \sum_{r\geq 0} \left( \sum_{s=0}^{r} c_r c_s F^{r-s} \otimes t^{s-r} F^s \right) \otimes E^r. \end{aligned} \tag{6.27}$$

For the left-hand side the following formula will be useful:

$$\Delta(F^r) = \sum_{s=0}^{r} q^{s(s-r)} \binom{r}{s}_q F^{r-s} \otimes t^{s-r} F^s \tag{6.28}$$

where for $r \in \mathbb{Z}_{\geq 0}$, $s \in \mathbb{Z}$, we have defined

$$\binom{r}{s}_q = \begin{cases} \frac{[r]_q!}{[s]_q![r-s]_q!} & \text{if } 0 \leq s \leq r, \\ 0 & \text{otherwise.} \end{cases} \tag{6.29}$$

The proof of (6.28) is an induction argument using the q-binomial identity

$$q^{s(s-r)} \binom{r}{s}_q + q^{(s+1)(s-r-1)} \binom{r}{s-1}_q = q^{s(s-r-1)} \binom{r+1}{s}_q. \tag{6.30}$$

We obtain that (6.25) is equivalent to

$$\sum_{r\geq 0} \left( \sum_{s=0}^{r} c_r q^{s(s-r)} \binom{r}{s}_q F^{s-r} \otimes t^{r-s} F^s \right) \otimes E^r = \sum_{r\geq 0} \left( \sum_{s=0}^{r} c_r c_s F^{s-r} \otimes t^{r-s} F^s \right) \otimes E^r. \tag{6.31}$$

By linear independence, we see that (6.25) is equivalent to

$$c_r q^{s(s-r)} \binom{r}{s}_q = c_r c_s. \tag{6.32}$$

Considering the explicit formula (6.23), this is true if and only if $c_0 = 1$. $\square$

6.6. **The universal R-matrix for $U_q\mathfrak{sl}_2$.** Now note that Propositions 6.4 and 6.6 combine to yield the desired coproduct result:

$$(\Delta \otimes \mathrm{id})(\mathcal{R}) = (\Delta \otimes \mathrm{id})(\kappa)(\Delta \otimes \mathrm{id})(\widetilde{\mathcal{R}}) = \kappa_{13}\kappa_{23}\mathrm{Ad}(\kappa_{23}^{-1})(\widetilde{\mathcal{R}}_{13})\widetilde{\mathcal{R}}_{23} = \kappa_{13}\widetilde{\mathcal{R}}_{13}\kappa_{23}\widetilde{\mathcal{R}}_{23} = \mathcal{R}_{13}\mathcal{R}_{23}. \tag{6.33}$$

Putting everything together, we have obtained the following main result, which establishes the quasitriangularity of $U_q\mathfrak{sl}_2$ up to completion:

**Theorem 6.7.** *For the Hopf algebra $U_q\mathfrak{sl}_2$, the invertible element*

$$\mathcal{R} := \kappa \sum_{r \geq 0} \frac{(q-q^{-1})^r}{[r]_q!} q^{r(r-1)/2} F^r \otimes E^r \in \widehat{U}^{(2)} \tag{6.34}$$

*satisfies the axioms (3.2-3.4) of a universal R-matrix.*

*Remark* 6.8. It follows that the larger algebra $\widehat{U}$ is quasitriangular up to completion as well. It is still not quasitriangular in the strict sense since $\mathcal{R} \notin \widehat{U} \otimes \widehat{U}$. $\varnothing$

A large range of matrix solutions to the Yang-Baxter equation (3.25) now arises naturally. Recall the $n$-dimensional representation $(\pi^{(n)}, V^{(n)})$ of $U_q(\mathfrak{sl}_2)$ defined in (5.8). By evaluating $(\pi^{(m)} \otimes \pi^{(n)})(\mathcal{R})$ for various $m, n$ we obtain linear maps on $V^{(m)} \otimes V^{(n)}$ which satisfy (3.25) in $V^{(l)} \otimes V^{(m)} \otimes V^{(n)}$ for various $l, m, n$. For instance, with respect to the basis $(v_1^{(2)}, v_2^{(2)})$ the 2-dimensional representation $\pi^{(2)}$ can be written as

$$\pi^{(2)}(E) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad \pi^{(2)}(F) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad \pi^{(2)}(t) = \begin{pmatrix} q & 0 \\ 0 & q^{-1} \end{pmatrix}. \tag{6.35}$$

With respect to the basis $(v_1^{(2)} \otimes v_1^{(2)}, v_1^{(2)} \otimes v_2^{(2)}, v_2^{(2)} \otimes v_1^{(2)}, v_2^{(2)} \otimes v_2^{(2)})$ of $V^{(2)} \otimes V^{(2)}$, we obtain

$$(\pi^{(2)} \otimes \pi^{(2)})(\kappa) = \begin{pmatrix} q^{1/2} & 0 & 0 & 0 \\ 0 & q^{-1/2} & 0 & 0 \\ 0 & 0 & q^{-1/2} & 0 \\ 0 & 0 & 0 & q^{1/2} \end{pmatrix} \qquad (\pi^{(2)} \otimes \pi^{(2)})(\widetilde{R}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & q-q^{-1} & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{6.36}$$

(in this case we have $d = 2$) and hence the following nontrivial solution of the Yang-Baxter equation:

$$R := (\pi^{(2)} \otimes \pi^{(2)})(\mathcal{R}) = q^{-1/2} \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & q-q^{-1} & 1 & 0 \\ 0 & 0 & 0 & q \end{pmatrix}. \tag{6.37}$$

6.7. **Generalization to other Drinfeld-Jimbo quantum groups.** The construction of $\widetilde{R}$ can be generalized, in a fairly explicit way, to arbitrary $U_q\mathfrak{g}$, which we now outline. Consider the quantum analogues $U_q\mathfrak{b}^\pm$ of the upper and lower Borel subalgebras, i.e. $U_q\mathfrak{b}^+ = \langle E_i, t_i, t_i^{-1} \,|\, i \in I \rangle$ and $U_q\mathfrak{b}^- = \langle F_i, t_i, t_i^{-1} \,|\, i \in I \rangle$. There exists a unique $k(q^{1/d})$-bilinear pairing

$$\langle \cdot, \cdot \rangle : U_q\mathfrak{b}^- \times U_q\mathfrak{b}^+ \to k(q^{1/d}) \tag{6.38}$$

such that

$$\begin{aligned} \langle Y, XX' \rangle &= \langle \Delta(Y), X' \otimes X \rangle & \langle YY', X \rangle &= \langle Y \otimes Y', \Delta(X) \rangle, \\ \langle t_i, t_j \rangle &= q^{-(\alpha_i, \alpha_j)} & \langle F_i, E_j \rangle &= \frac{\delta_{ij}}{q_i^{-1} - q_i}, \\ \langle t_i, E_j \rangle &= 0, & \langle F_i, t_j \rangle &= 0 \end{aligned} \tag{6.39}$$

for all $X, X' \in U_q\mathfrak{b}^+$, $Y, Y' \in U_q\mathfrak{b}^-$, $i, j \in I$, see e.g. [Lu94, Ch. 1]. Here we have defined the canonical extension of the pairing to $U_q\mathfrak{g}^{\otimes 2}$: $\langle a_1 \otimes a_2, b_1 \otimes b_2 \rangle = \langle a_1, b_1 \rangle \langle a_2, b_2 \rangle$. Then the restriction of the pairing to $(U_q\mathfrak{n}^-)_{-\nu} \times (U_q\mathfrak{n}^+)_\mu$ is nondegenerate if $\mu = \nu$ and vanishes otherwise.

Now, for arbitrary $\mu \in Q^+$, choose a basis $(Y_{\mu,r})_r$ for $(U_q\mathfrak{n}^-)_{-\mu}$ and let $(X_{\mu,r})_r$ be the basis of $(U_q\mathfrak{n}^+)_\mu$, dual with respect to the bilinear pairing $\langle \, , \, \rangle$. Now set

$$\widetilde{R} = \sum_{\mu \in Q^+} \widetilde{R}_\mu \in \widehat{U}^{(2)}, \qquad \widetilde{R}_\mu = \sum_r Y_{\mu,r} \otimes X_{\mu,r} \in U_q\mathfrak{n}^-_{-\mu} \otimes U_q\mathfrak{n}^+_\mu. \tag{6.40}$$

In other words, $\widetilde{R}$ is the "canonical element" of the pairing $\langle \, , \, \rangle$. One can then prove that $\widetilde{R}$ satisfies both (6.15) and (6.25) using Lusztig's formalism of left- and right skew derivations, certain linear maps $\ell_i, r_i$ :

$U_q\mathfrak{n}^- \to U_q\mathfrak{n}^-$, see [Lu94, Sec. 1.2 and 3.1]. The key point is that these skew derivations can be characterized in terms of relations involving elements of $U_q\mathfrak{n}^-$ and $E_i$, generalizing formula (6.21), in terms of the coproduct formulas for arbitrary elements of $U_q\mathfrak{n}^-$, generalizing formula (6.28), and in terms of the bilinear pairing $\langle\,,\,\rangle$, allowing us to connect with the definition of $\widetilde{R}$.

## References


[CP95] V. Chari, A.N. Pressley (1995), *A guide to quantum groups.* Cambridge University Press.

[Dr85] V. Drinfeld (1985), *Hopf algebras and the quantum Yang-Baxter equation.* Soviet Math. Dokl. **32**, 254–258.

[Ja96] J.C. Jantzen (1996), *Lectures on quantum groups.* Grad. Stud. Math., vol. 6, Amer. Math. Soc.

[Ji86] M. Jimbo (1986), *A q-analogue of $U(\mathfrak{gl}(N+1))$, Hecke algebra and the Yang-Baxter equation.* Lett. Math. Phys. **11** (1986), no. 3, 247–252.

[KR83] P.P. Kulish, N.Yu. Reshetikhin (1983), *Quantum linear problem for the sine-Gordon equation and higher representations.* J. Sov. Math. **23**, 2345–2441.

[Lu94] G. Lusztig (1994), *Introduction to quantum groups.* Birkhäuser, Boston.

[RT90] N.Yu. Reshetikhin, V.G. Turaev (1990), *Ribbon graphs and their invariants derived from quantum groups.* Commun. Math. Phys. **127**, 1–26.

[Sw69] M.E. Sweedler (1969), *Hopf algebras*, Benjamin, New York.

[Ta92] T. Tanisaki (1992), *Killing forms, Harish-Chandra isomorphisms, and universal R-matrices for quantum algebras.* Int. J. Mod. Phys. A **7**, no. supp01b, 941–961.